MARTINGALES ON RIEMANNIAN MANIFOLDS WITH PRESCRIBED LIMIT

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Running head. Manifold-valued martingales

Abstract. We are given a random variable on a Riemannian manifold and we study the set of manifold-valued martingales converging to this variable; more precisely we are interested in uniqueness and existence theorems. We use stochastic calculus tools; for the existence, we restrict ourselves to Wiener probability spaces and we apply the differential calculus which has been developed on them. The relations with some nonlinear partial differential equations are discussed.

0. Introduction

It is well-known that real-valued martingales can be used in the study of some partial differential equations, such as the heat equation and the Dirichlet problem; let us describe briefly how this can be done. Consider a smooth manifold U and a second order differential operator $\mathcal{L}$ on U without term of order zero; if ϕ is a real-valued function defined on U, the heat equation consists of finding a real-valued function f defined on $(-\infty, 0] \times U$ and satisfying

$$\frac{\partial f}{\partial t} + \mathcal{L} f = 0, \qquad f(0, .) = \phi. \tag{0.1}$$

The probabilistic solution of (0.1) is as follows; let Ω be the canonical space of continuous paths from $(-\infty, 0]$ into U, let X_t be the canonical process and let $\mathbb{P}^{s,x}$ be the law of the Markov process with infinitesimal generator $\mathcal{L}$ and initial value x at time s; then a smooth function f is solution of (0.1) if and only if for any (s, x), the process $f(t, X_t)$, $t \geq s$ is a local $\mathbb{P}^{s,x}$-martingale with final value $\phi(X_0)$. Now let $\overline{U}$ be a manifold with boundary ∂U and interior U; if ϕ is a real-valued function defined on ∂U, the Dirichlet problem consists of finding a real-valued function f defined on $\overline{U}$ and satisfying

$$\mathcal{L} f = 0 \quad \text{on} \quad U, \qquad f = \phi \quad \text{on} \quad \partial U. \tag{0.2}$$

This problem has also a probabilistic counterpart; let Ω be the canonical space of continuous paths from $[0, \infty)$ into $\overline{U}$ stopped when they quit U, let X_t be the canonical process, let

$$\tau = \inf\{t \geq 0, \quad X_t \in \partial U\}, \tag{0.3}$$

let $\mathbb{P}^x$ be the law of the Markov process with generator $\mathcal{L}$ stopped at τ with initial value x at time 0, and suppose that $\tau < \infty$ P^x-almost surely for any x; then a smooth function f is solution of (0.2) if and only if for any x, the process $f(X_t)$ is a local $\mathbb{P}^x$-martingale converging to $\phi(X_\tau)$. Thus both problems can be solved by considering local martingales converging to some fixed random variable.

Let V be a complete Riemannian manifold; the Riemannian metric defines a Hessian operator; if g is a smooth real-valued function defined on V, its Hessian at point x will be denoted by $g''(x)$; it is a bilinear form on $T_x V$; if U and $\mathcal{L}$ are as above and if f is a smooth function from U into V, one can define a field $\mathcal{L}_V f$ by the following formula; it is a section of the tangent space TV satisfying

$$\big(\mathcal{L}_V f(x), (g' \circ f)(x)\big) = \mathcal{L}(g \circ f)(x) \tag{0.4}$$

for any smooth real-valued function g satisfying $(g'' \circ f)(x) = 0$ (according to the second-order differential geometry [11], $\mathcal{L}_V f$ is the first-order part of the second-order vector field $g \mapsto \mathcal{L}(g \circ f)$); $\mathcal{L}_V f$ is called the tension field of f and the equation $\mathcal{L}_V f = 0$ characterizes $\mathcal{L}$-harmonic maps. By replacing $\mathcal{L}$ by $\mathcal{L}_V$, we can consider equations similar to (0.1) and (0.2), but with ϕ and f taking their values in V; they are nonlinear partial differential equations which can be studied with analytical methods (see [4], [5]). These equations have also equivalent probabilistic problems provided that one uses the notion of V-valued martingale; if M_t is a continuous adapted V-valued process, we will say that M_t is a martingale (see [2]) if for any smooth real-valued bounded function g on V and for any stopping times $T_1 \leq T_2$ such that $g''(M_t) \geq 0$ on $\{T_1 \leq t < T_2\}$, then $g(M_t)$, $T_1 \leq t \leq T_2$, is a submartingale. As in the real case, we have to study the set of V-valued martingales converging almost surely to some fixed random variable; the aim of this work is to study existence and uniqueness of processes in this set; this program was already completed for the circle in [14]; for general manifolds some results are proved in [6], [9]; we will check that they are implied by our results. The uniqueness will rely on a maximum principle. For the existence, we will suppose that the probability space is a Wiener space and will consider two types of terminal values: the variables which live in a small enough convex domain of V, and the variables which may be unbounded but are differentiable in the sense of the differential calculus on the Wiener space.

