Observers for populations dynamics

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Outline

1. The model
2. Spectral properties
3. Detectability
4. Application: observer design for populations dynamics
5. Numerical approximation
Outline

1. The model
2. Spectral properties
3. Detectability
4. Application: observer design for populations dynamics
5. Numerical approximation
The model

Spectral properties
Detectability
Application : observer design for populations dynamics
Numerical approximation

Model

\[
\begin{aligned}
\partial_t p(a, x, t) &+ \partial_a p(a, x, t) \\
&= -\mu(a)p(a, x, t) + k\Delta p(a, x, t), \quad a \in (0, a^*), \ x \in \Omega, \ t > 0,
\end{aligned}
\]

\[
p(a, x, t) = 0, \quad a \in (0, a^*), \ x \in \partial\Omega, \ t > 0,
\]

\[
p(a, x, 0) = p_0(a, x), \quad a \in (0, a^*), \ x \in \Omega,
\]

\[
p(0, x, t) = \int_0^{a^*} \beta(a)p(a, t, x) \, da, \quad x \in \Omega, \ t > 0.
\]

- $p(a, x, t)$ denotes the distribution density of the population of age $a$ at spatial position $x$ at time $t$;
- $p_0$ denotes the initial distribution and $a^*$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates (which are supposed to be independent of $x$);
- $\Omega$ is a smooth bounded domain, $k$ is a positive constant diffusion coefficient and $\Delta$ the laplacian with respect to $x$. 

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Typical assumptions on $\beta$ and $\mu$ are the following:

- $\beta \in L^2(0, a^*) \cap L^\infty(0, a^*)$, $\beta \geq 0$ a.e. in $(0, a^*)$;
- $\mu \in L^1_{\text{loc}}(0, a^*)$, $\mu \geq 0$ a.e. in $(0, a^*)$ and

$$\lim_{a \to a^*} \int_0^a \mu(s) \, ds = +\infty.$$

We also introduce the function

$$\pi(a) := \exp \left( - \int_0^a \mu(s) \, ds \right)$$

which represents the probability to survive at age $a > 0$. Note that $\lim_{a \to a^*} \pi(a) = 0$.  

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Assume that $p_0$ is unknown.

The aim of this talk is to design an observer for (1). More precisely, we want to recover the distribution density of the population $p(a, x, t)$ for arbitrary $a$ and $x$ and for $t$ large enough, from the knowledge of $p(a, x, t)$ for arbitrary $t$ but only for an age interval $(a_1, a_2)$ (where $0 \leq a_1 < a_2 \leq a^*$) and a subdomain $\mathcal{O} \subset \Omega$. In other words, the available output is $y(t) = p|_{(a_1, a_2) \times \mathcal{O}}$ where $t \in (0, T)$ ($T$ has to be chosen large enough).
Some references on population dynamics

- Semigroup properties: Song et al., Chan, Guo, Li et al., Langlais, Walker
- Controllability problems: Ainseba, Anita, Iannelli, Langlais, Traoré, Kavian
- Inverse problems: Traoré, Rundell, Filin, Perasso, Picart
- Numerical aspects: Lopez, Trigiante, Douglas, Milner, Huyer, Guo, Gerardo-Giorda
First order system

We introduce the Hilbert space $X := L^2((0, a^*) \times \Omega)$, and define the following unbounded operator $A$ of $X$:

$$D(A) = \{ \varphi \in X \mid \varphi(a, \cdot) \in H^2(\Omega) \cap H^1_0(\Omega) \text{ for almost all } a \in (0, a^*);$$

$$\varphi(0, x) = \int_0^{a^*} \beta(a) \varphi(a, x) \, da \text{ for almost all } x \in \Omega;$$

$$-\partial_a \varphi - \mu \varphi + k \Delta \varphi \in X \}$$

$$A\varphi = -\partial_a \varphi - \mu \varphi + k \Delta \varphi, \quad \forall \varphi \in D(A).$$

Then problem (1) reads in the compact form

$$\begin{cases}
\dot{p}(t) := Ap(t), & t > 0 \\
p(0) = p_0.
\end{cases}$$

(2)
Theorem

The operator $A$ is the infinitesimal generator of a $C_0$ semigroup $e^{tA}$ on $X$.

Consequently, if $p_0 \in X$, there exists a unique solution $p \in C([0, \infty), X)$ of (2), and if $p_0 \in D(A)$, there exists a unique solution $p \in C([0, \infty), D(A)) \cap C^1([0, \infty), X)$ of (2).
Outline

1. The model

2. **Spectral properties**
   - The diffusion free population dynamics
   - The population dynamics with diffusion

3. Detectability

4. Application: observer design for populations dynamics

5. Numerical approximation
The diffusion free case is described by the so-called McKendrick–Von Foerster model (1959):

\[
\begin{cases}
    \frac{\partial}{\partial t} p(a, t) + \frac{\partial}{\partial a} p(a, t) = -\mu(a) p(a, t), & a \in (0, a^*), \ t > 0, \\
    p(a, 0) = p_0(a), & a \in (0, a^*), \\
    p(0, t) = \int_0^{a^*} \beta(a) p(a, t) \, da, & t > 0.
\end{cases}
\]
First order system

The population operator $A_0$ corresponding to the above system is defined as follows

$$D(A_0) = \left\{ \varphi \in H^1(0, a^*) \mid \varphi(0) = \int_0^{a^*} \beta(a) \varphi(a) \, da; \right.$$ \[ \begin{align*}
- \frac{d\varphi}{da} - \mu \varphi & \in L^2(0, a^*) \end{align*} \right\}.

