Controllability of the linear 1D wave equation with inner moving forces

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Sevilla, January 15, 2014

joint work with Carlos Castro (Madrid) and Nicolae Cîndea (Clermont-Ferrand)
Problem statement

\[ QT = (0, 1) \times (0, T), \quad q_T \subset Q_T, \quad V := H_0^1(0, 1) \times L^2(0, 1), \quad a, b \in C([0, T], [0, 1]) \]

\[
\begin{cases}
  y_{tt} - y_{xx} = v_1 q_T, \\
  y = 0, \\
  (y(\cdot, 0), y_t(\cdot, 0)) = (y_0, y_1) \in V,
\end{cases}
\]

\[(x, t) \in Q_T \quad (x, t) \in \partial \Omega \times (0, T) \quad x \in (0, 1).\]

\[ q_T = \left\{ (x, t) \in Q_T; \ a(t) < x < b(t), \ t \in (0, T) \right\} \]

Goals of the works -

- For some \( T > 0 \) and \( q_T \), prove the existence of uniform null \( L^2(q_T) \)-controls.
- Approximate numerically the control of minimal \( L^2(q_T) \)-norm.

Dependent domains \( q_T \) included in \( Q_T \).
Problem statement

\[ Q_T = (0, 1) \times (0, T), \quad q_T \subset Q_T, \quad V := H^1_0(0, 1) \times L^2(0, 1), \quad a, b \in C([0, T], ]0, 1[) \]

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\begin{cases}
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  (x, t) \in \partial \Omega \times (0, T)
\end{cases}
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\[ x \in (0, 1). \]

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Combination of two works

This contribution is a combination of two recent works:

- C. Castro: Exact controllability of the 1D wave equation from a moving interior point, COCV - 2013
  \[
  \begin{cases}
  y_{tt} - y_{xx} = v(x, t) \mathbf{1}_{x=\gamma(t)}, & (x, t) \in Q_T, \\
  \gamma \in C^1([0, T], (0, 1)), & 0 < |\gamma'(t)| < 1, t \in (0, T).
  \end{cases}
  \]
  Existence of $H^{-1}(\cup_{t \in (0, T)} \gamma(t) \times (0, T))$ null controls for $(y_0, y_1) \in L^2(0, 1) \times H^{-1}(0, 1)$, $T > 2$

- N. Cîndea and AM: A mixed formulation for the direct approximations of the control of minimal $L^2$-norm for linear type wave equations
  \[
  y_{tt} - (a(x)y_x)_x + b(x, t)y = v1_\omega, \quad (x, t) \in Q_T
  \]
  Robust numerical approximation of the control of minimal $L^2(\omega \times (0, T))$-norm using a space-time formulation, well-adapted to our non cylindrical case.
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\begin{align*}
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\end{align*}
\]

Existence of \( H^{-1}(\bigcup_{t \in (0, T)} \gamma(t) \times (0, T)) \) null controls for \((y_0, y_1) \in L^2(0, 1) \times H^{-1}(0, 1), \ T > 2\)

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Generalized Observability inequality

$q_T$-non-cylindrical domain, $L\varphi = \varphi_{tt} - \varphi_{xx}$. We define the Hilbert space ("by completion")

$$\Phi = \left\{ \varphi : \varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi \in L^2(0, T; H^{-1}(0, 1)) \right\}.$$ 

endowed, for any $\eta > 0$, with the following inner product

$$(\varphi, \overline{\varphi})_\Phi = \iint_{q_T} \varphi(x, t)\overline{\varphi}(x, t) \, dx \, dt + \eta \int_0^T <L\varphi, L\overline{\varphi}>_{H^{-1}(0, 1), H^{-1}(0, 1)} \, dt,$$

**Proposition (Castro, Cîndea, Münch)**

Assume that $T > 2$ and $q_T$ contains a $C^1$-curve $\gamma : [0, T] \to (0, 1)$ such that

1. $\gamma(t) \in (a(t), b(t)) \forall t \in [0, T]$, i.e. $\gamma \subset q_T$
2. $0 < |\gamma'(t)| < 1 \forall t \in [0, T]$.

Set $H = L^2(0, 1) \times H^{-1}(0, 1)$. There exists $C > 0$ such that

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2 \leq C \left( \|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0, T; H^{-1}(0, 1))} \right), \quad \forall \varphi \in \Phi. \quad (1)$$

Arnaud Münch  Controllability of the linear 1D wave equation with inner moving forces
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**Proposition** (Castro, Cîndea, Münch)

Assume that \( T > 2 \) and \( q_T \) contains a \( C^1 \)-curve \( \gamma : [0, T] \to (0, 1) \) such that

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Set \( H = L^2(0, 1) \times H^{-1}(0, 1) \). There exists \( C > 0 \) such that

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\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2 \leq C \left( \|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0, T; H^{-1}(0, 1))}^2 \right), \quad \forall \varphi \in \Phi.
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Set $H = L^2(0, 1) \times H^{-1}(0, 1)$. There exists $C > 0$ such that

$$||\varphi(\cdot, 0), \varphi_t(\cdot, 0)||_H^2 \leq C \left(||\varphi||_{L^2(q_T)}^2 + ||L\varphi||_{L^2(0, T; H^{-1}(0, 1))}^2\right), \quad \forall \varphi \in \Phi.$$ (1)
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$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2 \leq C \left( \|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0, T; H^{-1}(0, 1))}^2 \right), \quad \forall \varphi \in \Phi. \quad (1)$$
Set $W = \{ \varphi : \varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T \text{ such that } L\varphi = 0 \in L^2(0, T; H^{-1}(0, 1)) \}$. $W \subset \Phi$.

**Step 1:** We write an observability inequality for initial data in $V$, when the observation is taken on the curve $\gamma \subset q_T$ and $L\varphi = 0$. For $T > 2$, the following inequality is proved in [Castro, 2013]:

$$\exists C > 0 : \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_V \leq C \int_0^T \| \frac{d}{dt} \varphi(\gamma(t), t) \|^2 dt, \quad \forall \varphi \in W. \quad (2)$$
Step 2. We extend the observation in (2) from $\gamma$ to $q_T$. More precisely, we show that for some constant $C > 0$,

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq C \left( \|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2 \right),$$

for any $\varphi \in W$ and initial data in $V$.

Let us consider $\delta_0 > 0$ small enough such that $\gamma(t) + \delta_0 \in (a(t), b(t))$ for all $t \in [0, T]$. In this case, we can define small translations of the curve $\gamma$, i.e. $\gamma_\delta = \gamma + \delta$ in such a way that $\gamma_\delta \subset q_T$ for all $\delta < \delta_0$. $\gamma_\delta : [0, T] \to (0, 1)$ satisfies the same properties stated for $\gamma$ in the Step 1 and (2) holds for all such curves with the same constant. In particular, we have

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq \frac{C}{2\delta_0} \int_{-\delta_0}^{\delta_0} \int_0^T \| d\varphi(\gamma(t) + \delta, t) \|^2 dt d\delta$$

$$\leq \frac{C}{2\delta_0} \int_{q_T} \|\varphi_t(x, t) + \gamma'(t)\varphi_x(x, t)\|^2 dx dt$$

$$\leq \frac{C}{2\delta_0} \left( 1 + \max_{t \in [0, T]} |\gamma'(t)|^2 \right) \left( \|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2 \right).$$
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$$

(3)

for any $\varphi \in W$ and initial data in $V$.

Let us consider $\delta_0 > 0$ small enough such that $\gamma(t) + \delta_0 \in (a(t), b(t))$ for all $t \in [0, T]$. In this case, we can define small translations of the curve $\gamma$, i.e. $\gamma_\delta = \gamma + \delta$ in such a way that $\gamma_\delta \subset q_T$ for all $\delta < \delta_0$. $\gamma_\delta : [0, T] \to (0, 1)$ satisfies the same properties stated for $\gamma$ in the Step 1 and (2) holds for all such curves with the same constant. In particular, we have

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\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq \frac{C}{2\delta_0} \int_{-\delta_0}^{\delta_0} \int_0^T \|\frac{d}{dt}\varphi(\gamma(t) + \delta, t)\|^2 dt \, d\delta
$$

$$
\leq \frac{C}{2\delta_0} \int_{q_T} \|\varphi_t(x, t) + \gamma'(t)\varphi_x(x, t)\|^2 dx \, dt
$$

$$
\leq \frac{C}{2\delta_0} (1 + \max_{t \in [0, T]} |\gamma'(t)|^2) \left( \|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi_x\|_{L^2(q_T)}^2 \right).
$$
Step 3. We show that we can substitute \( \varphi_x \) by \( \varphi \) in the right hand side of (3), i.e.

\[
\| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_{V}^2 \leq C \left( \| \varphi_t \|_{L^2(q_T)}^2 + \| \varphi \|_{L^2(q_T)}^2 \right),
\]

for any \( \varphi \in W \) and initial data in \( V \).