We will first give some preliminary results concerning the geodesic distance and the differential calculus on the Wiener space. In §2, we will study the uniqueness problem; we will describe classes of martingales in which two martingales converging to the same value are indistinguishable; the basic tool is a maximum principle which will be proved by means of Itô's formula. In §3 we will prove an existence theorem when the final value is differentiable in the sense of the differential calculus on the Wiener space; in the proof we will need a stochastic implicit function theorem. In §4 we will obtain a general existence theorem for martingales taking values in a domain of V satisfying some assumptions; regular geodesic balls (see [9]) are a particular case of such domains; in particular, when the sectional curvatures are non-positive, we will obtain the existence for any integrable final value. Finally, in §5, from these results we will deduce probabilistic solutions to the nonlinear heat equation and Dirichlet problem.

Throughout this work, we will assume that V is a complete Riemannian manifold (without boundary) of dimension d; in each tangent space T_xV, the Riemannian norm and product will be denoted by $|.|$ and $(.,.)$. We will always assume that the injectivity radius R of V is positive and that its sectional curvatures are bounded above; we will let K be the smallest non-negative number dominating all the sectional curvatures. If f is a smooth real function defined on V, its derivative $f'(x)$ is a linear form on T_xV and its value on a tangent vector u will be denoted by $f'(x)\langle u\rangle$; similarly, the Hessian $f''(x)$ is a bilinear form the value of which is denoted by $f''(x)\langle u,\overline{u}\rangle$; if f is defined on $V \times V$ endowed with the product Riemannian metric, if $x = (x_0, x_1)$ is a point, $u = (u_0, u_1)$ a tangent vector, we will consider the partial derivatives

$$f''_{00}(x)\langle u_0,\overline{u}_0\rangle = f''(x)\langle (u_0,0),(\overline{u}_0,0)\rangle, \tag{0.5}$$

$$f''_{01}(x)\langle u_0,u_1\rangle = f''(x)\langle (u_0,0),(0,u_1)\rangle \tag{0.6}$$

and f''_{11} defined similarly. We will always assume that we are given a probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ satisfying the usual conditions; for some results, we will suppose that Ω is a Wiener space. The quadratic variation of a V-valued martingale M_t (computed for the Riemannian metric) will be denoted by $\langle\!\langle M\rangle\!\rangle_t$. We will say that a sequence of processes X_t^k converges uniformly in probability to a process X_t if $\sup_t |X_t^k - X_t|$ converges in probability to 0. Different constant numbers will often be denoted by the same letter C. Geometric results which are used in this work can be found in [1], [10], whereas the theory of real-valued martingales and the stochastic differential geometry are respectively dealt with in [3] and [6] (among other books and articles).

1. Preliminary results

We will first prove some useful geometrical estimates for the distance function on $V \times V$. Then we will briefly describe the differential calculus on Wiener spaces.

§1.1 Estimates on the derivatives of the distance

If x_0 and x_1 are two points of V, denote by $\delta(x_0, x_1)$ the geodesic distance between them; the function δ is defined on $V \times V$ and is smooth except on the cut locus and the diagonal $\{x_0 = x_1\}$. We want to estimate its first and second derivatives when $V \times V$ is endowed with the product Riemannian metric. Let $x = (x_0, x_1)$ be a point which is not in the cut locus or the diagonal; there exists a unique minimizing geodesic $c(t)$, $0 \le t \le 1$ from x_0 to x_1; if u_t is a vector of $T_{c(t)}V$, we can decompose u_t as $v_t + w_t$ where v_t is the orthogonal projection of u_t on $c'(t)$; the vectors v_t and w_t will be respectively called the parallel and orthogonal components of u_t. If $u = (u_0, u_1)$ is a vector of $T_x(V \times V)$, (v_0, v_1) and (w_0, w_1) will also be called its parallel and orthogonal components.