Then the McKendrick–Von Foerster model reads in the compact form

$$\begin{cases}
\dot{p}(t) := A_0 p(t), & t > 0 \\
p(0) = p_0 \end{cases}.$$
Theorem

The operator $A_0$ has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity. The eigenvalues $(\mu_n)_{n \geq 1}$ of $A_0$ (counted without multiplicity) are the solutions of the characteristic equation

$$F(\mu) := \int_0^{a^*} \beta(a)e^{-\mu a} \pi(a) \, da = 1.$$  

The eigenvalues $(\mu_n)_{n \geq 1}$ are of geometric multiplicity one, the eigenspace associated to $\mu_n$ being the one-dimensional subspace of $L^2(0, a^*)$ generated by the function

$$\psi_n(a) = e^{-\mu_n a} \pi(a) = e^{-\mu_n a - \int_0^a \mu(s) \, ds}.$$
Spectral properties [Song et al., 1982]

Theorem

Finally, every vertical strip of the complex plane \( \alpha_1 \leq \Re(z) \leq \alpha_2 \), \( \alpha_1, \alpha_2 \in \mathbb{R} \), contains a finite number of eigenvalues of \( A_0 \).
Spectral properties [Song et al., 1982]

Theorem

The operator $A_0$ has a unique real eigenvalue $\mu_1$. Moreover, we have the following properties

1. $\mu_1$ is of algebraic multiplicity one;
2. $\mu_1 > 0$ (reps. $\mu_1 < 0$) if and only if $F(0) > 1$ (resp. $F(0) < 1$);
3. $\mu_1$ is a real dominant eigenvalue:

$$\mu_1 > \Re(\mu_n), \forall n \geq 2.$$
Recall:

\[ \mathcal{D}(A) = \{ \varphi \in X \mid \varphi(a, \cdot) \in H^2(\Omega) \cap H^1_0(\Omega) \text{ for almost all } a \in (0, a^*) ; \]
\[ \varphi(0, x) = \int_0^{a^*} \beta(a) \varphi(a, x) \, da \text{ for almost all } x \in \Omega; -\partial_a \varphi - \mu \varphi + k \Delta \varphi \in X \} \]
\[ A\varphi = -\partial_a \varphi - \mu \varphi + k \Delta \varphi, \quad \forall \varphi \in \mathcal{D}(A). \]

Let \( 0 < \lambda^D_1 < \lambda^D_2 \leq \lambda^D_3 \leq \cdots \) be the increasing sequence of eigenvalues of \(-k\Delta\) with Dirichlet boundary conditions and let \((\varphi_n)_{n \geq 1}\) be a corresponding orthonormal basis of \(L^2(\Omega)\).

Let \((\mu_n)_{n \geq 1}\) and \((\psi_n)_{n \geq 1}\) be respectively the sequence of eigenvalues and eigenfunctions of the free diffusion operator \(A_0\).
Spectral properties [Chan and Guo, 1989]

Theorem

1. The operator $A$ has compact resolvent and its eigenvalues are

$$\sigma(A) = \left\{ \mu_i - \lambda_j^D \mid i, j \in \mathbb{N}^* \right\}$$

2. $A$ has a real dominant eigenvalue:

$$\lambda_1 = \mu_1 - \lambda_1^D > \text{Re}(\lambda), \forall \lambda \in \sigma(A), \lambda \neq \lambda_1.$$  

$\lambda_1$ is a simple eigenvalue, the corresponding eigenspace being generated by

$$\Phi_1(a, x) := \psi_1(a) \varphi_1(x) = e^{-\mu_1 a} \pi(a) \varphi_1(x).$$

3. The eigenspace associated to an eigenvalue $\lambda$ of $A$ is given by

$$\text{Span} \left\{ \psi_i(a) \varphi_j(x) = e^{-\mu_i a} \pi(a) \varphi_j(x) \mid \mu_i - \lambda_j^D = \lambda \right\}.$$
Figure: The spectrums of the free diffusion operator $A_0$ (green) and of $-k \Delta$ (red).
Compactness [Chan and Guo, 1989]

**Proposition**

The semigroup $e^{tA}$ generated on $X$ by $A$ is compact for $t \geq a^*$.  

According to Zabczyk, this implies in particular that

$$\omega_a(A) = \omega_0(A)$$

where $\omega_a(A) := \lim_{t \to +\infty} t^{-1} \ln \|e^{tA}\|$ denotes the type of $e^{tA}$ and $\omega_0(A) := \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}$ the spectral bound of $A$. It is worth noticing that the above condition ensures that the exponential stability of $e^{tA}$ is equivalent to the condition $\omega_0(A) < 0$. 