This requires to extend slightly the observation zone \( q_T \). Instead, we first argue that (3) must hold for a slightly smaller open set. Let \( \varepsilon > 0 \) small enough so that \( T - 2\varepsilon > 2 \) and it exists \( \tilde{q}_T \) defined as

\[
\tilde{q}_T = \left\{ (x, t) \in Q_T; \tilde{a}(t) < x < \tilde{b}(t), \ t \in (\varepsilon, T - \varepsilon) \right\}
\]

with \( (\gamma(t) - \delta_0, \gamma(t) + \delta_0) \subset (\tilde{a}(t) - \varepsilon, \tilde{b}(t) + \varepsilon) \subset (a(t), b(t)) \) for all \( t \in [0, T] \). Therefore, (3) holds when considering \( \tilde{q}_T \) instead of \( q_T \). Now we introduce

\[
\eta(x, t) = \begin{cases} \frac{t(T - t)(x - a(t))^2(x - b(t))^2}{2}, & \text{if } (x, t) \in q_T \\ 0, & \text{otherwise.} \end{cases}
\]

Obviously, \( \eta \in C^1 \) is supported in \( q_T \) and there exists a constant \( C_1 \) such that

\[
\| \eta_t \|_{L^\infty} \leq C_1, \| \eta_x^2 / \eta \| \leq C_1.
\]

Moreover \( \eta > 0 \) and it is uniformly bounded below by a constant \( C_2 > 0 \) in \( \tilde{q}_T \).
Step 3. We show that we can substitute $\varphi_x$ by $\varphi$ in the right hand side of (3), i.e.

$$
\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq C \left( \|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi\|_{L^2(q_T)}^2 \right), \tag{4}
$$

for any $\varphi \in W$ and initial data in $V$.
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$$
\tilde{q}_T = \left\{ (x, t) \in Q_T; \tilde{a}(t) < x < \tilde{b}(t), \ t \in (\varepsilon, T - \varepsilon) \right\}
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with $(\gamma(t) - \delta_0, \gamma(t) + \delta_0) \subset (\tilde{a}(t) - \varepsilon, \tilde{b}(t) + \varepsilon) \subset (a(t), b(t))$ for all $t \in [0, T]$. Therefore, (3) holds when considering $\tilde{q}_T$ instead of $q_T$. Now we introduce

$$
\eta(x, t) = \left\{ \begin{array}{ll}
t(T - t)(x - a(t))^2(x - b(t))^2, & \text{if } (x, t) \in q_T \\
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\end{array} \right.
$$

Obviously, $\eta \in C^1$ is supported in $q_T$ and there exists a constant $C_1$ such that $\|\eta_t\|_{L^\infty} \leq C_1$, $\|\eta_x^2/\eta\| \leq C_1$. Moreover $\eta > 0$ and it is uniformly bounded below by a constant $C_2 > 0$ in $\tilde{q}_T$. 
Proof

Multiplying the equation of \( \varphi \) by \( \eta \varphi \) and integrating by parts we easily obtain

\[
\iint_{q_T} \eta |\varphi_x|^2 \, dx \, dt = \iint_{q_T} \eta |\varphi_t|^2 \, dx \, dt + \iint_{q_T} (\eta_t \varphi \varphi_t - \eta_x \varphi \varphi_x) \, dx \, dt
\]
\[
\leq \iint_{q_T} \eta |\varphi_t|^2 \, dx dt + \frac{\|\eta_t\|_{L^\infty(q_T)}}{2} \iint_{q_T} (|\varphi|^2 + |\varphi_t|) \, dx \, dt
\]
\[
+ \frac{1}{2} \iint_{q_T} \left( \frac{\eta_x^2}{\eta} \varphi^2 + \eta \varphi_x^2 \right) \, dx \, dt.
\]

Therefore,

\[
\iint_{q_T} \eta |\varphi_x|^2 \, dx \, dt \leq C \iint_{q_T} (|\varphi_t|^2 + |\varphi|^2) \, dx \, dt,
\]

for some constant \( C > 0 \), and we obtain

\[
\| \varphi_x \|_{L^2(\tilde{q}_T)}^2 \leq C_2^{-1} \iint_{q_T} \eta |\varphi_x|^2 \, dx \, dt \leq C_2^{-1} C \iint_{q_T} (|\varphi_t|^2 + |\varphi|^2) \, dx \, dt.
\]

This combined with (3) for \( \tilde{q}_T \) provides (4).
Step 4. Here we prove that we can remove the second term in the right hand side of (4), i.e.

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq C\|\varphi_t\|_{L^2(q_T)}^2,$$  \hspace{1cm} (5)

for any $\varphi \in W$ and initial data in $V$. 

Note that, for each time $t \in [0, T]$ and each $\omega \subset \Omega$ we have the following regularity estimate

$$\int_{a(t)}^{b(t)} |\varphi(x, t)|^2 dx \leq \|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2,$$  \hspace{1cm} for all $t \in [0, T]$.

Therefore, integrating in time, we obtain

$$\|\varphi\|_{L^2(q_T)}^2 \leq T\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2.$$

We now substitute this inequality in (4)

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_V^2 \leq C \left(\|\varphi_t\|_{L^2(q_T)}^2 + \|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2\right).$$

Inequality (5) is finally obtained by contradiction. Assume that it is not true. Then, there exists a sequence $(\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0))_{k>0} \in V$ such that

$$\|\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)\|_V^2 = 1, \hspace{0.5cm} \forall k > 0, \hspace{0.5cm} \|\varphi^k_t\|_{L^2(q_T)}^2 \to 0, \text{ as } k \to \infty.$$

There exists a subsequence such that $(\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)) \rightharpoonup (\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0))$ weakly in $V$ and strongly in $H$. Passing to the limit in the equation we see that the solution associated to $(\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0))$, $\varphi^*$ must vanish at $q_T$ and therefore, by (4), $\varphi^* = 0$. 

Arnaud Münch

Controllability of the linear 1D wave equation with inner moving forces
Step 4. Here we prove that we can remove the second term in the right hand side of (4), i.e.

\[ \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_V \leq C \| \varphi_t \|^2_{L^2(q_T)}, \tag{5} \]

for any \( \varphi \in W \) and initial data in \( V \).

Note that, for each time \( t \in [0, T] \) and each \( \omega \subset \Omega \) we have the following regularity estimate

\[ \int_{a(t)}^{b(t)} |\varphi(x, t)|^2 \, dx \leq \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_H, \quad \text{for all } t \in [0, T] \]

Therefore, integrating in time, we obtain

\[ \| \varphi \|^2_{L^2(q_T)} \leq T \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_H. \]

We now substitute this inequality in (4)

\[ \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_V \leq C \left( \| \varphi_t \|^2_{L^2(q_T)} + \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|^2_H \right). \]

Inequality (5) is finally obtained by contradiction. Assume that it is not true. Then, there exists a sequence \( (\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)) \) \( k > 0 \) \( \in V \) such that

\[ \| \varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0) \|^2_V = 1, \quad \forall k > 0, \quad \| \varphi^k_t \|^2_{L^2(q_T)} \to 0, \text{ as } k \to \infty. \]

There exists a subsequence such that \( (\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)) \to (\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0)) \) weakly in \( V \) and strongly in \( H \). Passing to the limit in the equation we see that the solution associated to \( (\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0)) \), \( \varphi^* \) must vanish at \( q_T \) and therefore, by (4), \( \varphi^* = 0 \).

Arnaud Münch
Controllability of the linear 1D wave equation with inner moving forces
**Step 4.** Here we prove that we can remove the second term in the right hand side of (4), i.e.

\[ \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_V^2 \leq C \| \varphi_t \|_{L^2(q_T)}^2, \tag{5} \]

for any \( \varphi \in W \) and initial data in \( V \).

Note that, for each time \( t \in [0, T] \) and each \( \omega \subset \Omega \) we have the following regularity estimate

\[ \int_{a(t)}^{b(t)} |\varphi(x, t)|^2 dx \leq \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_H^2, \quad \text{for all } t \in [0, T] \]

Therefore, integrating in time, we obtain

\[ \| \varphi \|_{L^2(q_T)}^2 \leq T \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_H^2. \]

We now substitute this inequality in (4)

\[ \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_V^2 \leq C \left( \| \varphi_t \|_{L^2(q_T)}^2 + \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_H^2 \right). \]

Inequality (5) is finally obtained by contradiction. Assume that it is not true. Then, there exists a sequence \((\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)))_{k>0} \in V \) such that

\[ \| \varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0) \|_V^2 = 1, \quad \forall k > 0, \quad \| \varphi^k_t \|_{L^2(q_T)}^2 \to 0, \text{ as } k \to \infty. \]

There exists a subsequence such that \((\varphi^k(\cdot, 0), \varphi^k_t(\cdot, 0)) \to (\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0))\) weakly in \( V \) and strongly in \( H \). Passing to the limit in the equation we see that the solution associated to \((\varphi^*(\cdot, 0), \varphi^*_t(\cdot, 0))\), \( \varphi^* \) must vanish at \( q_T \) and therefore, by (4), \( \varphi^* = 0. \)
Step 5. We now write (5) with respect to the weaker norm. In particular, we obtain

\[ \| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \|_{H^2}^2 \leq C \| \varphi \|_{L^2(q_T)}^2, \]  

for any $\varphi \in \Phi$ with $L \varphi = 0$.

