Existence and approximation for a steady fluid-structure interaction problem using fictitious domain approach with penalization

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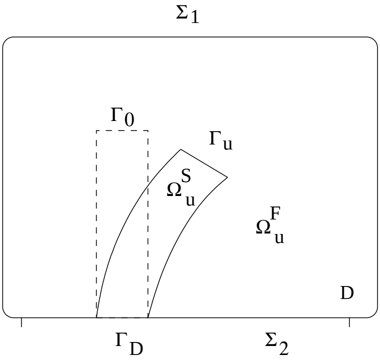
Geometrical configuration



$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$$

The deformed domain $\Omega_u^S = \varphi\left(\Omega_0^S\right)$.

Strong formulation

Find $\mathbf{u} : \overline{\Omega}_0^S \to \mathbb{R}^2$, $\mathbf{v} : \overline{\Omega}_u^F \to \mathbb{R}^2$ and $p : \overline{\Omega}_u^F \to \mathbb{R}$ such that

$$\begin{aligned}
-\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}) &= \mathbf{f}^S, \quad \text{in } \Omega_0^S && (1)\\
\mathbf{u} &= 0, \quad \text{on } \Gamma_D && (2)\\
-\nabla \cdot \sigma^F(\mathbf{v}, p) &= \mathbf{f}^F, \quad \text{in } \Omega_u^F && (3)\\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^F && (4)\\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_1 && (5)\\
\mathbf{v} &= 0, \quad \text{on } \Sigma_2 \setminus \Gamma_D && (6)\\
\mathbf{v} &= 0, \quad \text{on } \Gamma_u && (7)\\
\omega\left(\sigma^F(\mathbf{v}, p)\,\mathbf{n}^F\right) \circ \varphi &= -\sigma^S(\mathbf{u})\,\mathbf{n}^S, \quad \text{on } \Gamma_0 && (8)
\end{aligned}$$

Fictitious domain approach using penalization

$$\begin{aligned} -\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\mathcal{P}(\mathbf{v}) &= \mathbf{f}^F, \quad \text{in } \Omega_u^S \\ \nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^S \end{aligned}$$

where $\epsilon > 0$ is a penalization parameter,

$$\mathcal{P}(\mathbf{v}) = \left(|v_1|^{\alpha-1} sgn(v_1), |v_2|^{\alpha-1} sgn(v_2) \right)$$

where $\mathbf{v} = (v_1, v_2)$ and $1 < \alpha < 2$ is a real number.

$$\chi_u^S(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \overline{\Omega}_u^S \\ 0, & \mathbf{x} \in D \setminus \overline{\Omega}_u^S \end{cases} \quad \text{and} \quad \chi_u^F = 1 - \chi_u^S.$$

$$\begin{aligned} -\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\chi_u^S \mathcal{P}(\mathbf{v}) &= \mathbf{f}^F, \quad \text{in } D \\ \nabla \cdot \mathbf{v} &= 0, \quad \text{in } D. \end{aligned}$$

Weak formulation in the structure domain (I)

Structure equations in the initial domain

$$\int_{\Omega_0^S} \sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X} + \int_{\Gamma_0} \sigma^S(\mathbf{u}) \, \mathbf{n}^S \cdot \mathbf{w}^S dS.$$

Fluid equations in the deformed structure domain

We define $\widetilde{\mathbf{w}}^S : \Omega_u^S \to \mathbb{R}^2$, $\widetilde{\mathbf{w}}^S = \mathbf{w}^S \circ \varphi^{-1}$

$$\int_{\Omega_u^S} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) : \nabla \widetilde{\mathbf{w}}^S \, d\mathbf{x} + \frac{1}{\epsilon} \int_{\Omega_u^S} \mathcal{P}(\mathbf{v}_\epsilon) \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x}$$
$$= \int_{\Omega_u^S} \mathbf{f}^F \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x} - \int_{\Gamma_u} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \, \mathbf{n}^F \cdot \widetilde{\mathbf{w}}^S \, ds$$