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Abstract framework

Let $A : \mathcal{D}(A) \to X$ be a linear operator with compact resolvent on a Hilbert space $X$ generating a $C_0$-semigroup in $X$, and let $C \in \mathcal{L}(X, Y)$, where $Y$ is another Hilbert space. We assume

(A1.) $A$ admits $M$ eigenvalues (counted without multiplicities) with real part greater than 0. More precisely we can reorder the eigenvalues $(\lambda_n)_{n \in \mathbb{N}^*}$ of $A$ so that the sequence $(\text{Re } \lambda_n)_{n \in \mathbb{N}^*}$ is nonincreasing, and we suppose that $M \in \mathbb{N}^*$ is such that

$$\cdots \leq \text{Re } \lambda_{M+1} < 0 \leq \text{Re } \lambda_M \leq \cdots \leq \text{Re } \lambda_2 \leq \text{Re } \lambda_1.$$

(A2.) We have the equality

$$\omega_a(A) = \omega_0(A).$$
The pair \((A, C)\) is detectable if there exists \(H \in \mathcal{L}(Y, X)\) such that \((A + HC)\) generates an exponentially stable semigroup. Such an operator \(H\) is called a stabilizing output injection operator for \((A, C)\).

In this section, our goal is to show that the detectability of the infinite dimensional system \((A, C)\) can be deduced from the detectability of the finite dimensional system corresponding to the unstable part of \(A\) (namely the projection on the finite dimensional space associated with the unstable eigenvalues \(\lambda_1, \cdots, \lambda_M\)) provided a Hautus type assumption is satisfied.
We set \( \Sigma_+ := \{\lambda_1, \ldots, \lambda_M\} \) and let \( \Gamma_+ \) be a positively oriented curve enclosing \( \Sigma_+ \) but no other point of the spectrum \( \sigma(A) \) of \( A \). Let \( P_+ : X \to X \) be the projection operator defined by

\[
P_+ := -\frac{1}{2\pi i} \int_{\Gamma_+} (\xi - A)^{-1} d\xi.
\]
Splitting

We set \( X_+ := P_+ X \) and \( X_- := (I - P_+)X \), and then \( P_+ \) provides the following decomposition of \( X \)

\[
X = X_+ \oplus X_-.
\]

Following [Triggiani, Raymond, Thevenet, Badra & Takahashi], we can decompose our system into two subsystems:

- a finite dimension one to be stabilized,
- a stable infinite dimensional one.

More precisely, \( X_+ \) and \( X_- \) are invariant subspaces under \( A \) (since \( A \) and \( P_+ \) commute) and the spectra of the restricted operators \( A\big|_{X_+} \) and \( A\big|_{X_-} \) are respectively \( \Sigma_+ \) and \( \sigma(A) \setminus \Sigma_+ \). We also set

\[
A_+ := A\big|_{\mathcal{D}(A) \cap X_+} : \mathcal{D}(A) \cap X_+ \to X_+,
\]

\[
A_- := A\big|_{\mathcal{D}(A) \cap X_-} : \mathcal{D}(A) \cap X_- \to X_-.
\]
The space $X_+$ is the finite dimensional space spanned by the generalized eigenfunctions of $A$ associated to the unstable eigenvalues:

$$X_+ = \bigoplus_{k=1}^{M} \text{Ker} \left( A - \lambda_k \right)^{m^P_k}$$

where $m^P_k$ is the multiplicity of the pole $\lambda_k$ in the resolvent. If $m^A_k := \text{dim} \text{Ker} \left( A - \lambda_k \right)^{m^P_k}$ denotes the algebraic multiplicity of $\lambda_k$, $k = 1, \ldots, M$, then the dimension of $X_+$ is the sum of the algebraic multiplicities:

$$\text{dim} \ X_+ = \sum_{k=1}^{M} m^A_k.$$ 

If $A_+ := A|_{D(A) \cap X_+}$ is diagonalizable, then $m^G(\lambda_k) = m^A(\lambda_k)$ for all positive eigenvalues $\lambda_k$ of $A$. This implies in particular that the unstable space is $X_+ = \bigoplus_{k=1}^{M} \text{Ker} \left( A - \lambda_k \right)$, since $m^A(\lambda_k) = 1$, for all $k = 1, \ldots, M$. 

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Theorem

Let $Q_+ : Y \to Y_+ := CX_+$ be the orthogonal projection operator from $Y$ to $Y_+$, $i_{X_+} : X_+ \to X$ be the injection operator from $X_+$ into $X$ and let

$$C_+ = Q_+ C i_{X_+} \in \mathcal{L}(X_, Y_+).$$

Assume that the finite dimensional projected system $(A_+, C_+)$ is detectable through a stabilizing output injection operator $H_+ \in \mathcal{L}(Y_+, X_+)$. Then, the infinite dimensional system $(A, C)$ is detectable through the stabilizing output injection operator

$$H = i_{X_+} H_+ Q_+ \in \mathcal{L}(Y, X).$$  \hspace{1cm} (3)
Proof: splitting

For $H \in \mathcal{L}(Y, X)$, consider the system

$$\dot{z}(t) = Az(t) + HCz(t). \quad (4)$$

If we write $z = z_+ + z_-$ where $z_+ := P_+ z$ and $z_- := (I - P_+) z$, by applying $P_+$ and $(I - P_+)$ to (4), there is a corresponding splitting of (4) into two equations satisfied by $z_+$ and $z_-$ respectively

$$\dot{z}_+(t) = A_+ z_+(t) + P_+ HCz(t) \quad \text{and} \quad \dot{z}_-(t) = A_- z_-(t) + (I - P_+) HCz(t).$$