Let $\eta \in \Phi$ be defined by $\eta(x, t) = \eta(x, 0) + \int_0^t \varphi(x, s) \, ds$, for all $(x, t) \in Q_T$ such that

\[ (\eta(\cdot, 0), \eta_t(\cdot, 0)) = (\Delta^{-1} \varphi_t(\cdot, 0), \varphi(\cdot, 0)) \in V \]

where $\Delta$ designates the Dirichlet Laplacian in $(0, 1)$. Then $L \eta = 0$ in $Q_T$.

Then, inequality (5) on $\eta$ and the fact that $\Delta$ is an isomorphism from $H^1_0(0, 1)$ to $L^2(0, 1)$, provide

\[
\left\| \varphi(\cdot, 0), \varphi_t(\cdot, 0) \right\|_{H^2}^2 = \left\| \Delta^{-1} \varphi_t(\cdot, 0), \varphi(\cdot, 0) \right\|_V^2 \\
= \left\| (\eta(\cdot, 0), \eta_t(\cdot, 0)) \right\|_V^2 \\
\leq C \left\| \eta_t \right\|_{L^2(q_T)}^2 = C \| \varphi \|_{L^2(q_T)}^2.
\]
Proof

Step 5. We now write (5) with respect to the weaker norm. In particular, we obtain

\[ \|\varphi(\cdot,0), \varphi_t(\cdot,0)\|^2_{H} \leq C\|\varphi\|^2_{L^2(q_T)}, \]  

(6)

for any \( \varphi \in \Phi \) with \( L\varphi = 0 \).

Let \( \eta \in \Phi \) be defined by \( \eta(x,t) = \eta(x,0) + \int_0^t \varphi(x,s) \, ds \), for all \((x,t) \in Q_T\) such that

\[ (\eta(\cdot,0), \eta_t(\cdot,0)) = (\Delta^{-1}\varphi_t(\cdot,0), \varphi(\cdot,0)) \in V \]

where \( \Delta \) designates the Dirichlet Laplacian in \((0, 1)\). Then \( L\eta = 0 \) in \( Q_T \).

Then, inequality (5) on \( \eta \) and the fact that \( \Delta \) is an isomorphism from \( H^1_0(0, 1) \) to \( L^2(0, 1) \), provide

\[ \|\varphi(\cdot,0), \varphi_t(\cdot,0)\|^2_{H} = \|\Delta^{-1}\varphi_t(\cdot,0), \varphi(\cdot,0)\|^2_{V} \]
\[ = \|\eta(\cdot,0), \eta_t(\cdot,0)\|^2_{V} \]
\[ \leq C\|\eta_t\|^2_{L^2(q_T)} = C\|\varphi\|^2_{L^2(q_T)}. \]
Step 6. Here we finally obtain (1). Given $\varphi \in \Phi$ we can decompose it as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2 \in \Phi$ solve

$$\begin{cases} L\varphi_1 = L\varphi, \\ \varphi_1(\cdot, 0) = (\varphi_1)_t(\cdot, 0) = 0 \end{cases} \quad \begin{cases} L\varphi_2 = 0, \\ \varphi_2(\cdot, 0) = \varphi(\cdot, 0), \quad (\varphi_2)_t(\cdot, 0) = \varphi_t(\cdot, 0). \end{cases}$$

From Duhamel's principle, we can write

$$\varphi_1(\cdot, t) = \int_0^t \psi(\cdot, t - s, s)ds$$

where $\psi(x, t, s)$ solves, for each value of the parameter $s \in (0, t),$

$$\begin{cases} L\psi(\cdot, \cdot, s) = 0, \\ \psi(\cdot, 0, s) = 0, \quad \psi_t(\cdot, 0, s) = L\varphi(\cdot, s). \end{cases}$$

Therefore,

$$\|\varphi_1\|^2_{L^2(q_T)} \leq \int_0^T \|\psi(\cdot, \cdot, s)\|^2_{L^2(q_T)}ds \leq C \int_0^T \|\psi(\cdot, 0, s), \psi_t(\cdot, 0, s)\|^2_Hds$$

$$\leq C\|L\varphi\|^2_{L^2(0, T; H^{-1}(0, 1))} \quad (7)$$

Combining (7) and estimate (6) for $\varphi_2$ we obtain

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|^2_H = \|\varphi_2(\cdot, 0), (\varphi_2)_t(\cdot, 0)\|^2_H \leq C\|\varphi_2\|^2_{L^2(q_T)}$$

$$\leq C \left(\|\varphi\|^2_{L^2(q_T)} + \|\varphi_1\|^2_{L^2(q_T)}\right) \leq C \left(\|\varphi\|^2_{L^2(q_T)} + \|L\varphi\|^2_{L^2(0, T; H^{-1})}\right).$$
**Step 6.** Here we finally obtain (1). Given \( \varphi \in \Phi \) we can decompose it as \( \varphi = \varphi_1 + \varphi_2 \) where \( \varphi_1, \varphi_2 \in \Phi \) solve

\[
\begin{align*}
L \varphi_1 &= L \varphi, \\
\varphi_1(\cdot, 0) &= (\varphi_1)_t(\cdot, 0) = 0
\end{align*}
\]

and

\[
\begin{align*}
L \varphi_2 &= 0, \\
\varphi_2(\cdot, 0) &= \varphi(\cdot, 0), \\
(\varphi_2)_t(\cdot, 0) &= \varphi_t(\cdot, 0).
\end{align*}
\]

From Duhamel's principle, we can write

\[
\varphi_1(\cdot, t) = \int_0^t \psi(\cdot, t-s, s) ds
\]

where \( \psi(x, t, s) \) solves, for each value of the parameter \( s \in (0, t) \),

\[
\begin{align*}
L \psi(\cdot, \cdot, s) &= 0, \\
\psi(\cdot, 0, s) &= \psi(\cdot, 0), \\
\psi_t(\cdot, 0, s) &= L \varphi(\cdot, s).
\end{align*}
\]

Therefore,

\[
\|\varphi_1\|_{L^2(qT)}^2 \leq \int_0^T \|\psi(\cdot, \cdot, s)\|_{L^2(qT)}^2 ds \leq C \int_0^T \|\psi(\cdot, 0, s), \psi_t(\cdot, 0, s)\|_{\mathcal{H}}^2 ds
\]

\[
\leq C \|L \varphi\|_{L^2(0, T; H^{-1})(0,1)}^2
\]

Combining (7) and estimate (6) for \( \varphi_2 \) we obtain

\[
\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_{\mathcal{H}}^2 = \|\varphi_2(\cdot, 0), (\varphi_2)_t(\cdot, 0)\|_{\mathcal{H}}^2 \leq C \|\varphi_2\|_{L^2(qT)}^2
\]

\[
\leq C \left( \|\varphi\|_{L^2(qT)}^2 + \|\varphi_1\|_{L^2(qT)}^2 \right) \leq C \left( \|\varphi\|_{L^2(qT)}^2 + \|L \varphi\|_{L^2(0, T; H^{-1})}^2 \right).
\]
Step 6. Here we finally obtain (1). Given $\varphi \in \Phi$ we can decompose it as $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2 \in \Phi$ solve

$$
\begin{cases}
L\varphi_1 = L\varphi, \\
\varphi_1(\cdot, 0) = (\varphi_1)_t(\cdot, 0) = 0
\end{cases}
\quad
\begin{cases}
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(\varphi_2)_t(\cdot, 0) = \varphi_t(\cdot, 0).
\end{cases}
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$$
\begin{cases}
L\psi(\cdot, \cdot, s) = 0, \\
\psi(\cdot, 0, s) = 0, \\
\psi_t(\cdot, 0, s) = L\varphi(\cdot, s).
\end{cases}
$$