Lemma 1.1.1. *Let x be point of $V \times V$ which is not in the cut locus and such that $0 < \delta(x) < \pi/\sqrt{K}$. Let u be a vector of $T_x(V \times V)$ and let v and w be its parallel and orthogonal components. Then*

$$\big|\delta'(x)\langle u\rangle\big| \le \sqrt{2}|v| \tag{1.1.1}$$

and

$$\delta''(x)\langle u, u\rangle \ge -\sqrt{K}\tan\Big(\frac{\sqrt{K}\delta(x)}{2}\Big)|w|^2. \tag{1.1.2}$$

Proof. Let $g(s)$, $s \ge 0$ be the geodesic line on $V \times V$ satisfying $g(0) = x$ and $g'(0) = u$; then the quantities that we have to estimate are

$$\delta'(x)\langle u\rangle = \frac{d}{ds}\delta(g(s))\Big|_{s=0}, \qquad \delta''(x)\langle u, u\rangle = \frac{d^2}{ds^2}\delta(g(s))\Big|_{s=0}. \tag{1.1.3}$$

For s small enough, let $c(s,t)$, $0 \le t \le 1$ be the unique minimizing geodesic joining the two components of $g(s)$; put $c(t) = c(0,t)$. Let $J(t)$ be the derivative of $c(s,t)$ with respect to s at $s = 0$; it is the Jacobi field on $c(t)$ satisfying $J(0) = u_0$ and $J(1) = u_1$; let $J'(t)$ be the covariant derivative of J with respect to t. We also define $K(t)$ to be the sectional curvature at the point $c(t)$ for the plane generated by $c'(t)$ and $J(t)$ (zero if these two vectors do not generate a plane); if K_0 is the curvature tensor of V, then

$$\Big(K_0\big(J(t),c'(t)\big)c'(t),J(t)\Big) = K(t)\Big[\big|J(t)\big|^2\big|c'(t)\big|^2 - \big(J(t),c'(t)\big)^2\Big]. \tag{1.1.4}$$

By noticing that $\delta(g(s))$ is the norm of $\frac{\partial}{\partial t}c(s,t)$, one can compute the expressions of (1.1.3) (see the proof of Proposition 3.8.1 of [10]) and obtain

$$\delta'(x)\langle u\rangle = \frac{\big(c'(t),J'(t)\big)}{\big|c'(t)\big|} \tag{1.1.5}$$

for any $0 \le t \le 1$ and

$$\begin{aligned}\delta''(x)\langle u,u\rangle &= \int_0^1 \frac{\big|J'(t)\big|^2\big|c'(t)\big|^2 - \big(J'(t),c'(t)\big)^2}{\big|c'(t)\big|^3}dt \\ &\quad - \int_0^1 \frac{\Big(K_0\big(J(t),c'(t)\big)c'(t),J(t)\Big)}{\big|c'(t)\big|}dt \\ &= \int_0^1 \frac{\big|J'(t)\big|^2\big|c'(t)\big|^2 - \big(J'(t),c'(t)\big)^2}{\big|c'(t)\big|^3}dt \\ &\quad - \int_0^1 K(t)\frac{\big|J(t)\big|^2\big|c'(t)\big|^2 - \big(J(t),c'(t)\big)^2}{\big|c'(t)\big|}dt.\end{aligned} \tag{1.1.6}$$