Using the definition $H = iX_+ H_+ Q_+$ and the facts that $P_+ iX_+ = \text{Id}_{X_+}$ and $(I - P_+) iX_+ = 0$, we obtain

$$\dot{z}_+(t) = A_+ z_+(t) + H_+ Q_+ Cz(t) \quad \text{and} \quad \dot{z}_-(t) = A_- z_-(t).$$
It follows from assumption (A2.) that $z_-$ is exponentially stable:

$$\|z_-(t)\| \leq Ke^{-\omega_- t} \|z(0)\|$$

where $\omega_- = -\text{Re} \lambda_{N+1} > 0$. On the other hand, by using the definition $C_+ = Q_+Ci_{X_+}$ and since $i_{X_+}z_+ = z_+$, we have

$$\dot{z}_+(t) = A_+z_+(t) + H_+Q_+C(z_+(t) + z_-(t))$$
$$= A_+z_+(t) + H_+Q_+Ci_{X_+}z_+(t) + H_+Q_+Cz_-(t)$$
$$= (A_+ + H_+C_+)z_+(t) + H_+Q_+Cz_-(t).$$
Proof: Duhamel’s formula

Using Duhamel’s formula, we get

\[ z_+(t) = T^+_t z_+(0) + \int_0^t T^+_t H_+ Q_+ C z_-(s) ds, \]

where \( T^+_t \) is the semigroup generated by \( (A_+ + H_+ C_+) \), which is exponentially stable by the detectability assumption, i.e. there exists \( \omega_+ > 0 \) such that

\[
\| T^+_t x \| \leq K e^{-\omega_+ t} \| x \| \quad \forall x \in X_+, \forall t > 0.
\]

Combined with exponential stability of \( z_- \), this yields

\[
\| z_+(t) \| \leq K \left\{ e^{-\omega_+ t} \| z_+(0) \| + \| H_+ \| \| C \| \int_0^t e^{-\omega_+ (t-s)} e^{-\omega_- s} \| z_-(0) \| ds \right\},
\]

and consequently

\[
\| z_+(t) \| \leq K \left( e^{-\omega_+ t} + \| H_+ \| \| C \| \frac{e^{-\omega_+ t} - e^{-\omega_- t}}{\omega_- - \omega_+} \right) \| z_0 \|.
\]
It is then sufficient to choose $\omega_+$ small enough such that $0 < \omega_+ < \omega_-$ to have the exponential decay of $t \mapsto z_+(t)$:

$$
\|z_+(t)\| \leq Ke^{-\omega_+ t} \|z_0\|, \quad t > 0.
$$

(6)

Relations (5) and (6) yield immediately the exponential decay of $z = z_+ + z_-$. ■
Hautus test

The following result provide a Hautus type sufficient condition for the detectability of the projected system \((A_+, C_+)\).

**Proposition**

*If the Hautus test*

\[
(\varphi \in \mathcal{D}(A) \mid A\varphi = \lambda \varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0) \implies \varphi = 0
\]

*is satisfied, then \((A_+, C_+)\) is detectable.*

**Proof:** Since \(C_+ z_+ = C z_+\) for any \(z_+ \in X_+\) by definition of \(Y_+\) and \(Q_+\), if the Hautus criterion is satisfied, then it is clear that the following Hautus test is also satisfied:

\[
(\varphi \in \mathcal{D}(A) \cap X_+ \mid A_+\varphi = \lambda \varphi \text{ and } C_+\varphi = 0) \implies \varphi = 0.
\]

As the above system is finite dimensional, \((A_+, C_+)\) is detectable. ■
Combining the Theorem and Proposition shows that \((A, C)\) is detectable via the stabilizing output injection operator \(H\) defined as previously provided that the Hautus test is satisfied.

Classically, the stabilizing output injection operator \(H^+\) of the finite dimensional projected system \((A^+, C^+)\) can be determined by solving a (finite dimensional) Riccati equation.

It is worth noticing that the matrices \(A^+\) and \(C^+\) are in practice of small size (their dimensions are respectively \(\dim X_+ \times \dim X_+\) and \(\dim Y_+ \times \dim X_+\)), making the computation of the Riccati matrix affordable.
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1. The model
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Goal

Going back to the population system (1)

\[
\begin{aligned}
\partial_t p(a, x, t) + \partial_a p(a, x, t) &= -\mu(a)p(a, x, t) + k\Delta p(a, x, t), \quad a \in (0, a^*), x \in \Omega, t > 0 \\
p(a, x, t) &= 0, \quad a \in (0, a^*), x \in \partial\Omega, t > 0 \\
p(a, x, 0) &= p_0(a, x), \quad a \in (0, a^*), x \in \Omega, \\
p(0, x, t) &= \int_0^{a^*} \beta(a)p(a, t, x) da, \quad x \in \Omega, t > 0,
\end{aligned}
\]

our aim is to design an observer for (1).

More precisely, we want to recover the distribution density of the population \( p(a, x, t) \) for arbitrary \( a \) and \( x \) and for \( t \) large enough, from the knowledge of \( p(a, x, t) \) for arbitrary \( t \) but only for an age interval \( (a_1, a_2) \) (where \( 0 \leq a_1 < a_2 \leq a^* \)) and a subdomain \( O \subset \Omega \).