Therefore,

$$
\|\varphi_1\|^2_{L^2(qT)} \leq \int_0^T \|\psi(\cdot, \cdot, s)\|^2_{L^2(qT)}ds \leq C \int_0^T \|\psi(\cdot, 0, s), \psi_t(\cdot, 0, s)\|^2_H ds
$$

$$
\leq C\|L\varphi\|^2_{L^2(0,T;H^{-1}(0,1))}
$$

Combining (7) and estimate (6) for $\varphi_2$ we obtain

$$
\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|^2_H = \|\varphi_2(\cdot, 0), (\varphi_2)_t(\cdot, 0)\|^2_H \leq C\|\varphi_2\|^2_{L^2(qT)}
$$

$$
\leq C\left(\|\varphi\|^2_{L^2(qT)} + \|\varphi_1\|^2_{L^2(qT)}\right) \leq C\left(\|\varphi\|^2_{L^2(qT)} + \|L\varphi\|^2_{L^2(0,T;H^{-1})}\right).
$$
**Theorem** (Castro, Cîndea, Münch)

Set $H = L^2(0, 1) \times H^{-1}(0, 1)$. Let $T > 0$. Assume that $q_T$ satisfies the geometric optic condition. Then, there exists $C > 0$ such that

$$\|\varphi(\cdot, 0), \varphi_t(\cdot, 0)\|_H^2 \leq C \left( \|\varphi\|_{L^2(q_T)}^2 + \|L\varphi\|_{L^2(0, T; H^{-1}(0, 1))}^2 \right), \quad \forall \varphi \in \Phi.$$
Control of minimal $L^2$-norm: a mixed formulation

$$\min_{(\varphi_0, \varphi_1) \in H} J^*(\varphi_0, \varphi_1) = \frac{1}{2} \int_0^1 \int_{q_T} |\varphi|^2 \, dx \, dt + \langle \varphi_1, y_0 \rangle_{H^{-1}(0,1), H^1_0(0,1)} - \int_0^1 \varphi_0 \, y_1 \, dx.$$ 

where $L \varphi = 0$ in $Q_T$; $\varphi = 0$ on $\Sigma_T$, $(\varphi, \varphi_t)(\cdot, 0) = (\varphi_0, \varphi_1)$ and

$$\langle \varphi_1, y_0 \rangle_{H^{-1}(0,1), H^1_0(0,1)} = \int_0^1 \partial_x ((-\Delta)^{-1} \varphi_1)(x) \partial_x y_0(x) \, dx$$

where $-\Delta$ is the Dirichlet Laplacian in $(0, 1)$.

Since the variable $\varphi$ is completely and uniquely determined by $(\varphi_0, \varphi_1)$, the idea of the reformulation is to keep $\varphi$ as variable and consider the following extremal problem:

$$\min_{\varphi \in W} \hat{J}^*(\varphi) = \frac{1}{2} \int_0^1 \int_{q_T} |\varphi|^2 \, dx \, dt + \langle \varphi_t(\cdot, 0), y_0 \rangle_{H^{-1}(0,1), H^1_0(0,1)} - \int_0^1 \varphi(\cdot, 0) \, y_1 \, dx,$$

$$W = \left\{ \varphi : \varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T, L \varphi = 0 \in L^2(0, T; H^{-1}(0,1)) \right\}.$$

From (1), the property $\varphi \in W$ implies that $(\varphi(\cdot, 0), \varphi_t(\cdot, 0)) \in H$, so that the functional $\hat{J}^*$ is well-defined over $W$. (8)
Control of minimal $L^2$-norm: a mixed formulation

$$
\min_{(\varphi_0, \varphi_1) \in H} J^*(\varphi_0, \varphi_1) = \frac{1}{2} \iint_{q_T} |\varphi|^2 \, dx \, dt + <\varphi_1, y_0 >_{H^{-1}(0,1), H^1_0(0,1)} - \int_0^1 \varphi_0 y_1 \, dx.
$$

where $L \varphi = 0$ in $Q_T$; $\varphi = 0$ on $\Sigma_T$, $(\varphi, \varphi_t)(\cdot, 0) = (\varphi_0, \varphi_1)$ and

$$
<\varphi_1, y_0 >_{H^{-1}(0,1), H^1_0(0,1)} = \int_0^1 \partial_x ((-\Delta)^{-1} \varphi_1)(x) \partial_x y_0(x) \, dx
$$

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$$

$W = \left\{ \varphi : \varphi \in L^2(q_T), \varphi = 0 \text{ on } \Sigma_T, L \varphi = 0 \in L^2(0, T; H^{-1}(0,1)) \right\}$.

From (1), the property $\varphi \in W$ implies that $(\varphi(\cdot, 0), \varphi_t(\cdot, 0)) \in H$, so that the functional $\hat{J}^*$ is well-defined over $W$. 

(8)
The main variable is now \( \varphi \) submitted to the constraint equality \( L\varphi = 0 \) as an \( L^2(0, T; H^{-1}(0, 1)) \) function. This constraint is addressed introducing a Lagrangian multiplier \( \lambda \in L^2(0, T; H^1_0(\Omega)) \):

We consider the following problem: find \( (\varphi, \lambda) \in \Phi \times L^2(0, T; H^1_0(0, 1)) \) solution of

\[
\begin{align*}
  a_r(\varphi, \bar{\varphi}) + b(\varphi, \lambda) &= l(\varphi), \quad \forall \varphi \in \Phi \\
  b(\varphi, \bar{\lambda}) &= 0, \quad \forall \bar{\lambda} \in L^2(0, T; H^1_0(0, 1)),
\end{align*}
\]

where \( (r \geq 0 - \text{augmentation parameter}) \)

\[
\begin{align*}
  a_r : \Phi \times \Phi &\rightarrow \mathbb{R}, \quad a_r(\varphi, \bar{\varphi}) = \int\int_{Q_T} \varphi \bar{\varphi} \,dx \,dt + r \int_0^T <L\varphi, L\bar{\varphi}>_{H^{-1},H^{-1}} \,dt \\
  b : \Phi \times L^2(0, T; H^1_0(0, 1)) &\rightarrow \mathbb{R}, \quad b(\varphi, \lambda) = \int_0^T <L\varphi, \lambda>_{H^{-1}(0,1),H^1_0(0,1)} \,dt \\
  &\quad = \int\int_{Q_T} \partial_x (-\Delta^{-1}(L\varphi)) \cdot \partial_x \lambda \,dx \,dt \\
  l : \Phi &\rightarrow \mathbb{R}, \quad l(\varphi) = -<\varphi_t(\cdot,0), y_0>_{H^{-1}(0,1),H^1_0(0,1)} + \int_0^1 \varphi(\cdot, 0) \,y_1 \,dx.
\end{align*}
\]
Theorem

1. The mixed formulation (9) is well-posed.
2. The unique solution \((\varphi, \lambda) \in \Phi \times L^2(0, T; H^1_0(0, 1))\) is the unique saddle-point of the Lagrangian \(\mathcal{L} : \Phi \times L^2(0, T; H^1_0(0, 1)) \to \mathbb{R}\) defined by

\[
\mathcal{L}(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi).
\]

3. The optimal function \(\varphi\) is the minimizer of \(\hat{J}^*\) over \(\Phi\) while the optimal function \(\lambda \in L^2(0, T; H^1_0(0, 1))\) is the state of the controlled wave equation in the weak sense (associated to the control \(-\varphi 1_{q_T}\)).

The well-posedness of the mixed formulation is a consequence of two properties [FORTIN-BREZZI’91]:

- \(a\) is coercive on \(\text{Ker}(b) = \{\varphi \in \Phi \text{ such that } b(\varphi, \lambda) = 0 \text{ for every } \lambda \in L^2(0, T; H^1_0(0, 1))\}\).
- \(b\) satisfies the usual "inf-sup" condition over \(\Phi \times L^2(0, T; H^1_0(0, 1))\): there exists \(\delta > 0\) such that

\[
\inf_{\lambda \in L^2(0, T; H^1_0(0, 1))} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi} \|\lambda\|_{L^2(0, T; H^1_0(0, 1))}} \geq \delta. \quad (10)
\]
Well-posedness of the mixed formulation

**Theorem**

1. **The mixed formulation** (9) **is well-posed.**
2. The unique solution $(\varphi, \lambda) \in \Phi \times L^2(0, T; H^1_0(0, 1))$ **is the unique saddle-point** of the Lagrangian $\mathcal{L} : \Phi \times L^2(0, T; H^1_0(0, 1)) \to \mathbb{R}$ **defined by**
   \[
   \mathcal{L}(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi).
   \]
3. The optimal function $\varphi$ is the minimizer of $\hat{J}^*$ over $\Phi$ while the optimal function $\lambda \in L^2(0, T; H^1_0(0, 1))$ **is the state of the controlled wave equation in the weak sense (associated to the control $-\varphi \mathbf{1}_{q_T}$).**

The well-posedness of the mixed formulation is a consequence of two properties [Fortin-Brezzi'91]:

- $a$ is coercive on $\text{Ker}(b) = \{ \varphi \in \Phi \text{ such that } b(\varphi, \lambda) = 0 \text{ for every } \lambda \in L^2(0, T; H^1_0(0, 1)) \}$.
- $b$ satisfies the usual "inf-sup" condition over $\Phi \times L^2(0, T; H^1_0(0, 1))$: there exists $\delta > 0$ such that
  \[
  \inf_{\lambda \in L^2(0, T; H^1_0(0, 1))} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi} \|\lambda\|_{L^2(0, T; H^1_0(0, 1))}} \geq \delta. \tag{10}
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Well-posedness of the mixed formulation

**Theorem**

1. The mixed formulation (9) is well-posed.

2. The unique solution \((\varphi, \lambda) \in \Phi \times L^2(0, T; H_0^1(0, 1))\) is the unique saddle-point of the Lagrangian \(L : \Phi \times L^2(0, T; H_0^1(0, 1)) \to \mathbb{R}\) defined by

   \[
   L(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - I(\varphi).
   \]

3. The optimal function \(\varphi\) is the minimizer of \(\hat{J}^*\) over \(\Phi\) while the optimal function \(\lambda \in L^2(0, T; H_0^1(0, 1))\) is the state of the controlled wave equation in the weak sense (associated to the control \(-\varphi 1_{qT}\)).