Now the decomposition $u = v + w$ induces a decomposition $J = J_v + J_w$ into two Jacobi fields which are its parallel and orthogonal components. Thus it follows from (1.1.5) that $\delta'(x)\langle u\rangle$ depends only on the component v; moreover one immediately deduces from the triangle inequality that δ is $\sqrt{2}$-Lipschitz so (1.1.1) is checked. Similarly, $\delta''(x)\langle u,u\rangle$ depends only on the component w; moreover if $K = 0$, the right-hand side of (1.1.6) is non-negative so we only have to consider the case $K > 0$. Formula (1.1.6) can be written as

$$\delta''(x)\langle u,u\rangle = \frac{1}{\delta(x)}\int_0^1 \big|J'_w(t)\big|^2 dt - \delta(x)\int_0^1 K(t)\big|J_w(t)\big|^2 dt. \tag{1.1.7}$$

On the other hand, we can deduce from the Jacobi equation

$$J''(t) + K_0\big(J(t),c'(t)\big)c'(t) = 0 \tag{1.1.8}$$

that

$$\big(J(t),J''(t)\big) + K(t)\Big(\big|J(t)\big|^2\big|c'(t)\big|^2 - \big(J(t),c'(t)\big)^2\Big) = 0 \tag{1.1.9}$$

so for $J = J_w$, this yields

$$\big(J_w(t), J_w''(t)\big) + K(t)\delta^2(x)\big|J_w(t)\big|^2 = 0. \tag{1.1.10}$$

We deduce from (1.1.7) and (1.1.10) that

$$\delta''(x)\langle u, u\rangle = \frac{1}{2\delta(x)} \int_0^1 \frac{d^2}{dt^2}\big|J_w(t)\big|^2 dt \tag{1.1.11}$$

and that

$$\begin{aligned} \frac{d^2}{dt^2}\big|J_w(t)\big| &= \frac{\big|J_w(t)\big|^2\big|J_w'(t)\big|^2 - \big(J_w(t), J_w'(t)\big)^2}{\big|J_w(t)\big|^3} - K(t)\delta^2(x)\big|J_w(t)\big| \\ &\geq -K\delta^2(x)\big|J_w(t)\big| \end{aligned} \tag{1.1.12}$$

at points t such that $J_w(t) \neq 0$. Consider the function

$$j(t) = \alpha \sin\big(\sqrt{K}\delta(x)t + \beta\big) \tag{1.1.13}$$

where $\alpha > 0$ and $0 \leq \beta \leq \pi - \sqrt{K}\delta(x)$ are defined by

$$\alpha \sin\beta = |w_0|, \qquad \alpha \sin\big(\sqrt{K}\delta(x) + \beta\big) = |w_1|. \tag{1.1.14}$$

In particular

$$\alpha^2 = \frac{|w|^2}{\sin^2\beta + \sin^2\big(\sqrt{K}\delta(x) + \beta\big)}. \tag{1.1.15}$$

By comparing (1.1.12) with the corresponding differential equation with boundary conditions, one deduces since $\sqrt{K}\delta(x) < \pi$ that

$$\frac{d}{dt}\big|J_w(t)\big|\bigg|_{t=0} \leq j'(0), \qquad \frac{d}{dt}\big|J_w(t)\big|\bigg|_{t=1} \geq j'(1). \tag{1.1.16}$$

Moreover, $|J_w(0)|$ and $|J_w(1)|$ are respectively $j(0)$ and $j(1)$. Thus from (1.1.11),

$$\begin{aligned} \delta''(x)\langle u, u\rangle &\geq \frac{1}{\delta(x)}\big(j(1)j'(1) - j(0)j'(0)\big) \\ &= |w|^2\sqrt{K}\frac{\sin\big(\sqrt{K}\delta(x) + \beta\big)\cos\big(\sqrt{K}\delta(x) + \beta\big) - \sin\beta\cos\beta}{\sin^2\beta + \sin^2\big(\sqrt{K}\delta(x) + \beta\big)}. \end{aligned} \tag{1.1.17}$$

The derivative with respect to β of the above ratio is

$$\frac{\cos\big(2\sqrt{K}\delta(x) + 2\beta\big) - \cos\big(2\beta\big)}{\Big[\sin^2\beta + \sin^2\big(\sqrt{K}\delta(x) + \beta\big)\Big]^2}$$

so the minimum with respect to β of the right-hand side of (1.1.17) is obtained for $\beta = (\pi - \sqrt{K}\delta(x))/2$ and is exactly the right-hand side of (1.1.2). $\square$