In other words, the available output is \( y(t) = p|_{(a_1, a_2) \times O} \) where \( t \in (0, T) \) (\( T \) has to be chosen large enough).
More precisely, with the notation of Section 1, let $A$ be defined in Section 1 and let $p_0 \in X$ be given. We consider the abstract system (2), i.e.

$$\begin{align*}
\dot{p}(t) &= Ap(t), \quad t \in (0, T) \\
p(0) &= p_0,
\end{align*}$$

with the output

$$y(t) = Cp(t), \quad t \in (0, T),$$

where the observation operator $C \in \mathcal{L}(X, Y)$, $Y := L^2((a_1, a_2) \times \mathcal{O})$ is defined by

$$C\varphi := \varphi|_{(a_1, a_2) \times \mathcal{O}} \text{ for all } \varphi \in X.$$
We introduce the observer

\[
\begin{align*}
\dot{\hat{p}}(t) &= A\hat{p}(t) + HC\hat{p}(t) - Hy(t), \quad t \in (0, T) \\
\hat{p}(0) &= 0,
\end{align*}
\]

where \( H \in \mathcal{L}(Y, X) \) is a linear operator to be defined.

Then the error \( e := \hat{p} - p \) satisfies

\[
\begin{align*}
\dot{e}(t) &= (A + HC)e(t), \quad t \in (0, T) \\
e(0) &= -p_0.
\end{align*}
\]
Assumptions (A1.) and (A2.) given in Section 3 are satisfied by the above system and the problem of determining the stabilizing output injection operator $H$ for $(A, C')$ fits into the framework described in Section 3. Therefore we can apply this approach.

Let $N$ denote the number of (distinct) eigenvalues of $A$ with positive real part and $\Sigma_+$ the set of these unstable eigenvalues. Let $\Gamma_+$ be a positively oriented curve enclosing $\Sigma_+$ but no other point of the spectrum of $A$, and let $P_+ : X \rightarrow X$ be the projection operator defined as previously.
It remains to verify that the Hautus test is satisfied for our system $(A, C)$:

**Lemma**

If $\Phi \in \mathcal{D}(A)$ satisfies $A\Phi = \lambda \Phi$ for $\lambda \in \Sigma_+$ and $C\Phi = 0$, then $\Phi$ vanishes identically.
Proof of the Hautus test

Let $\lambda$ be an unstable eigenvalue of $A$ and $\Phi$ an associated eigenfunction: $A\Phi = \lambda \Phi$, $\Phi \in \mathcal{D}(A)$, $\Phi \neq 0$ and $\Re \lambda \geq 0$. Then, we have

$$\Phi(a, x) = \sum_{i,j\mid \lambda = \mu_i - \lambda_j^D} \alpha_j \Phi_{ij}(a, x) = \sum_{i,j\mid \lambda = \mu_i - \lambda_j^D} \alpha_j e^{-\mu_i a} \pi(a) \varphi_j(x).$$

Assume now that $C\Phi = \Phi\big|_{(a_1, a_2) \times \mathcal{O}} = 0$, i.e. that

$$\sum_{i,j\mid \lambda = \mu_i - \lambda_j^D} \alpha_j e^{-\mu_i a} \pi(a) \varphi_j\big|_{\mathcal{O}} = 0, \quad a \in (a_1, a_2).$$

As the eigenvalues $(\mu_i)_{i \in \mathbb{N}}$ are distinct, this equation reduces to

$$\sum_{j\mid \lambda + \lambda_j^D \in \sigma(A_0)} \alpha_j e^{-\mu_{i(j)} a} \varphi_j\big|_{\mathcal{O}}(x) = 0, \quad a \in (a_1, a_2),$$

where $i(j) \in \mathbb{N}^*$ is the unique index s.t. $\mu_i = \lambda + \lambda_j^D$. Since the eigenfunctions of $-k\Delta$ with Dirichlet boundary cond. are analytic, $\Phi = 0$. 

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Main result

This immediately leads to the following result.

**Theorem**

Let $p_0 \in X$ and $p$ the solution of (1). Assume that $y(t) = p|_{(a_1, a_2) \times \mathcal{O}} (t > 0)$ is known. Let $\hat{p}$ the observer defined by (7), where $H \in \mathcal{L}(Y, X)$ is the (finite dimensional) stabilizing output injection operator defined by (3). Then, there exists $C > 0$ and $\omega > 0$ such that

$$\|\hat{p}(t) - p(t)\| \leq C e^{-\omega t} \|p_0\|, \quad t > 0.$$
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   - Rescaling the problem
   - Finite element discretization in space
   - Finite difference discretization in age
   - Detectability at the discrete level
From now on, we make the following assumptions for the sake of simplicity:

1. The birth rate $\beta$ belongs to $C^1([0, a^*])$ and $\beta(0) = 0$.

2. $A_+ := A|_{\mathcal{D}(A) \cap X_+}$ is diagonalizable, i.e. $m^G(\lambda_k) = m^A(\lambda_k)$ for all positive eigenvalues $\lambda_k$ of $A$. This implies in particular that the unstable space is $X_+ = \bigoplus_{k=1}^{M} \text{Ker} (A - \lambda_k)$, since $m^A(\lambda_k) = 1$, for all $k = 1, \ldots, M$. 

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Rescaling the problem

In order to overcome the difficulties due to unboundedness of the coefficient $\mu$, we introduce the auxiliary variable

$$u(a, x, t) = \frac{p(a, x, t)}{\pi(a)} = \exp \left( \int_0^a \mu(s) \, ds \right) p(a, x, t).$$

which satisfies

$$\begin{cases} 
\partial_t u(a, x, t) + \partial_a u(a, x, t) - \Delta u(a, x, t) = 0, & a \in (0, a^*), \ x \in \Omega, \ t > 0, \\
u(a, x, t) = 0, & a \in (0, a^*), \ x \in \partial\Omega, \ t > 0, \\
u(a, x, 0) = u_0(a, x), & a \in (0, a^*), \ x \in \Omega, \\
u(0, x, t) = \int_0^{a^*} m(a)u(a, x, t) \, da, & x \in \Omega, \ t > 0 
\end{cases}$$

where we have set $u_0(a, x) = p_0(a, x)/\pi(a)$ and $m(a) = \beta(a)\pi(a)$ stands for the maternity function.