The well-posedness of the mixed formulation is a consequence of two properties [Fortin-Brezzi'91]:

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The well-posedness of the mixed formulation is a consequence of two properties [Fortin-Brezzi’91]:

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\[
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Well-posedness of the mixed formulation

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\[
\inf_{\lambda \in L^2(0, T; H_0^1(0, 1))} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi} \|\lambda\|_{L^2(0, T; H_0^1(0, 1))}} \geq \delta.
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Well-posedness of the mixed formulation

1. The mixed formulation (9) is well-posed.
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\[
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- \(b\) satisfies the usual "inf-sup" condition over \(\Phi \times L^2(0, T; H^1_0(0, 1))\): there exists \(\delta > 0\) such that

\[
\inf_{\lambda \in L^2(0, T; H^1_0(0, 1))} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi} \|\lambda\|_{L^2(0, T; H^1_0(0, 1))}} \geq \delta.
\]

Arnaud Münch
Controllability of the linear 1D wave equation with inner moving forces
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1. \(a\) is coercive on 
   \[
   \text{Ker}(b) = \{\varphi \in \Phi \text{ such that } b(\varphi, \lambda) = 0 \text{ for every } \lambda \in L^2(0, T; H^1_0(0, 1))\}.
   \]

2. \(b\) satisfies the usual "inf-sup" condition over \(\Phi \times L^2(0, T; H^1_0(0, 1))\): there exists \(\delta > 0\) such that
   \[
   \inf_{\lambda \in L^2(0, T; H^1_0(0,1))} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_{\Phi} \|\lambda\|_{L^2(0,T,H^1_0(0,1))}} \geq \delta. \quad (10)
   \]
For any $\lambda_0 \in L^2(H^1_0)$, we define the (unique) element $\varphi_0$ such that

$$L\varphi_0 = -\Delta \lambda_0 \quad Q_T, \quad \varphi_0(\cdot, 0) = \varphi_0, t(\cdot, 0) = 0 \quad \Omega, \quad \varphi_0 = 0 \quad \Sigma_T$$

From the direct inequality,

$$\|\varphi_0\|_{L^2(Q_T)} \leq C_{\Omega, T} \| -\Delta \lambda_0 \|_{L^2(0,T;H^{-1}(0,1))} \leq C_{\Omega, T} \| \lambda_0 \|_{L^2(0,T;H^1_0(0,1))}$$

we get that $\varphi_0 \in \Phi$. In particular, $b(\varphi_0, \lambda_0) = \| \lambda_0 \|_{L^2(0,T;H^1_0(0,1))}^2$ and

$$\sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda_0)}{\|\varphi\|_{\Phi} \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{b(\varphi_0, \lambda_0)}{\|\varphi_0\|_{\Phi} \|\lambda_0\|_{L^2(Q_T)}}$$

$$= \frac{\| \lambda_0 \|_{L^2(0,T;H^1_0(0,1))}^2}{\left( \| \varphi_0 \|_{L^2(Q_T)}^2 + \eta \| \lambda_0 \|_{L^2(0,T;H^1_0(0,1))}^2 \right)^{1/2}} \| \lambda_0 \|_{L^2(0,T;H^1_0(0,1))}$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi_0 \in \Phi} \frac{b(\varphi_0, \lambda_0)}{\|\varphi_0\|_{\Phi} \|\lambda_0\|_{L^2(0,T;H^1_0(0,1))}} \geq \frac{1}{\sqrt{C_{\Omega, T}^2 + \eta}}$$

and, hence, (10) holds with $\delta = \left( C_{\Omega, T}^2 + \eta \right)^{-1/2}$. 

Arnaud Münch  
Controllability of the linear 1D wave equation with inner moving forces
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we get that $\varphi_0 \in \Phi$. In particular, $b(\varphi_0, \lambda_0) = \|\lambda_0\|^2_{L^2(0, T; H^1_0(0, 1))}$ and
\[
\sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda_0)}{\|\varphi\|_{\Phi} \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{b(\varphi_0, \lambda_0)}{\|\varphi_0\|_{\Phi} \|\lambda_0\|_{L^2(Q_T)}} = \frac{\|\lambda_0\|^2_{L^2(0, T; H^1_0(0, 1))}}{\left(\|\varphi_0\|^2_{L^2(Q_T)} + \eta \|\lambda_0\|^2_{L^2(0, T; H^1_0(0, 1))}\right)^{\frac{1}{2}} \|\lambda_0\|_{L^2(0, T; H^1_0(0, 1))}}.
\]
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\]
and, hence, (10) holds with $\delta = \left(C_{\Omega, T}^2 + \eta\right)^{-\frac{1}{2}}$. 

Arnaud Münch

Controllability of the linear 1D wave equation with inner moving force
Lemma

Let $A_r$ be the linear operator from $L^2(H^1_0)$ into $L^2(H^1_0)$ defined by

$$A_r\lambda := -\Delta^{-1}(L\varphi), \quad \forall \lambda \in L^2(H^1_0) \quad \text{where} \quad \varphi \in \Phi \quad \text{solves} \quad a_r(\varphi, \varphi) = b(\varphi, \lambda), \quad \forall \varphi \in \Phi.$$  

For any $r > 0$, the operator $A_r$ is a strongly elliptic, symmetric isomorphism from $L^2(H^1_0)$ into $L^2(H^1_0)$.

Theorem

$$\sup_{\lambda \in L^2(H^1_0)} \inf_{\varphi \in \Phi} L_r(\varphi, \lambda) = -\inf_{\lambda \in L^2(0,T,H^1_0)} J^{**}(\lambda) + L_r(\varphi_0, 0)$$

where $\varphi_0 \in \Phi$ solves $a_r(\varphi_0, \varphi) = I(\varphi), \forall \varphi \in \Phi$ and $J^{**} : L^2(H^1_0) \to \mathbb{R}$ defined by

$$J^{**}(\lambda) = \frac{1}{2} \int_0^T \int_{Q_T} A_r\lambda(x,t)\lambda(x,t) \, dx \, dt - b(\varphi_0, \lambda)$$
Dual ...... of the dual problem ("UZAWA" type algorithm)

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where $\varphi_0 \in \Phi$ solves $a_r(\varphi_0, \overline{\varphi}) = l(\overline{\varphi}), \forall \overline{\varphi} \in \Phi$ and $J^{**} : L^2(H^1_0) \to \mathbb{R}$ defined by

$$J^{**}(\lambda) = \frac{1}{2} \iint_{Q_T} A_r\lambda(x,t)\lambda(x,t) \, dx \, dt - b(\varphi_0, \lambda)$$
Conformal approximation

Let then $\Phi_h$ and $M_h$ be two finite dimensional spaces parametrized by the variable $h$ such that

$$\Phi_h \subset \Phi, \quad M_h \subset L^2(0, T; H_0^1(0, 1)), \quad \forall h > 0.$$ 

Then, we can introduce the following approximated problems: find $(\varphi_h, \lambda_h) \in \Phi_h \times M_h$ solution of

$$\begin{cases}
  a_r(\varphi_h, \overline{\varphi}_h) + b(\varphi_h, \lambda_h) = l(\varphi_h), & \forall \varphi_h \in \Phi_h \\
  b(\varphi_h, \overline{\lambda}_h) = 0, & \forall \lambda_h \in M_h.
\end{cases} \tag{11}$$