Lemma 1.2.1. *Let x be a point of $V \times V$ which is not in the cut locus and such that $0 < \delta(x) < \pi/\sqrt{K}$. Let (u_0, u_1) be a tangent vector and denote by (v_0, v_1) and (w_0, w_1) its parallel and orthogonal components. Then*

$$\left|\delta_0'(x)\langle u_0\rangle\right| = |v_0|, \tag{1.2.1}$$

$$\delta_{00}''(x)\langle u_0, u_0\rangle \geq \sqrt{K}\cot\bigl(\sqrt{K}\delta(x)\bigr)|w_0|^2 \tag{1.2.2}$$

and

$$\left|\delta_{01}''(x)\langle u_0, u_1\rangle\right| \leq \frac{\sqrt{K}}{\sin\bigl(\sqrt{K}\delta(x)\bigr)}|w_0|\,|w_1| \tag{1.2.3}$$

where the coefficients in (1.2.2) and (1.2.3) are defined to be $1/\delta(x)$ if $K = 0$.

Proof. We will use the notations defined in the proof of Lemma 1.1.1. It is sufficient to prove the lemma for $K > 0$; the case $K = 0$ is obtained by applying the result for $K > 0$ and letting $K \downarrow 0$. Note also that the linear form $\delta_0'(x)$ is the scalar product by $c'(0)$ so we immediately deduce (1.2.1). Put $u = (u_0, 0)$ and $\overline{u} = (0, u_1)$. We can apply (1.1.17) to our vector u with $j(0) = |w_0|$ and $j(1) = 0$ so that

$$\beta = \pi - \sqrt{K}\delta(x), \qquad \alpha = |w_0|/\sin\beta. \tag{1.2.4}$$

We deduce (1.2.2). Let us now prove (1.2.3). Use the decomposition $u = v + w$ and $\overline{u} = \overline{v} + \overline{w}$; by polarizing (1.1.11) we obtain

$$\begin{aligned}\delta''(x)\langle u, \overline{u}\rangle &= \frac{1}{2\delta(x)}\int_0^1 \frac{d^2}{dt^2}\bigl(J_w(t), J_{\overline{w}}(t)\bigr)dt \\ &= \frac{1}{2\delta(x)}\Bigl[\frac{d}{dt}\bigl(J_w(t), J_{\overline{w}}(t)\bigr)\Big|_{t=1} - \frac{d}{dt}\bigl(J_w(t), J_{\overline{w}}(t)\bigr)\Big|_{t=0}\Bigr].\end{aligned} \tag{1.2.5}$$

We have

$$\Bigl|\bigl(J_w(t), J_{\overline{w}}(t)\bigr)\Bigr| \leq \left|J_w(t)\right|\left|J_{\overline{w}}(t)\right| \tag{1.2.6}$$

and these terms are 0 for $t = 0$ or 1, so

$$\left|\frac{d}{dt}\bigl(J_w(t), J_{\overline{w}}(t)\bigr)\right| \leq \left|\frac{d}{dt}\Bigl(\left|J_w(t)\right|\left|J_{\overline{w}}(t)\right|\Bigr)\right| \tag{1.2.7}$$

for $t = 0$ or 1. Thus

$$\left|\delta''(x)\langle u, \overline{u}\rangle\right| \le \frac{1}{2\delta(x)}\Big[|w|\;\frac{d}{dt}\big|J_{\overline{w}}(t)\big|\Big|_{t=0} - |\overline{w}|\;\frac{d}{dt}\big|J_w(t)\big|\Big|_{t=1}\Big]. \tag{1.2.8}$$

We deduce from (1.1.16) that we can estimate the derivative of $|J_w(t)|$ at $t = 1$ by $j'(1)$ where $j(t)$ is given by (1.1.13) and (1.2.4); thus

$$-\frac{d}{dt}\big|J_w(t)\big|\Big|_{t=1} \le |w|\frac{\sqrt{K}\delta(x)}{\sin\big(\sqrt{K}\delta(x)\big)}. \tag{1.2.9}$$