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Variational formulation

The variational formulation of the above problem leads to the following problem: Find \( u(t) \in L^2(0, a^*; H^1_0(\Omega)) \) such that for all \( v \in H^1_0(\Omega) \)

\[
\frac{d}{dt} \int_{\Omega} u(a, x, t)v(x) \, dx + \frac{d}{da} \int_{\Omega} u(a, x, t)v(x) \, dx + \int_{\Omega} \nabla u(a, x, t) \cdot \nabla v(x) \, dx = 0
\]

with the initial conditions

\[
\begin{cases}
  u(a, x, 0) = u_0(a, x), & a \in (0, a^*), \ x \in \Omega, \\
  u(0, x, t) = \int_0^{a^*} m(a)u(a, x, t) \, da, & x \in \Omega, \ t > 0.
\end{cases}
\]
Finite element discretization in space

We consider a $\mathbb{P}^1$ finite element approximation of $u(a, \cdot, t)$, that is we seek for the solution $u_h(a, \cdot, t) \in V_h$ such that for all $v_h \in V_h$

\[
\frac{d}{dt} \int_{\Omega} u_h(a, x, t)v_h(x) \, dx + \frac{d}{da} \int_{\Omega} u_h(a, x, t)v_h(x) \, dx \\
+ \int_{\Omega} \nabla u_h(a, x, t) \cdot \nabla v_h(x) \, dx = 0
\]

where

$V_h = \{ v_h \in C^0(\Omega); \, v_h|_{T_\ell} \in \mathbb{P}^1(T_\ell), \, \forall \ell = 1, \ldots, L, \, v_h|_{\partial \Omega} = 0 \}$ is the finite element space associated to the triangulation $\Omega = (T_\ell)_{\ell=1}^L$.

We denote by $(w_i)_{i=1}^{N_h}$ the finite element basis of the finite dimensional approximation space $V_h$. 

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Plugging the formula $u_h(a, x, t) = \sum_{i=1}^{N_h} u_i(a, t) w_i(x)$ in the above relation leads to the following problem for $U(a, t) := \begin{bmatrix} u_1(a, t) \\ \vdots \\ u_{N_h}(a, t) \end{bmatrix}$:

$$
\begin{cases}
\mathbb{M} \frac{dU}{dt}(a, t) + \mathbb{M} \frac{dU}{da}(a, t) + \mathbb{K} U(a, t) = 0, \\
U(a, 0) = U_{0h}(a) \\
U(0, t) = \int_0^{a^*} m(a) U(a, t) \, da,
\end{cases}
$$

(10)

where

- $\mathbb{M}$ and $\mathbb{K}$ denote respectively the mass and the stiffness matrices ($\mathbb{M}_{ij} = \int_\Omega w_i w_j$, $\mathbb{K}_{ij} = \int_\Omega \nabla w_i \cdot \nabla w_j$),

- $U_{0h}(a) = \begin{bmatrix} u_0(a, x_1) \\ \vdots \\ u_0(a, x_{N_h}) \end{bmatrix}$, where $(x_i)_{i=1}^{N_h}$ denote the nodes of the mesh.

Note that $\mathbb{M}$ and $\mathbb{K}$ are symmetric matrices and that $\mathbb{M}$ is invertible.
Let us denote by $u_i^k(t)$ an approximation of $u_i(k\Delta a, t)$ and by

$$U^k(t) := \begin{bmatrix} u_1^k(t) \\ \vdots \\ u_{N_h}^k(t) \end{bmatrix}$$

an approximation of $U(k\Delta a, t)$.

The approximation of $u(k\Delta a, x, t)$ is then given by

$$u(k\Delta a, x, t) \simeq \sum_{i=1}^{N_h} u_i^k(t) w_i(x).$$
Equation for $U^k(t), k = 1, \cdots, K$

Using a Crank-Nicholson scheme to approximate the solution of problem (10), we can move from age $(k - 1)\Delta a$ to age $k\Delta a$ following the scheme

$$
\frac{d}{dt} \mathbf{M} \left( \frac{U^k(t) + U^{k-1}(t)}{2} \right) + \frac{1}{\Delta a} \mathbf{M} \left( U^k(t) - U^{k-1}(t) \right) + K \left( \frac{U^k(t) + U^{k-1}(t)}{2} \right) = 0
$$

with the initial condition

$$
U^k(0) = U_{0h}(k\Delta a), \quad \forall k = 1, \ldots, K.
$$

This implies that, for any $k \in \{1, \ldots, K\}$, we have

$$
\frac{d}{dt} \mathbf{M} \left( U^k(t) + U^{k-1}(t) \right) = - \left( \frac{2}{\Delta a} \mathbf{M} + K \right) U^k(t) + \left( \frac{2}{\Delta a} \mathbf{M} - K \right) U^{k-1}(t).
$$
In order to obtain an equation for $U^0(t)$, we consider the continuous equation satisfied at age $a = 0$:

$$u(0, x, t) = \int_0^{a^*} m(a)u(a, x, t) \, da, \quad x \in \Omega, \ t > 0.$$  

Taking the time derivative of the above equation, we obtain after integration by parts that

$$\partial_t u(0, x, t) = \int_0^{a^*} [m(a)\Delta u(a, x, t) - m'(a)u(a, x, t)] \, da, \quad x \in \Omega, \ t > 0.$$  