Weak formulation in the structure domain (II)

Fluid equations in the undeformed structure domain

$$\int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X} + \frac{1}{\epsilon}\int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\, d\mathbf{X}$$
$$= \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\, d\mathbf{X} - \int_{\Gamma_0}\omega\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\,\mathbf{n}^F\circ\varphi\right)\cdot\mathbf{w}^S\, dS.$$

Subtracting from the structure equations, we get

$$\int_{\Omega_0^S}\sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X} - \int_{\Omega_0^S}\mathbf{f}^S\cdot\mathbf{w}^S\, d\mathbf{X}$$
$$= \int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X}$$
$$+\frac{1}{\epsilon}\int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\, d\mathbf{X} - \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\, d\mathbf{X}$$

Constitutive relations

$$\epsilon(\mathbf{w}) = \frac{1}{2}\left(\nabla \mathbf{w} + (\nabla \mathbf{w})^T\right).$$

Linear elasticity

$$a_S\left(\mathbf{u}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\lambda^S (\nabla \cdot \mathbf{u}) \left(\nabla \cdot \mathbf{w}^S\right) + 2\mu^S \epsilon(\mathbf{u}) : \epsilon\left(\mathbf{w}^S\right)\right) \, d\mathbf{X}.$$

Stokes equations

$$\begin{aligned} a_F(\mathbf{v}, \mathbf{w}) &= \int_D 2\mu^F \epsilon(\mathbf{v}) : \epsilon(\mathbf{w}) \, d\mathbf{x} \\ b_F(\mathbf{w}, p) &= -\int_D (\nabla \cdot \mathbf{w}) \, p \, d\mathbf{x}. \end{aligned}$$

Parametrization of the characteristic function

Let $j \in W^{1,\infty}(D)$ be a parametrization of $\Omega_0^S \subset D$, i.e. :

$$
\begin{array}{rcl}
j(\mathbf{x}) & > & 0,\ \mathbf{x} \in \Omega_0^S, \\
j(\mathbf{x}) & < & 0,\ \mathbf{x} \in D \setminus \overline{\Omega}_0^S, \\
j(\mathbf{x}) & = & 0,\ \mathbf{x} \in \partial\Omega_0^S.
\end{array}
$$

$\Omega_u^S = \varphi(\Omega_0^S)$, where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$

$$
j_u(\mathbf{y}) = \begin{cases} j(\mathbf{x}), & \mathbf{y} = \varphi(\mathbf{x}) \in \Omega_u^S \\ 0, & \mathbf{y} \in \partial\Omega_u^S \\ -dist(\mathbf{y}, \overline{\Omega}_u^S), & \mathbf{y} \notin \overline{\Omega}_u^S \end{cases}
$$

is a parametrization of Ω_u^S, $j_u \in W^{1,\infty}(D)$.

Regularization of the characteristic function

If H is the Heaviside function $H : \mathbb{R} \to \{0, 1\}$,

$$H(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

then $H(j_u(\cdot))$ is the characteristic function of Ω_u^S.

$$\Omega_0^\epsilon \subset\subset \Omega_0^S, \quad \Omega_u^\epsilon = (\mathbf{id} + \mathbf{u})\,(\Omega_0^\epsilon)$$

There exits $\mu_\epsilon > 0$ such that $j(\mathbf{x}) \geq \mu_\epsilon > 0$, for all $\mathbf{x} \in \Omega_0^\epsilon$.
Then we take $\widetilde{H} = H^{\mu_\epsilon}$, the Yosida regularization of H

$$H^{\mu_\epsilon}(r) = \begin{cases} 1, & r \geq \mu_\epsilon \\ \frac{r}{\mu_\epsilon} & 0 \leq r < \mu_\epsilon \\ 0, & r < 0 \end{cases}$$

It follows that $H^{\mu_\epsilon}(j_u(\mathbf{x})) = 1$ for all $\mathbf{x} \in \Omega_u^\epsilon$.
$\widetilde{H}(j_u)$ is Lipschitz and $0 \leq \widetilde{H}(j_u(\mathbf{x})) \leq 1$ for all $\mathbf{x} \in \overline{D}$.