The derivative of $\left|J_{\overline{w}}(t)\right|$ is estimated in a similar way so that we check

$$\left|\delta''(x)\langle u, \overline{u}\rangle\right| \le |w|\,|\overline{w}|\frac{\sqrt{K}}{\sin\big(\sqrt{K}\delta(x)\big)}. \tag{1.2.10}$$

This is exactly (1.2.3). □

§*1.3 The differential calculus on Wiener spaces*

In this subsection, we suppose that the probability space is a Wiener space, we review briefly the differential calculus which can be developed on it (see for instance [12]) and apply it to V-valued variables. So let Ω be the space of continuous functions from $[0, \infty)$ into some Euclidean space $\mathbb{R}^m$, let $\mathbb{P}$ be the standard Wiener measure and W_t be the canonical process: it is a m-dimensional standard Wiener process. Then let $\mathcal{S}(\mathbb{R})$ be the set of real-valued smooth functionals $X = f(W_{t_1}, \ldots, W_{t_k})$ which are bounded as well as all their derivatives. For such a smooth functional, define

$$D_t X = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(W_{t_1}, \ldots, W_{t_k}) 1_{\{t \le t_i\}} \tag{1.3.1}$$

considered as a m-dimensional row vector and put

$$\big\|X\big\|_{\mathcal{D}}^2 = \mathbb{E}X^2 + \mathbb{E}\int_0^\infty |D_t X|^2 dt. \tag{1.3.2}$$

One can prove that $\mathcal{S}(\mathbb{R})$ is closable for this norm and we denote by $\mathcal{D}(\mathbb{R})$ the resulting completion; then $\mathcal{D}(\mathbb{R})$ is a Hilbert space and if X is in it, the process $D_t X$ is well defined almost everywhere. If X is in $\mathcal{D}(\mathbb{R})$ and if ϕ is uniformly Lipschitz, then $\phi(X)$ is in

$\mathcal{D}(\mathbb{R})$ (this can be proved by approximating ϕ by smooth functions); if moreover ϕ is smooth, then the derivative of $\phi(X)$ is given by the classical chain rule; more generally, if X_i are p variables in $\mathcal{D}(\mathbb{R})$ and if ϕ is Lipschitz on $\mathbb{R}^p$ endowed with the sup norm, then $\phi(X_1, \ldots, X_p)$ is in $\mathcal{D}(\mathbb{R})$ and

$$\big|D\phi(X_1, \ldots, X_p)\big| \leq \|\phi\|_{\mathrm{lip}} \sup_i |DX_i|. \tag{1.3.3}$$

A fundamental property of variables of $\mathcal{D}(\mathbb{R})$ is the representation formula

$$X = \mathbb{E}X + \sum_{j=1}^{m} \int_0^\infty \mathbb{E}\big[D_t^j X \mid \mathcal{F}_t\big] dW_t^j. \tag{1.3.4}$$

We will need two other properties of $\mathcal{D}(\mathbb{R})$.

Lemma 1.3.1. *Let X be a variable of $\mathcal{D}(\mathbb{R})$ which takes its values in a discrete subset of $\mathbb{R}$; then X is almost surely constant.*

Proof. Denote the discrete subset by A; there exists a smooth bounded function ϕ with bounded derivative such that $\phi' = 0$ on A and ϕ is injective. Thus $\phi(X)$ is in $\mathcal{D}(\mathbb{R})$ and $\phi'(X) = 0$ implies $D\phi(X) = 0$; thus, from (1.3.4), $\phi(X)$ is almost surely constant and we can conclude. $\square$

Lemma 1.3.2. *Let X^n be a sequence of variables of $\mathcal{D}(\mathbb{R})$ converging in probability to X; suppose that DX^n is bounded in $\mathbb{L}^2([0,\infty] \times \mathbb{P})$. Then X is in $\mathcal{D}(\mathbb{R})$.*

Proof. Since $\mathcal{D}(\mathbb{R})$ is a Hilbert space, it is sufficient to check that X^n is bounded for the norm (1.3.2), and therefore that it is bounded in $\mathbb{L}^2$; but we deduce from (1.3.4) that $X^n - \mathbb{E}X^n$ is bounded in $\mathbb{L}^2$ and if $\mathbb{E}X^n$ is unbounded, X^n cannot converge in probability. $\square$