The above relation suggests the following equation for $U^0(t)$

$$\frac{d}{dt}MU^0(t) = -\sum_{k=0}^{K} \omega_k [m(k\Delta a)K + m'(k\Delta a)M] U^k(t).$$

where we have used the approximation

$$\int_0^{a^*} f(a) \, da \simeq \sum_{k=0}^{K} \omega_k f(k\Delta a).$$
Consequently, setting

\[ B^k := -\omega_k \left[ m(k\Delta a)K + m'(k\Delta a)M \right] \]

\[ K^- := -\left( \frac{2}{\Delta a} M + K \right) \quad \quad \quad K^+ := \frac{2}{\Delta a} M - K, \]

we have

\[
\begin{aligned}
\frac{d}{dt} M U^0(t) &= \sum_{k=0}^{K} B^k U^k(t), \\
\frac{d}{dt} M (U^k(t) + U^{k-1}(t)) &= K^- U^k(t) + K^+ U^{k-1}(t), \quad \forall k = 1, \ldots, K \\
U^k(0) &= U_{0h}(k\Delta a), \quad \forall k = 0, \ldots, K.
\end{aligned}
\]
This can be written as

\[ M \dot{Z}(t) = \begin{bmatrix}
    B^0 & B^1 & \cdots & \cdots & \cdots & B^K \\
    K^+ & K^- & 0 & \cdots & \cdots & 0 \\
    0 & K^+ & K^- & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
    0 & \cdots & \cdots & 0 & K^+ & K^- \\
\end{bmatrix} \begin{bmatrix} Z(t) \end{bmatrix}, \]

where

\[ M = \begin{bmatrix}
    M & 0 & \cdots & 0 \\
    M & M & \ddots & \vdots \\
    \vdots & \ddots & \ddots & 0 \\
    0 & \cdots & M & M \\
\end{bmatrix} \quad \text{and} \quad Z(t) = \begin{bmatrix} U^0(t) \\
    U^1(t) \\
    \vdots \\
    U^K(t) \end{bmatrix} \]

is a vector of size \((K + 1)N_h\).
Discrete system

This is equivalent to

\[
\begin{cases}
E \dot{Z}(t) = AZ(t) \\
Z(0) = Z_0,
\end{cases}
\]

(11)

where

\[
E = M^T M, \quad A = M^T \\
\begin{bmatrix}
B^0 & B^1 & \cdots & \cdots & \cdots & B^K \\
K^+ & K^- & 0 & \cdots & \cdots & 0 \\
0 & K^+ & K^- & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & K^+ & K^- \\
\end{bmatrix}, \quad Z_0 = \begin{bmatrix}
U_{0h}(0) \\
U_{0h}(\Delta a) \\
\vdots \\
U_{0h}(K\Delta a)
\end{bmatrix}.
\]

Note that \( E \) and \( A \) are squared matrices of size \((K + 1)N_h\) and that \( E \) is symmetric definite positive (since \( M \) is invertible).

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Given a discretization \( C \in \mathbb{R}^{m \times (K+1)N_h} \) of \( C \), our goal here is to show that the detectability of the system \((A, C)\) can be deduced from the detectability of the smaller finite dimensional system corresponding to the unstable part of \( A \) (namely the projection on the finite dimensional space associated with the unstable eigenvalues \( \lambda_1, \cdots, \lambda_N \)).

To achieve this, we adapt the ideas developed in Section 3 for the continuous case, using arguments introduced in the PhD thesis of L. Thevenet.
We recall that the complex $\lambda$ is an eigenvalue of the matrix pencil $(A, E)$ if there exists $Z \neq 0$ such that $AZ = \lambda EZ$. We assume that $(A, E)$ admits $N$ eigenvalues (counted with multiplicities) with real part greater than 0 and, having in mind the estimation problem in population dynamics, we assume that there is only one real eigenvalue, which is moreover dominant. More precisely we reorder the eigenvalues $(\lambda_n)_n$ of $A$ so that the sequence $(\text{Re } \lambda_n)_n$ is nonincreasing, and we suppose that $N \in \mathbb{N}^*$ is such that

$$
\cdots \leq \text{Re } \lambda_{N+1} < 0 \leq \text{Re } \lambda_N \leq \cdots \leq \text{Re } \lambda_2 < \lambda_1.
$$