Functional spaces

$$
\begin{aligned}
W^S &= \left\{ \mathbf{w}^S \in \left(H^1\left(\Omega_0^S\right)\right)^2;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D \right\}, \\
W &= \left(H_0^1(D)\right)^2, \\
Q &= L_0^2(D) = \{q \in L^2(D);\ \int_D q\, dx = 0\}.
\end{aligned}
$$

$\mathbf{f}^F \in \left(L^2(D)\right)^2$, $\mathbf{f}^S \in \left(L^2(\Omega_0^S)\right)^2$, $\mathbf{g} \in \left(H^{1/2}(\partial D)\right)^2$, such that $\mathbf{g} = 0$ on Σ_2 and $\int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F \, ds = 0$.

For a given $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$, such that $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1$ and $\mathbf{u} = 0$ on Γ_D, we define:

- fluid velocity $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2,
- fluid pressure $p_\epsilon \in Q$,
- structure displacement $\mathbf{u}_\epsilon \in W^S$

Coupled system

$$
\begin{aligned}
a_F(\mathbf{v}_\epsilon, \mathbf{w}) + b_F(\mathbf{w}, p_\epsilon) & \\
+\frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\cdot\mathbf{w}\,dx &= \int_D \mathbf{f}^F\cdot\mathbf{w}dx, \quad \forall\mathbf{w}\in W \qquad (9)\\
b_F(\mathbf{v}_\epsilon, q) &= 0, \quad \forall q\in Q \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
a_S\left(\mathbf{u}_\epsilon, \mathbf{w}^S\right) &= \int_{\Omega_0^S}\mathbf{f}^S\cdot\mathbf{w}^S\,d\mathbf{X}\\
&+ \int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T}:\nabla_{\mathbf{X}}\mathbf{w}^S\,d\mathbf{X}\\
&+ \frac{1}{\epsilon}\int_{\Omega_0^S} J\widetilde{H}(j_u\circ\varphi)\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\,d\mathbf{X}\\
&- \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\,d\mathbf{X},\ \forall\mathbf{w}^S\in W^S \qquad (11)
\end{aligned}
$$

where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$, $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{X})$, $J(\mathbf{X}) = \det\mathbf{F}(\mathbf{X})$.

Sobolev spaces

Denote by $\|\cdot\|_{m,s,\Omega}$ the usual norm of the Sobolev space $W^{m,s}(\Omega)$.

$$\frac{1}{\alpha}+\frac{1}{\alpha'}=1, \quad 1<\alpha<2, \quad 2<\alpha'$$

If $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, then $\mathbf{v}_\epsilon \in (L^\alpha(D))^2$ since $\alpha<2$. It follows that

$$\int_D \left(|\mathbf{v}_\epsilon|^{\alpha-1}\right)^{\alpha'} d\mathbf{x} = \int_D |\mathbf{v}_\epsilon|^\alpha \, d\mathbf{x} < \infty$$

and consequently $|\mathbf{v}_\epsilon|^{\alpha-1} \in \left(L^{\alpha'}(D)\right)^2$. Since the coefficient $\widetilde{H}(j_u)$ is Lipschitz and $0 \le \widetilde{H}(j_u(\mathbf{x})) \le 1$ for all $\mathbf{x} \in \overline{D}$, we obtain that

$$\frac{1}{\epsilon}\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon) \in \left(L^{\alpha'}(D)\right)^2.$$

Definition of the nonlinear operator

$$\left\{ \mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S) \right)^2 ;\ \|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1,\ \mathbf{u} = 0 \text{ on } \Gamma_D \right\}$$