Then let $\mathcal{D}(V)$ be the set of V-valued variables X such that for any function $\phi : V \to \mathbb{R}$ which is smooth and uniformly Lipschitz, the variable $\phi(X)$ is in $\mathcal{D}(\mathbb{R})$; by using the Nash embedding theorem, the manifold V can be considered as a submanifold of a Euclidean space $\mathbb{R}^r$, and saying that X is in $\mathcal{D}(V)$ is equivalent to saying that each component of X in

$\mathbb{R}^r$ is in $\mathcal{D}(\mathbb{R})$. Moreover if X is in $\mathcal{D}(V)$, we can consider a process $(X, D_t^1 X, \dots, D_t^m X)$ defined $\mathbb{P} \otimes dt$ almost everywhere, such that $D_t^j X \in T_X V$ and

$$D_t^j \phi(X) = \big(\phi'(X), D_t^j X\big) \qquad \mathbb{P} \otimes dt \quad \text{a.e.} \tag{1.3.5}$$

If X_t is a measurable process, we will say that X is in $\mathcal{D}^\star(V)$ if X is the uniform limit in probability of a sequence of V-valued step processes Y^n (associated to deterministic subdivisions) satisfying

$$\limsup_{n\to\infty} \mathbb{E} \sum_{j=1}^m \int_0^\infty \sup_t \big| D_s^j Y_t^n \big|^2 ds < \infty. \tag{1.3.6}$$

This implies in particular that for any t, X_t is in $\mathcal{D}(V)$.

2. The uniqueness property

Fix some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose that M_t and $\overline{M}_t$ are two martingales on the Riemannian manifold V converging almost surely to the same variable as $t \to \infty$; we want to find conditions ensuring that $M = \overline{M}$ almost surely. If $V = \mathbb{R}$, the process $M_t - \overline{M}_t$ is a local martingale converging to 0; it is well known that it is not necessarily zero; however if $\mathcal{T}$ denotes the set of $\mathcal{F}_t$ stopping times and if M_t and $\overline{M}_t$ are in the class of processes X_t such that $(X_\tau, \tau \in \mathcal{T})$ is uniformly integrable, then M_t and $\overline{M}_t$ are uniformly integrable martingales, so they coincide. More generally if V is a Cartan-Hadamard manifold (a simply connected manifold with non-positive sectional curvatures) then the geodesic distance δ is convex on $V \times V$ so $\delta(M_t, \overline{M}_t)$ is a local submartingale; thus if O is some fixed point of V and M_t and $\overline{M}_t$ are in the class of processes X_t such that $(\delta(O, X_\tau), \tau \in \mathcal{T})$ is uniformly integrable, we can again conclude that M and $\overline{M}$ coincide. On the other hand, this property does not hold for more general manifolds: consider for instance the circle ([14]). The aim of this section is to describe classes of martingales satisfying this property; the basic tool will be a maximum principle.

§*2.1 A maximum principle*

In this subsection, we prove a maximum principle and give simple applications of it. More precisely, the maximum principle says that if a subset of V satisfies a convexity property and if M_t is a V-valued martingale converging to a variable in the subset, then the whole martingale lives in the subset; actually this will generally not be valid for all martingales but only for those in some class which we now define.

Definition 2.1.1. *If λ is a non-negative measurable function defined on V, let $\mathcal{E}_\lambda$ be the set of V-valued martingales M_t satisfying*

$$\mathbb{E}\exp\int_0^\infty \lambda(M_t)d\langle\langle M\rangle\rangle_t < \infty. \tag{2.1.1}$$

First note that if λ is locally bounded and if M_t is a martingale which converges almost surely, then it follows from [16] that $\int \lambda(M_t)d\langle\langle M\rangle\rangle_t$ is almost surely finite. On the other hand, we now verify by a simple application of Itô's formula that the class $\mathcal{E}_\lambda$ may contain all Δ-valued martingales, for some small enough domains Δ.