Moreover, we assume that $(A, E)$ admits $N$ eigenvectors, denoted by $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_N$, corresponding to the $N$ unstable eigenvalues $\lambda_1, \ldots, \lambda_N$. Similarly, we denote by $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_N$ the $N$ eigenvectors of $(A^T, E)$ corresponding to the $N$ unstable eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of $A^T$. 
Let
\[ \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N), \]
be the diagonal matrix corresponding to the unstable eigenvalues of \((A, E)\) and \(\Lambda^*\) its conjugate matrix. We denote by
\[ \tilde{\Phi} = \left[ \tilde{\Phi}_1 \mid \cdots \mid \tilde{\Phi}_N \right] \in \mathbb{C}^{(K+1)N_h \times N}, \]
the matrix of \(N\) complex eigenvectors of \((A, E)\) and by
\[ \tilde{\Psi} = \left[ \tilde{\Psi}_1 \mid \cdots \mid \tilde{\Psi}_N \right] \in \mathbb{C}^{(K+1)N_h \times N}, \]
the matrix of \(N\) complex eigenvectors of \((A^T, E)\). In other words, we have
\[ A \tilde{\Phi} = E \tilde{\Phi} \Lambda \quad \text{and} \quad A^T \tilde{\Psi} = E \tilde{\Psi} \Lambda^*. \]
We choose \(\tilde{\Phi}\) and \(\tilde{\Psi}\) satisfying
\[ \tilde{\Phi}^* E \tilde{\Psi} = I_{(K+1)N_h}. \]
Real vectors

In order to work with real subspaces, we introduce

$$\mathbf{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \mathbf{S} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{S} \end{bmatrix} \in \mathbb{C}^{N \times N},$$

where $\mathbf{U}$ is a unitary matrix constituted of $(N - 1)/2$ diagonal blocks of $\mathbf{S}$. We then set

$$\Phi = \tilde{\Phi} \mathbf{U}^* \in \mathbb{R}^{(K+1)N_h \times N} \quad \text{and} \quad \Psi = \tilde{\Psi} \mathbf{U}^* \in \mathbb{R}^{(K+1)N_h \times N},$$

This is equivalent to $\tilde{\Phi} = \Phi \mathbf{U}$ and $\tilde{\Psi} = \Psi \mathbf{U}$, since $\mathbf{U}$ is a unitary matrix.

As immediate consequence, we have

$$\Phi^T \mathbf{E} \Psi = \text{Id}_{(K+1)N_h}.$$
The projector on the unstable space

We define $P_+$ the projection onto the unstable subspace of the continuous operator $A$, parallel to the stable one and given by:

$$P_+ z = \sum_{m=1}^{N} (z, \psi_m) x \phi_m,$$

where $(\phi_1, \cdots, \phi_m)$ (resp. $(\psi_1, \cdots, \psi_m)$) is a basis of the unstable subspace associated to the operator $A$ (resp. $A^*$).

We can show that a discrete version of this operator is given by

$$P_+ = \Phi \Psi^T \begin{bmatrix} \omega_0 \mathbf{M} & & \\ & \ddots & \\ & & \omega_K \mathbf{M} \end{bmatrix} \in \mathbb{R}^{(K+1)N_h \times (K+1)N_h},$$

in the sense that as $h$ and $\Delta a$ tend to zero, $P_+ z \simeq P_+ Z$, $\forall z \in X$, where

$$Z = \begin{bmatrix} Z^0 \\ \vdots \\ Z^K \end{bmatrix} \in \mathbb{R}^{(K+1)N_h \times 1}, \quad Z^k = \begin{bmatrix} z(a^k_1, x_1) \\ \vdots \\ z(a^k_{N_h}, x_{N_h}) \end{bmatrix} \in \mathbb{R}^{N_h \times 1}.$$
Detectability of \((A, C)\) from the detectability of a smaller finite dimensional system

The pair \((A, C)\) is \(E\)-detectable if there exists \(L \in \mathbb{R}^{(K+1)Nh \times m}\) such that the solution \(Z\) of \(E\dot{Z}(t) = AZ(t) + LCZ(t)\) is exponentially stable. Such an operator \(L\) is called a stabilizing output injection operator for \((A, C)\).

**Theorem**

*Let*

\[
C_+ = C\Phi \in \mathbb{R}^{m \times N}
\]

*and*

\[
A_+ = \Psi^T A\Phi \in \mathbb{R}^{N \times N}.
\]

*Assume that the projected system \((A_+, C_+)\) of small dimension is detectable through a stabilizing output injection operator \(L_+ \in \mathbb{R}^{N \times m}\). Then, the high dimensional system \((A, C)\) is \(E\)-detectable through the stabilizing output injection operator

\[
L = E\Phi L_+ \in \mathbb{R}^{(K+1)Nh \times m}.
\]

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Observers for populations dynamics
Observer at the discrete level

Assume that $Y(t) = CZ(t)$, $(t \in (0, T))$ is known. Let $\hat{Z}$ the observer defined by

$$\begin{cases}
    E\dot{\hat{Z}}(t) = A\hat{Z}(t) + LC\hat{Z}(t) - LY(t) \\
    \hat{Z}(0) = 0,
\end{cases}$$

where $L \in \mathbb{R}^{(K+1)N_h \times m}$. Then the error $e = \hat{Z} - Z$ satisfies

$$\begin{cases}
    E\dot{e}(t) = (A + LC)e(t) \\
    e(0) = -Z_0.
\end{cases}$$

Consequently, if $(A_+, C_+)$ is detectable through $L_+ \in \mathbb{R}^{N \times m}$, then $(A, C)$ is detectable through $L = E\Phi L_+$ and

$$\|\hat{Z}(t) - Z(t)\| \leq Ke^{-\omega t} \|Z_0\|.$$
To go further

1. **Approximation**
   - Convergence analysis and error estimates, uniform exponential stability (with respect to $\Delta a$ et $h$)
   - Numerical validation

2. **Other models**
   - Age dependant coefficients $k = k(a)$
   - Nonlinear models