The well-posedness is again a consequence of two properties: the coercivity of the bilinear form $a_r$ on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_h; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in M_h\}$. From the relation

$$a_r(\varphi, \varphi) \geq \frac{r}{\eta} \|\varphi\|_\Phi^2, \quad \forall \varphi \in \Phi$$

the form $a_r$ is coercive on the full space $\Phi$, and so a fortiori on $\mathcal{N}_h(b) \subset \Phi_h \subset \Phi$. The second property is a discrete inf-sup condition: there exists $\delta_h > 0$ such that

$$\inf_{\lambda_h \in M_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{M_h}} \geq \delta_h. \tag{12}$$

For any fixed $h$, the spaces $M_h$ and $\Phi_h$ are of finite dimension so that the infimum and supremum in (12) are reached: moreover, from the property of the bilinear form $a_r$, $\delta_h$ is strictly positive. Consequently, for any fixed $h > 0$, there exists a unique couple $(\varphi_h, \lambda_h)$ solution of (11).
Conformal approximation

Let then $\Phi_h$ and $M_h$ be two finite dimensional spaces parametrized by the variable $h$ such that

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Then, we can introduce the following approximated problems: find $(\varphi_h, \lambda_h) \in \Phi_h \times M_h$ solution of

$$\begin{cases}
    \mathcal{a}_r(\varphi_h, \bar{\varphi}_h) + b(\varphi_h, \lambda_h) = l(\bar{\varphi}_h), & \forall \bar{\varphi}_h \in \Phi_h \\
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Arnaud Münch
Controllability of the linear 1D wave equation with inner moving force
The space $\Phi_h$ must be chosen such that $L\varphi_h \in L^2(0, T, H^{-1}(0, 1))$ for any $\varphi_h \in \Phi_h$. This is guaranteed for instance as soon as $\varphi_h$ possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of $Q_T$ is obtained with spaces of functions continuously differentiable with respect to both $x$ and $t$.

We introduce a triangulation $T_h$ such that $Q_T = \bigcup_{K \in T_h} K$ and we assume that $\{T_h\}_{h>0}$ is a regular family. We note $h := \max\{\text{diam}(K), K \in T_h\}$.

We introduce the space $\Phi_h$ as follows:

$$\Phi_h = \{\varphi_h \in \Phi_h \in C^1(Q_T) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in T_h, \varphi_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in $x$ and $t$. We consider for $\mathbb{P}(K)$ the reduced Hsieh-Clough-Tocher $C^1$-element (Composite finite element and involves as degrees of freedom the values of $\varphi_h, \varphi_h, x, \varphi_h, t$ on the vertices of each triangle $K$).

We also define the finite dimensional space

$$M_h = \{\lambda_h \in C^0(Q_T), \lambda_h|_K \in \mathbb{P}_1(K) \quad \forall K \in T_h, \lambda_h = 0 \text{ on } \Sigma_T\}$$

For any $h > 0$, we have $\Phi_h \subset \Phi$ and $M_h \subset L^2(0, T; H^1_0(0, 1))$. 
Discretization

The space $\Phi_h$ must be chosen such that $L\phi_h \in L^2(0, T, H^{-1}(0, 1))$ for any $\phi_h \in \Phi_h$. This is guaranteed for instance as soon as $\phi_h$ possesses second-order derivatives in $L^2_{loc}(Q_T)$. A conformal approximation based on standard triangulation of $Q_T$ is obtained with spaces of functions continuously differentiable with respect to both $x$ and $t$.

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Arnaud Münch

Controllability of the linear 1D wave equation with inner moving fo
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For any $h > 0$, we have $\Phi_h \subset \Phi$ and $M_h \subset L^2(0, T; H^1_0(0, 1))$. 
[Bramble, Gunzburger]
Remark that if there exist two constants $C_0 > 0$ and $\alpha > 0$ such that

$$
\| \psi_h \|_{L^2(Q_T)}^2 \geq C_0 h^\alpha \| \psi_h \|_{L^2(0,T;H_0^1(0,1))}^2, \quad \forall \psi_h \in \Phi_h
$$

(13)

then a similar inequality it holds for weaker norms. More precisely, we have

$$
\| \varphi_h \|_{L^2(0,T;H^{-1}(0,1))}^2 \geq C_0 h^\alpha \| \varphi_h \|_{L^2(Q_T)}^2, \quad \forall \varphi_h \in \Phi_h.
$$

(14)

Indeed, to obtain (14) it suffices to take $\psi_h(\cdot, t) = (-\Delta)^{1/2} \varphi_h(\cdot, t)$ in (13). That gives

$$
\int_0^T \left\| (-\Delta)^{-1/2} \varphi_h(\cdot, t) \right\|_{L^2(0,1)}^2 dt \geq C_0 h^\alpha \int_0^T \left\| (-\Delta)^{-1/2} \varphi_h, x(\cdot, t) \right\|_{L^2(0,1)}^2 dt.
$$

Since $-\Delta$ is a self-adjoint positive operator and $\varphi_h \in \Phi_h \subset H_0^1(Q_T)$ we can integrate by parts in both hand-sides of the above inequality and hence we deduce estimate (14).

$C_0$ and $\alpha$ does not depend on $T$. 
[Bramble, Gunzburger]  
Remark that if there exist two constants $C_0 > 0$ and $\alpha > 0$ such that  
\[
\|\psi_h\|_{L^2(Q_T)}^2 \geq C_0 h^\alpha \|\psi_h\|_{L^2(0,T;H_0^1(0,1))}^2, \quad \forall \psi_h \in \Phi_h \tag{13}
\]
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Indeed, to obtain (14) it suffices to take $\psi_h(\cdot, t) = (-\Delta)^{1/2} \varphi_h(\cdot, t)$ in (13). That gives  
\[
\int_0^T \left\| (-\Delta)^{-1/2} \varphi_h(\cdot, t) \right\|_{L^2(0,1)}^2 \, dt \geq C_0 h^\alpha \int_0^T \left\| (-\Delta)^{-1/2} \varphi_{h,x}(\cdot, t) \right\|_{L^2(0,1)}^2 \, dt.
\]

Since $-\Delta$ is a self-adjoint positive operator and $\varphi_h \in \Phi_h \subset H_0^1(Q_T)$ we can integrate by parts in both hand-sides of the above inequality and hence we deduce estimate (14).

$C_0$ and $\alpha$ does not depend on $T$. 
Change of the norm $\| \cdot \|_{L^2(H^{-1})}$ over the discrete space $\Phi_h$

We consider, for any fixed $h > 0$, the following equivalent definitions of the form $a_{r,h}$ and $b_h$ over the finite dimensional spaces $\Phi_h \times \Phi_h$ and $\Phi_h \times M_h$ respectively:

\[
a_{r,h} : \Phi_h \times \Phi_h \to \mathbb{R}, \quad a_{r,h}(\varphi_h, \overline{\varphi}_h) = a(\varphi_h, \overline{\varphi}_h) + r C_0 h^\alpha \iint_{Q_T} L \varphi_h L \overline{\varphi}_h dxdt
\]

\[
b_h : \Phi_h \times M_h \to \mathbb{R}, \quad b_h(\varphi_h, \lambda_h) = C_0 h^\alpha \iint_{Q_T} L \varphi_h \lambda_h dxdt.
\]

Let $n_h = \dim \Phi_h$, $m_h = \dim M_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$ defined by

\[
a_{r,h}(\varphi_h, \overline{\varphi}_h) = \langle A_{r,h}\{\varphi_h\}, \{\overline{\varphi}_h\}\rangle_{\mathbb{R}^{n_h,\mathbb{R}^{n_h}}}, \quad \forall \varphi_h, \overline{\varphi}_h \in \Phi_h,
\]

where $\{\varphi_h\} \in \mathbb{R}^{n_h,1}$ denotes the vector associated to $\varphi_h$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n_h,\mathbb{R}^{n_h}}}$ the usual scalar product over $\mathbb{R}^{n_h}$. The problem reads: find $\{\varphi_h\} \in \mathbb{R}^{n_h,1}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h,1}$ such that

\[
\begin{pmatrix}
A_{r,h} & B_h^T \\
B_h & 0
\end{pmatrix}
\begin{pmatrix}
\{\varphi_h\} \\
\{\lambda_h\}
\end{pmatrix}
= \begin{pmatrix}
L_h \\
0
\end{pmatrix}
\quad \in \mathbb{R}^{n_h+m_h,1}.
\]

The matrix of order $m_h + n_h$ is symmetric but not positive definite. We use exact integration methods and the LU decomposition method.

From $\varphi_h$, an approximation $\nu_h$ of the control $\nu$ is given by $\nu_h = -\varphi_h^T q_T \in L^2(Q_T)$. 