Fluid problem

$$\mathbf{u} \longrightarrow \mathbf{v}_\epsilon \in \left(H_0^1(D) \right)^2 \text{ and } p_\epsilon \in L_0^2(D)$$

Structure problem

$$\mathbf{u},\ \mathbf{v}_\epsilon,\ p_\epsilon \longrightarrow \mathbf{u}_\epsilon \in \left\{ \mathbf{w}^S \in \left(H^1\left(\Omega_0^S \right) \right)^2 ;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D \right\}$$

Nonlinear operator

$$T_\epsilon(\mathbf{u}) = \mathbf{u}_\epsilon.$$

Existence and unicity for the fluid problem (I)

Define $\phi : L^2(D)^2 \to \mathbb{R}$ by

$$\phi(\mathbf{v}) = \frac{1}{\alpha}\left(|v_1|^\alpha + |v_2|^\alpha\right)$$

where $1 < \alpha < 2$ and $\mathbf{v} = (v_1, v_2)$. This is a convex continuous function.

$$V = \left\{\mathbf{w} \in \left(H_0^1(D)\right)^2 ; \ \nabla \cdot \mathbf{w} = 0 \text{ on } D\right\}$$

Let $\widetilde{\mathbf{g}} \in \left(H^1(D)\right)^2$, such that

$$\widetilde{\mathbf{g}} = \mathbf{g} \text{ on } \Sigma_1, \ \widetilde{\mathbf{g}} = 0 \text{ on } \Sigma_2, \ \nabla \cdot \widetilde{\mathbf{g}} = 0 \text{ in } D$$

For $\mathbf{w} \in \widetilde{\mathbf{g}} + V$

$$\mathbf{w} \to \frac{1}{2} a_F(\mathbf{w}, \mathbf{w}) - \int_D \mathbf{f}^F \cdot \mathbf{w} dx + \phi(\mathbf{w})$$

Subdifferential of a convex function

$$\phi : X \to \mathbb{R}$$

Here $\partial\phi(\cdot)$ is the subdifferential of ϕ at x defined to be the set of those $\xi \in X'$ satisfying

$$\phi(y) - \phi(x) \geq \langle \xi, y - x \rangle$$

$$\mathcal{P}\left(\mathbf{v}\right) = \left(\left| v_1 \right|^{\alpha-1} sgn\left(v_1\right), \left| v_2 \right|^{\alpha-1} sgn\left(v_2\right) \right)$$

The choice of the penalization operator is justified by the expression

$$\mathcal{P}\left(\mathbf{v}\right) \in \partial\phi\left(\mathbf{v}\right)$$

Existence and unicity for the fluid problem (II)

Proposition

There exists a unique solution of fluid problem such that $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ *on* ∂D *and* $p_\epsilon \in Q$*.*

Proof. We get the existence of a unique weak solution $\mathbf{v}_\epsilon \in \widetilde{\mathbf{g}} + V$ of

$$a_F\left(\mathbf{v}_\epsilon, \mathbf{w}\right) + \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}\right)\cdot \mathbf{w}\, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x},\ \forall \mathbf{w} \in V.$$

There exists a unique $p_\epsilon \in Q$

Regularity of the fluid problem

Proposition

Let D and Ω_0^S be open bounded sets of class $\mathcal{C}^2$. We assume that $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2 and $p_\epsilon \in Q$ are solutions of the fluid problem.
If $\mathbf{f}^F \in \left(L^{\alpha'}(D)\right)^2$ and $\mathbf{g} \in \left(W^{2-1/\alpha',\alpha'}(\partial D)\right)^2$ such that $\mathbf{g} = 0$ on Σ_2 and $\int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F ds = 0$, then $\mathbf{v}_\epsilon \in \left(W^{2,\alpha'}(D)\right)^2$ and $p_\epsilon \in W^{1,\alpha'}(D)$.

