Corrigendum to “On the existence of smooth densities for jump processes”

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January 27, 2010

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We correct in this note a mistake in an intermediate result of [1]. Let us first recall the structure of the main results of this article:

$$\text{Theorems 2.1 and 2.2} \Rightarrow \text{Theorem 3.1} \Rightarrow \text{Theorem 4.1} \Rightarrow \begin{cases} \text{Corollary 4.4} \\ \text{Corollary 4.5} \end{cases}$$

- Theorem 2.1 is the main result about the existence of a smooth density for real variables defined on a Poisson space; Theorem 2.2 is the extension to vector-valued variables.
- Theorem 3.1 deals with the case of functionals of a Lévy process; in particular, an analogue of the Malliavin matrix of the Wiener case is introduced.
- Theorem 4.1 is the particular case of the solution of a stochastic differential equation driven by the Lévy process.
- Corollaries 4.4 and 4.5 consider respectively the case of a “uniformly elliptic” equation, and of the stochastic Lévy area.

[*] The author is grateful to John J. A. Hosking who pointed out this error.

However, there is a problem in the proof of Theorem 3.1. Technically, Theorem 3.1 is deduced from Theorem 2.1, so we have to verify that the assumptions of this latter theorem are fulfilled. In particular, we have to prove that condition (2.3) is verified for almost any $\tau \in A(\rho)^k$. Let us look carefully at the sequence of inequalities following (3.11); as it has been said, these inequalities should be valid for almost any $\tau \in A(\rho)^k$. The first inequality is trivial since the random variable R' satisfies $R' \leq \rho$. In the second inequality, we want to apply (3.4). A first problem is that R' is random whereas (3.4) was written for a deterministic radius. Moreover and more importantly, even if R' were deterministic, the application of (3.4) would require $\tau \in A(R')^k$, and this is not necessarily true since $A(R')$ may be strictly included in $A(\rho)$. Thus the proof of the theorem is not correct.

Our aim in this note is to give a corrected statement for Theorem 3.1, and to consider the consequences on subsequent results. In particular, Corollary 4.4 and 4.5 are still valid; Corollary 4.4 is used for instance in [2] which therefore is not changed by this correction.

Correction of Theorem 3.1

Condition (b) of Theorem 3.1 has to be replaced by a modified condition (b'), and the theorem becomes the following one.

Theorem 3.1'. *Suppose that the Lévy measure μ satisfies the conditions of Corollary 1.2(b). Let $T > 0$ and let F be a $\mathbb{R}^d$ valued functional of $(X_t; 0 \leq t \leq T)$ satisfying*

(a) for any p and k, the condition (3.1) holds true;

(b') there exists a matrix-valued process ψ_t such that for $|x| \leq 1$, $p \geq 1$, $k \geq 0$ and $\tau \in A(1)^k$,

$$\left\| \big(D_{tx}F - \psi_t x\big) \circ \varepsilon_\tau^+ \right\|_p \leq C_{p,k}|x|^r \tag{3.2'}$$

for some $r > 1$, and

$$\left\| \Big(\det \int_0^T \psi_t \psi_t^\star dt\Big)^{-1} \circ \varepsilon_\tau^+ \right\|_p \leq C_{p,k}. \tag{3.3'}$$

Then F has a C_b^∞ density.

Proof. Let us emphasise the changes with respect to the original Theorem 3.1. We have to check that the condition (2.3) in Theorem 2.1 is satisfied. Actually, the original proof is correct without the operator ε_τ^+, or equivalently with $k = 0$ and $\tau = \emptyset$. For the case $k \geq 1$, we simply have to apply the original proof to the variable $F\circ\varepsilon_\tau^+$. This variable satisfies (3.1) from Lemma 3.2; it also satisfies (3.2) and (3.3) for the matrix $\psi_t\circ\varepsilon_\tau^+$; thus, since $D_u(F\circ\varepsilon_\tau^+)=(D_uF)\circ\varepsilon_\tau^+$ for almost any u, (2.3) is satisfied. Moreover, the constants involved in the estimations do not depend on τ. □

Consequence on Theorem 4.1

Theorem 4.1 is correct if we replace the non-degeneracy condition (3.3) by the above (3.3').

With respect to the original proof, we have to check that the condition (3.2') is satisfied with ε_τ^+. This means that in the computations we have to replace respectively the solution Y_t and the semi-flow ϕ_{st} by $Y_t\circ\varepsilon_\tau^+$ and $\phi_{st}\circ\varepsilon_\tau^+$ for $\tau=(t_1,x_1,\dots,t_k,x_k)$, $|x_i|\le 1$; the bounds have to be uniform in τ for k fixed. Proving that the moments of $Y_t\circ\varepsilon_\tau^+$ are bounded is similar to the original study of Y_t. On the other hand, we have to extend Lemma 4.3 and estimate $\sup_{s,t}\big|(\phi_{st}\circ\varepsilon_\tau^+)^{(k)}(y)\big|$. The original proof can be used after inserting the k jumps (t_i,x_i) in the path of the Lévy process X. The number of stopping times σ_j is not changed too much, and the theorem follows.

Consequence on Corollaries 4.4 and 4.5

The two corollaries remain correct. This means that (3.3') can be proved in the two cases.

For Corollary 4.4, it is not a restriction to assume that $\rho_0>1$, so that $y\mapsto y+a(y,x)$ is a diffeomorphism for $|x|\le 1$. Then the times σ_j of the original proof are not changed by ε_τ^+, $\tau\in A(1)^k$, and $(Z_t^{\sigma_j}\circ\varepsilon_\tau^+)^{-1}$ can be estimated similarly.

For the proof of Corollary 4.5, the main point is that the variables X_t-X_s, $|t-s|\ge\delta$, have uniformly bounded density; but a deterministic translation does not modify the bound on the density, so the densities of $(X_t-X_s)\circ\varepsilon_\tau^+$ are also bounded.

References

[1] J. Picard, *On the existence of smooth densities for jump processes*, Probab. Theory Relat. Fields **105** (1996), 481–511.

[2] J. Picard, *Density in small time at accessible points for jump processes*, Stochastic Process. Appl. **67** (1997), 251–279.