Arnaud Münch  
Controllability of the linear 1D wave equation with inner moving forces
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\[
a_{r,h} : \Phi_h \times \Phi_h \to \mathbb{R}, \quad a_{r,h}(\varphi_h, \overline{\varphi}_h) = a(\varphi_h, \overline{\varphi}_h) + rC_0 h^\alpha \iint_{Q_T} L\varphi_h L\overline{\varphi}_h dx dt
\]

\[
b_h : \Phi_h \times M_h \to \mathbb{R}, \quad b_h(\varphi_h, \lambda_h) = C_0 h^\alpha \iint_{Q_T} L\varphi_h \lambda_h dx dt.
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Let \( n_h = \dim \Phi_h, m_h = \dim M_h \) and let the real matrices \( A_{r,h} \in \mathbb{R}^{n_h,n_h} \) defined by

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a_{r,h}(\varphi_h, \overline{\varphi}_h) = \langle A_{r,h}\varphi_h, \overline{\varphi}_h \rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, \quad \forall \varphi_h, \overline{\varphi}_h \in \Phi_h,
\]

where \( \{\varphi_h\} \in \mathbb{R}^{n_h,1} \) denotes the vector associated to \( \varphi_h \) and \( \langle \cdot, \cdot \rangle_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} \) the usual scalar product over \( \mathbb{R}^{n_h} \). The problem reads: find \( \{\varphi_h\} \in \mathbb{R}^{n_h,1} \) and \( \{\lambda_h\} \in \mathbb{R}^{m_h,1} \) such that

\[
\begin{pmatrix}
A_{r,h} & B_h^T \\
B_h & 0
\end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix}
\{\varphi_h\} \\
\{\lambda_h\}
\end{pmatrix}_{\mathbb{R}^{n_h+m_h,1}} = \begin{pmatrix}
L_h \\
0
\end{pmatrix}_{\mathbb{R}^{n_h+m_h,1}}.
\]

The matrix of order \( m_h + n_h \) is symmetric but not positive definite. We use exact integration methods and the LU decomposition method.

From \( \varphi_h \), an approximation \( v_h \) of the control \( v \) is given by \( v_h = -\varphi_h 1_{q_T} \in L^2(Q_T) \).
In order to approximate the values of the constants $C_0, \alpha$ appearing in (13)-(14) we consider the following problem:

$$\text{find } \alpha > 0 \text{ and } C_0 > 0 \text{ such that } \sup_{\varphi_h \in \Phi_h} \frac{\|\varphi_h\|_{L^2(0,T;H^1_0(0,1))}^2}{\|\varphi_h\|_{L^2(Q_T)}^2} \leq \frac{1}{C_0h^{\alpha}}, \quad \forall h > 0.$$ 

Since $\dim \Phi_h < \infty$, the supremum is, for any fixed $h > 0$, the solution of the following eigenvalue problem:

$$\forall h > 0, \quad \gamma_h = \sup \left\{ \gamma : \mathbf{K}_h\{\psi_h\} = \gamma \mathbf{J}_h\{\psi_h\}, \quad \forall \{\psi_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}$$

We determine $C_0$ and $\alpha$ such that $C_0h^{\alpha} = \gamma_h^{-1}$. We obtain

$$C_0 \approx 1.48 \times 10^{-2}, \quad \alpha \approx 2.1993.$$ 

We check that the constant $\gamma_h$ (and so $C_0$ and $\alpha$) does not depend on $T$ nor on the controllability domain.
In order to approximate the values of the constants $C_0$, $\alpha$ appearing in (13)-(14) we consider the following problem:

\[
\text{find } \alpha > 0 \text{ and } C_0 > 0 \text{ such that } \sup_{\varphi_h \in \Phi_h} \frac{\|\varphi_h\|^2_{L^2(0,T;H^1_0(0,1))}}{\|\varphi_h\|^2_{L^2(Q_T)}} \leq \frac{1}{C_0 h^\alpha}, \quad \forall h > 0.
\]

Since $\dim \Phi_h < \infty$, the supremum is, for any fixed $h > 0$, the solution of the following eigenvalue problem:

\[
\forall h > 0, \quad \gamma_h = \sup \left\{ \gamma : K_h\{\psi_h\} = \gamma \tilde{J}_h\{\psi_h\}, \quad \forall \{\psi_h\} \in \mathbb{R}^{m_h} \setminus \{0\} \right\}
\]

We determine $C_0$ and $\alpha$ such that $C_0 h^\alpha = \gamma_h^{-1}$. We obtain

\[
C_0 \approx 1.48 \times 10^{-2}, \quad \alpha \approx 2.1993.
\]

We check that the constant $\gamma_h$ (and so $C_0$ and $\alpha$) does not depend on $T$ nor on the controllability domain.
In order to solve the mixed formulation (11), we first test numerically the discrete inf-sup condition (12). Taking $\eta = r > 0$ so that $a_{r,h}(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_\Phi$ for all $\varphi, \overline{\varphi} \in \Phi$, it is readily seen that the discrete inf-sup constant satisfies

$$
\delta_h := \inf \left\{ \sqrt{\delta} : B_h A_{r,h}^{-1} B_h^T \lambda_h \delta J_h \lambda_h \right\}.
$$

The matrix $B_h A_{r,h}^{-1} B_h^T$ is symmetric, positive definite so that $\delta_h > 0$ for any $h > 0$.

\[ \text{Figure: } \delta_h \text{ vs. } h \text{ for various control domains } q_T, \ T > 0 \text{ and } r = 10^{-1}. \]
In order to solve the mixed formulation (11), we first test numerically the discrete inf-sup condition (12). Taking $\eta = r > 0$ so that $a_{r,h}(\varphi, \overline{\varphi}) = (\varphi, \overline{\varphi})_{\Phi}$ for all $\varphi, \overline{\varphi} \in \Phi$, it is readily seen that the discrete inf-sup constant satisfies

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**Figure:** $\delta_h$ vs. $h$ for various control domains $q_T$, $T > 0$ and $r = 10^{-1}$.
The discrete inf-sup test

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![Figure: $\delta_h$ vs. $h$ for various control domains $q_T$, $T > 0$ and $r = 10^{-1}$.](image)

$\Rightarrow$ $(\Phi_h, M_h)$ passes the discrete inf-sup test!

**Figure:** $\delta_h$ vs. $h$ for various control domains $q_T$, $T > 0$ and $r = 10^{-1}$. 

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**Figure:** \( \delta_h \) vs. \( h \) for various control domains \( q_T, T > 0 \) and \( r = 10^{-1} \).
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![Graph](image.png)

**Figure:** $\delta_h$ vs. $h$ for various control domains $q_T$, $T > 0$ and $r = 10^{-1}$. 

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**Figure:** $\delta_h$ vs. $h$ for various control domains $q_T$, $T > 0$ and $r = 10^{-1}$.
Figure: Time dependent domains $q^i_T$, $i \in \{0, 1, 2, 3\}$. 
Triangular meshes

Figure: Meshes #1 associated with the domains $q^i_{T=2.2} : i = 0, 1, 2, 3$. 

Arnaud Münch

Controllability of the linear 1D wave equation with inner moving forces
$T = 2.; \quad y_0(x) = \sin(\pi x); \quad y_1 = 0; \quad q_T = q_2^2$

<table>
<thead>
<tr>
<th># Mesh</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$7.18 \times 10^{-2}$</td>
<td>$3.59 \times 10^{-2}$</td>
<td>$1.79 \times 10^{-2}$</td>
<td>$8.97 \times 10^{-3}$</td>
<td>$4.49 \times 10^{-3}$</td>
</tr>
<tr>
<td>$|v_h|_{L^2(q_T)}$</td>
<td>5.370</td>
<td>5.047</td>
<td>4.893</td>
<td>4.815</td>
<td>4.776</td>
</tr>
<tr>
<td>$|L\varphi_h|_{L^2(0,T;H^{-1}(0,1))}$</td>
<td>2.286</td>
<td>$9.43 \times 10^{-1}$</td>
<td>$3.76 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-1}$</td>
<td>$6.15 \times 10^{-2}$</td>
</tr>
<tr>
<td>$|v - v_h|_{L^2(q_T)}$</td>
<td>$2.45 \times 10^{-1}$</td>
<td>$9.65 \times 10^{-2}$</td>
<td>$4.32 \times 10^{-2}$</td>
<td>$2.29 \times 10^{-2}$</td>
<td>$1.10 \times 10^{-2}$</td>
</tr>
<tr>
<td>$|y - \lambda h|_{L^2(Q_T)}$</td>
<td>$5.63 \times 10^{-3}$</td>
<td>$1.57 \times 10^{-3}$</td>
<td>$4.04 \times 10^{-4}$</td>
<td>$1.03 \times 10^{-4}$</td>
<td>$2.61 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$2.46 \times 10^7$</td>
<td>$2.67 \times 10^8$</td>
<td>$2.96 \times 10^9$</td>
<td>$3.03 \times 10^{10}$</td>
<td>$3.08 \times 10^{11}$</td>
</tr>
</tbody>
</table>