Estimation for the fluid problem

There exist constants C_1, C_2 independent of ϵ and $\mathbf{u}$, such that

$$\|\mathbf{v}_\epsilon\|_{1,2,D} \leq C_1 \left(\left\| \mathbf{f}^F \right\|_{0,2,D} + \|\mathbf{g}\|_{1/2,2,\partial D} \right)$$

$$\frac{1}{\epsilon} \left\| \widetilde{H}(j_u) \mathcal{P}\left(\mathbf{v}_\epsilon\right) \right\|_{0,\alpha',D} \leq \frac{1}{\epsilon} \|\mathbf{v}_\epsilon\|_{0,\alpha,\Omega_u^S}^{\alpha-1} \leq \frac{1}{\epsilon} \|\mathbf{v}_\epsilon\|_{0,2,D}^{\alpha-1}$$

$$\begin{aligned} \|\mathbf{v}_\epsilon\|_{2,\alpha',D} + \|p_\epsilon\|_{1,\alpha',D} \leq{} & C_2 \left(\left\| \mathbf{f}^F \right\|_{0,\alpha',D} + \|\mathbf{g}\|_{2-1/\alpha',\alpha',\partial D} \right) \\ & + \frac{1}{\epsilon} C_2 \left(\left\| \mathbf{f}^F \right\|_{0,\alpha',D} + \|\mathbf{g}\|_{2-1/\alpha',\alpha',\partial D} \right)^{\alpha-1} \end{aligned}$$

for all $0 < \epsilon$ and $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} \leq \eta_\delta$, where η_δ is given by

$$1 - \delta \leq \det\left(\mathbf{I} + \nabla \mathbf{u}\right) \leq 1 + \delta, \quad \text{a.e. } \mathbf{x} \in \Omega_0^S$$

Regularity and estimations for the structure problem

Proposition

Let Ω_0^S *be an open bounded set of class* $\mathcal{C}^2$*. We assume that* $\mathbf{f}^S \in \left(L^{\alpha'}(\Omega_0^S)\right)^2$ *and* $\mathbf{f}^F \in \left(L^{\alpha'}(D)\right)^2$*.*

If $\mathbf{v}_\epsilon \in \left(W^{2,\alpha'}(D)\right)^2$ *and* $p_\epsilon \in W^{1,\alpha'}(D)$ *are solutions of the fluid problem, then the structure problem has a unique solution* $\mathbf{u}_\epsilon \in \left(W^{2,\alpha'}(\Omega_0^S)\right)^2$ *and there exists a constant* C_3 *independent of* ϵ *and* $\mathbf{u}$*, such that*

$$\|\mathbf{u}_\epsilon\|_{2,\alpha',\Omega_0^S} \leq C_3 \left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D} + \|\mathbf{g}\|_{2-1/\alpha',\alpha',\partial D} + \left\|\mathbf{f}^S\right\|_{0,\alpha',\Omega_0^S} \right) + \frac{1}{\epsilon} C_3 \left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D} + \|\mathbf{g}\|_{2-1/\alpha',\alpha',\partial D} \right)^{\alpha-1}$$

for all $0 < \epsilon$ *and* $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} \leq \eta_\delta$*.*

Existence of the penalized fluid-structure problem

$$B_\delta = \{u \in W^{1,\infty}(\Omega_0^S)^2; ||u||_{1,\infty,\Omega_0^S} \leq \eta_\delta,\ u = 0 \text{ on } \Gamma_D\}$$

Theorem

Let $\epsilon > 0$ fixed. If f^F, f^S and g are "small" in their own norms, then the nonlinear operator T_ϵ has at least one fixed point $\mathbf{u}_\epsilon$ in B_δ.

Estimations independent of ϵ

There exist constant C_1 independent of ϵ and $\mathbf{u}$, such that

$$\|\mathbf{v}_\epsilon\|_{1,2,D} \leq C_1 \left(\left\| \mathbf{f}^F \right\|_{0,2,D} + \|\mathbf{g}\|_{1/2,2,\partial D} \right)$$

$$\frac{1}{\epsilon} \|\mathbf{v}_\epsilon\|^{\alpha}_{0,\alpha,\Omega_u^\epsilon} \leq C_1 \left(\left\| \mathbf{f}^F \right\|_{0,2,D} + \|\mathbf{g}\|_{1/2,2,\partial D} \right)^2$$

$$\frac{1}{\epsilon} \left\| \widetilde{H}(j_u) \mathcal{P}(\mathbf{v}_\epsilon) \right\|_{0,\alpha',D} \leq \frac{1}{\epsilon} \|\mathbf{v}_\epsilon\|^{\alpha-1}_{0,\alpha,\Omega_u^S}$$

Problems: if $0 < x < 1$, then $x^\alpha < x^{\alpha-1}$ and $\Omega_u^\epsilon \subset\subset \Omega_u^S$

What happens when $\epsilon \longrightarrow 0$

Proposition

If $\|\mathbf{u}_\epsilon\|_{2,\alpha',\Omega_0^S}$, $\|\mathbf{v}_\epsilon\|_{2,\alpha',D}$, $\|p_\epsilon\|_{1,\alpha',D}$ are borned independent of ϵ, then on a sub-sequence, we have $u_\epsilon \longrightarrow u^*$ weakly in $W^{2,\alpha'}(\Omega_0^S)$, $v_\epsilon \longrightarrow v^*$ weakly in $W^{2,\alpha'}(D)$, $p_\epsilon \longrightarrow p^*$ weakly in $W^{1,\alpha'}(D)$. Moreover, v^* and p^* satisfy (3), (4) in $\Omega^F_{u^*}$, (7) on Γ_{u^*}, (5) and (6). The mapping u^* satisfies (1) and (2).

Continuity of the stress at the interface

$$-\int_{\Gamma_0} \omega \left(\sigma^F \left(\mathbf{v}_\epsilon, p_\epsilon \right) \mathbf{n}^F \circ \varphi \right) \cdot \mathbf{w}^S dS = \int_{\Gamma_0} \sigma^S \left(\mathbf{u}_\epsilon \right) \mathbf{n}^S \cdot \mathbf{w}^S dS$$

for all $\mathbf{w}^S \in W^S$. This shows that the approximating solutions $\mathbf{v}_\epsilon, p_\epsilon, \mathbf{u}_\epsilon$ satisfy (8) for any ϵ.

Partitioned procedures based on the fixed point iterations

The penalized term $\mathcal{P}(\mathbf{v}) = \left(|v_1|^{\alpha-1} sgn(v_1), |v_2|^{\alpha-1} sgn(v_2) \right)$, where $1 < \alpha < 2$ is non-linear in $\mathbf{v}$. But, if α is close to 2, we can approach $\mathcal{P}(\mathbf{v})$ by $\mathbf{v}$. We can also replace $\widetilde{H}(j_u)$ by the characteristic function χ_u^S.

Algorithm 1

Step 1. Given the initial displacement of the structure $\mathbf{u}^0 \in W^S$, compute the characteristic function $\chi_{u^0}^S$, put $k := 0$

Step 2. Find the velocity $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2 and the pressure $p_\epsilon^k \in Q$ by solving the fluid problem

$$a_F\left(\mathbf{v}_\epsilon^k, \mathbf{w}\right) + b_F\left(\mathbf{w}, p_\epsilon^k\right) + \frac{1}{\epsilon}\int_D \chi_{u_\epsilon^k}^S \mathbf{v}_\epsilon^k \cdot \mathbf{w}\, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x},\ \forall \mathbf{w} \in W$$

$$b_F\left(\mathbf{v}_\epsilon^k, q\right) = 0,\ \forall q \in Q$$

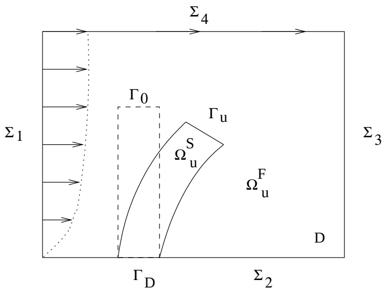
Step 3. Find the new displacement of the structure $\mathbf{u}_\epsilon^{k+1} \in W^S$ by solving