**Table:** Norms vs. $h$ for $r = 10^{-1}$.

$r = 10^{-1}$: $\|v - v_h\|_{L^2(q_T)} \approx O(h^{1.3})$, $\|L\varphi_h\|_{L^2(0,T;H^{-1}(0,1))} \approx O(h^{1.3})$, $\|y - \lambda h\|_{L^2(Q_T)} \approx O(h^{1.94})$

$r = 10^3$: $\|v - v_h\|_{L^2(q_T)} \approx O(h^{1.09})$, $\|L\varphi_h\|_{L^2(Q_T)} \approx O(h^{1.04})$, $\|y - \lambda h\|_{L^2(Q_T)} \approx O(h^{2.01})$.
Convergence as $h \to 0$

$$T = 2.; \quad y_0(x) = \sin(\pi x); \quad y_1 = 0; \quad q_T = q_{2}^{2}$$

Figure: $r = 10^{-1}; q_T = q_{2,2}^{2};$ Norms $\|v - v_h\|_{L^2(q_T)} (\bullet)$ and $\|y - \lambda_h\|_{L^2(Q_T)} (\diamond)$ vs. $h.$
Numerical illustration

\[ T = 2.2; \quad y_0(x) = e^{-500(x-0.8)^2}; \quad y_1 = 0; \quad q_T = q_{2.2}^2 \]

**Figure:** \(r = 10^{-1}; q_T = q_{2.2}^2\): Functions \(\varphi_h\) (Left) and \(\lambda_h\) (Right) over \(Q_T\).

\[ \|v - v_h\|_{L^2(Q_T)} \approx e^{5.85} h^{1.4}, \quad \|L\varphi_h\|_{L^2(Q_T)} \approx e^{7.96} h^{1.31}, \quad \|y - \lambda_h\|_{L^2(Q_T)} \approx e^{1.508} h^{1.62} \]
Numerical illustration

\[ T = 2.2; \quad y_0(x) = \frac{x}{\theta} 1_{(0,\theta)}(x) + \frac{1-x}{1-\theta} 1_{(\theta,1)}(x), \quad y_1(x) = 0, \quad \theta \in (0,1) \quad q_T = q^2_{2.2} \]

\[ \begin{align*}
  \|v - v_h\|_{L^2(q_T)} & \approx e^{1.54} h^{0.47}, \\
  \|L\varphi_h\|_{L^2(Q_T)} & \approx e^{2.91} h^{0.54}, \\
  \|y - \lambda_h\|_{L^2(Q_T)} & \approx e^{-1.52} h^{1.29}.
\end{align*} \]

**Figure:** Example EX3 with \( \theta = 1/3; \ r = 10^{-1}; \ q_T = q^2_{2.2} \) : Functions \( \varphi_h \) (Left) and \( \lambda_h \) (Right).
$T = 2.2; \quad y_0(x) = e^{-500(x-0.8)^2}; \quad y_1 = 0; \quad q_T = q_{2.2}^3$

**Figure:** Example EX2: $q_T = q_{2.2}^3$ - Function $\varphi_h$ (Left) and $\lambda_h$ (Right) over $Q_T$. 
Numerical illustration

\[ T = 2.2; \quad y_0(x) = \frac{x}{\theta} \cdot 1_{(0, \theta)}(x) + \frac{1 - x}{1 - \theta} \cdot 1_{(\theta, 1)}(x), \quad y_1(x) = 0, \quad \theta \in (0, 1) \quad q_T = q_{3.2}^3 \]

Figure: Example EX3, \( \theta = 1/3 \): \( q_T = q_{3.2}^3 \). Function \( \varphi_h \) (Left) and \( \lambda_h \) (Right) over \( Q_T \).
Numerical illustration : $q_T \to \bigcup_{t \in (0, T)} \gamma(t) \times \{t\}$

$T = 2.2; \quad y_0(x) = \sin(\pi x), \quad y_1(x) = 0, \quad \theta \in (0, 1) \quad q_T = q_2^2$

<table>
<thead>
<tr>
<th>$\delta_0$</th>
<th>$10^{-1}$</th>
<th>$10^{-1}/2$</th>
<th>$10^{-1}/2^2$</th>
<th>$10^{-1}/2^3$</th>
<th>$10^{-1}/2^4$</th>
<th>$10^{-1}/2^5$</th>
<th>$10^{-1}/2^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td># triangles</td>
<td>68 740</td>
<td>68 464</td>
<td>68 402</td>
<td>68 728</td>
<td>68 422</td>
<td>68 966</td>
<td>68 368</td>
</tr>
<tr>
<td>$|v_h|_{L^2(q_T)}$</td>
<td>4.8308</td>
<td>7.3308</td>
<td>11.5743</td>
<td>18.8056</td>
<td>29.7354</td>
<td>47.3157</td>
<td>123.9704</td>
</tr>
<tr>
<td>$|v_h|_{L^2(H-1)}$</td>
<td>0.0035</td>
<td>0.0042</td>
<td>0.0066</td>
<td>0.0107</td>
<td>0.0170</td>
<td>0.0270</td>
<td>0.0704</td>
</tr>
</tbody>
</table>

Table: Example **EX1**; $q_T = q_2^2$; Norms of the control $v_h$ obtained for the **EX1** for control domains $q_2^2$ for different values of $\delta_0$. 
Non constant velocity

\[ c(x) = \begin{cases} 
1, & x \in [0, 0.45] \\
\in [1, 5], & (c'(x) > 0), \\
5, & x \in (0.45, 0.55) \\
5, & x \in [0.55, 1]. 
\end{cases} \]

Figure: \[ r = 10^{-1} : \text{Example EX3, } \theta = 1/3: q_T = 2 \] for a non-constant velocity of propagation - Function \( \varphi_h \) (Left) and \( \lambda_h \) (Right) over \( Q_T \).
Figure: Example EX3, $\theta = 1/3$: $q_T = q_1^2$ - Function $\varphi_h$ (Left) and $\lambda_h$ (Right) over $Q_T$. 
Minimization of $J^{**}$

Figure: Example EX3. Evolution of the residue $\|g^n\|_{L^2(0,T;H^1_0(0,1))}/\|g^0\|_{L^2(0,T;H^1_0(0,1))}$ w.r.t. the iterate $n$.

$$g^n = -\Delta^{-1}(L\varphi^n)$$

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</tr>
<tr>
<td># iterate</td>
<td>87</td>
<td>105</td>
<td>119</td>
<td>140</td>
<td>166</td>
</tr>
<tr>
<td>$|\lambda_h - y|_{L^2(Q_T)}$</td>
<td>$1.15 \times 10^{-1}$</td>
<td>$5.2 \times 10^{-2}$</td>
<td>$1.65 \times 10^{-2}$</td>
<td>$6.03 \times 10^{-3}$</td>
<td>$2.89 \times 10^{-3}$</td>
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Table: Conjugate gradient algorithm. EX3 with $\theta = 1/3$, for control domain $q_2^2$ and $r = 10^3$.}

Arnaud Münch | Controllability of the linear 1D wave equation with inner moving forces
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Arnaud Münch

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![Diagram](image-url)

Time-Space Refinement of the mesh according to the gradient of $\lambda_h$ (from [Cîndea, Münch, 2014])
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The approach may be adapted to treat the heat equation (in progress with D. A. de Souza), etc.

This work allows now to consider the optimization of the controls with respect to $q_T$:

$$\forall (y_0, y_1) \in H, \ T > 0 \text{ and } L \in (0, 1), \text{ the problem reads :}$$

$$\inf_{q_T \in C_L} \|v_{q_T}\|_{L^2(q_T)}, \quad C_L = \{q_T : q_T \subset Q_T, |q_T| = L|Q_T| \text{ and such that (1) holds}\}$$

where $v_{q_T}$ denotes the control of minimal $L^2(q_T)$ norm for the wave eq. distributed over $q_T$. 

Arnaud Münch

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ADAPATION OF THE METHOD TO SOLVE INVERSE PROBLEMS VIA SPACE-TIME FORMULATION

Given the observation $z \in L^2(q_T)$, find $y \in Y$ such that

$$
\begin{cases}
Ly = 0 & \text{in } Q_T, \\
y = z & \text{in } q_T, \\
y = 0 & \text{on } \Sigma_T
\end{cases}
$$

Set $Y = \{y \in L^2(q_T), Ly = 0 \text{ in } L^2(0, T, H^{-1}(\Omega)), y = 0 \text{ on } \Sigma_T\}$, solve the Least-Squares problem :

$$
\inf_{y \in Y} \frac{1}{2} \int_0^T \int_{q_T} (y - z)^2 \, dx \, dt
$$

........

NADA MAS! THANK YOU FOR YOUR ATTENTION
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