$$a_S\left(\mathbf{u}_\epsilon^{k+1}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\mathbf{f}^S - \mathbf{f}^F\right) \cdot \mathbf{w}^S \, d\mathbf{x} + \int_{\Omega_0^S} 2\mu^F \epsilon\left(\mathbf{v}_\epsilon^k\right) : \epsilon\left(\mathbf{w}^S\right) \, d\mathbf{x}$$
$$- \int_{\Omega_0^S} \left(\nabla \cdot \mathbf{w}^S\right) p_\epsilon^k \, d\mathbf{x} + \frac{1}{\epsilon} \int_{\Omega_0^S} \left(\mathbf{v}_\epsilon^k \circ \varphi_\epsilon^k\right) \cdot \mathbf{w}^S \, d\mathbf{x} \quad \forall \mathbf{w}^S \in W^S$$

where $\varphi_\epsilon^k(\mathbf{X}) = \mathbf{X} + \mathbf{u}_\epsilon^k(\mathbf{X})$.
Step 4. Stopping test: if $\left\|\mathbf{u}_\epsilon^k - \mathbf{u}_\epsilon^{k+1}\right\|_{0,\Omega_0^S} \leq tol$, then **Stop**
Step 5. Compute the characteristic function $\chi^S_{u_\epsilon^{k+1}}$, put $k := k+1$ and **Go to Step 2.**

Under the assumption of small displacements for the structure, we can approach the Jacobian determinant J by 1 and the gradient of the deformation $\mathbf{F}$ by the identity matrix $\mathbf{I}$.

Deformation of a tall building under the action of wind



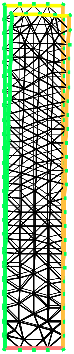
Physical parameters

The tall building: height $H = 180\ m$, length $L = 30\ m$, mass density $\rho^S = 160\ Kg/m^3$, Young modulus $E^S = 2.3 \times 10^8\ N/m^2$, Poisson's ratio $\nu^S = 0.25$, the applied volume forces on the structure $\mathbf{f}^S : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{f}^S = (0, -9.81\rho^S)\ N/m^3$.

The fluid is the air with: mass density $\rho^F = 1.25\ Kg/m^3$, dynamic viscosity $\mu^F = 7.03 \times 10^{-2}\ N \cdot s/m^2$, the applied volume forces on the fluid $\mathbf{f}^S : D \to \mathbb{R}^2$, $\mathbf{f}^F = (0, -9.81\rho^F)\ N/m^3$. The inflow velocity profile is

$$\mathbf{g}(x_1, x_2) = 100\left(\frac{x_2}{H}\right)^{0.19}\ m/s.$$

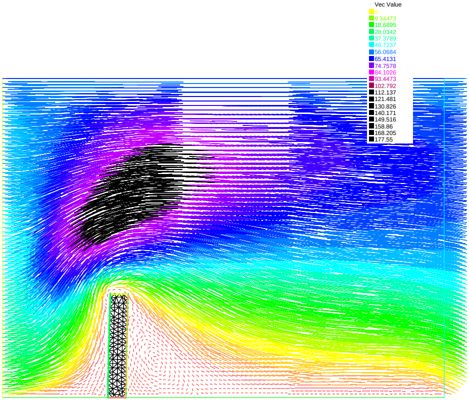
The initial and the deformed structure mesh



The fluid velocity in the fictitious domain

ϵ	$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}}$	$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} / \epsilon$
0.00100	0.009443	9.44397
0.00050	0.004723	9.44766
0.00010	0.000949	9.49400
0.00005	0.000607	12.1533

The fluid velocity around the final position of the structure



Thank you!