Proposition 2.1.2. *Let Δ be an open subset of V; suppose that we are given on Δ a non-negative locally bounded function λ and a C^2 function f satisfying $c \leq f(x) \leq C$ for some positive c and C; suppose moreover that $f'' + 2\lambda f \leq 0$ on Δ: this means that*

$$f''(x)\langle u, u\rangle + 2\lambda(x)f(x)|u|^2 \leq 0. \tag{2.1.2}$$

Then any Δ-valued martingale belongs to $\mathcal{E}_\lambda$.

Proof. Define

$$S_t = f(X_t)\exp\int_0^t \lambda(M_s)d\langle\langle M\rangle\rangle_s. \tag{2.1.3}$$

It follows immediately from Itô's formula and condition (2.1.2) that S_t is a local super-martingale; since it is non-negative, we deduce that $\mathbb{E}S_\infty \leq \mathbb{E}S_0$. By using the lower and upper bounds on f, we deduce (2.1.1). $\square$

We now state and prove the maximum principle for martingales.

Theorem 2.1.3. *Let f be a real-valued continuous function defined on V and consider the subsets $\Delta = \{f < 0\}$ and $F = \{f \leq 0\}$.*

(a) Suppose that f is C^2 and bounded on $V \setminus F$ and that there exists a non-negative

function λ defined on $V \setminus F$ and such that $f'' + 2\lambda f \geq 0$. Extend the function λ to V by putting $\lambda = 0$ on F. Then any martingale of $\mathcal{E}_\lambda$ converging to a variable in F lives in F.
(b) Suppose that f is C^2 on F and that there exists a non-positive locally bounded function defined on F and such that $f'' + 2\lambda f \geq 0$. Then any martingale living in F and converging to a variable in Δ lives in Δ.

Proof. Let M_t be a martingale of $\mathcal{E}_\lambda$ converging to $M_\infty \in F$. Fix some time $t_0 \geq 0$ and put

$$\tau = \inf\{t \geq t_0; M_t \in F\}. \tag{2.1.4}$$

The stopping time τ may be infinite but in any case, M_τ is in F. Define the process S_t by (2.1.3). On the event $\{M_{t_0} \notin F\}$, the process $(S_t, t_0 \leq t < \tau)$ is a non-negative local submartingale with positive initial value and zero limit; since f is bounded and M_t is in $\mathcal{E}_\lambda$, it is actually a uniformly integrable submartingale; we obtain a contradiction so the probability of the event $\{M_{t_0} \notin F\}$ is zero. Since t_0 is arbitrary and F is closed, we deduce (a). Under the assumptions of (b), let M_t be a martingale living in F; if S_t is again defined by (2.1.3), it is a non-positive local submartingale so the event $\{\sup_t S_t = 0\}$ is almost surely equal to $\{S_\infty = 0\}$. Now suppose that M_t converges to $M_\infty \in \Delta$; since λ is locally bounded, it follows from [16] that the integral in the exponential of (2.1.3) converges as $t \to \infty$, so $S_\infty < 0$; thus $\sup_t S_t < 0$ and (b) is proved. $\square$

Remark. We have assumed that f is C^2 in order to apply Itô's formula but we can also use less regular functions by an approximation technique. For instance, in case (a), if f is the uniform limit of a uniformly bounded sequence of C^2 functions f_n and if f_n'' is uniformly lower bounded, the theorem still holds with f'' replaced by $\liminf f_n''$ (apply Fatou's lemma).

Before the main applications, let us give two consequences of the maximum principle which will not be used subsequently. The first one can be compared with Theorem 3.1 of [9].

Corollary 2.1.4. *Consider a constant function $\lambda > 0$ such that $\lambda \geq \frac{K}{8} \vee \frac{1}{8}\left(\frac{\pi}{R}\right)^2$. Then any martingale of $\mathcal{E}_\lambda$ converging almost surely to a deterministic value is constant.*

Remark. In a Cartan-Hadamard manifold, the property holds for martingales of $\bigcup_{\lambda>0}\mathcal{E}_\lambda$; actually in this case, by using the convexity of the distance function, one can prove that if the sectional curvatures are bounded below, it is sufficient to assume that $\langle\langle M\rangle\rangle_\infty$ is integrable.

Proof. Let us consider martingales converging to a point x_0; define the function

$$f(x) = \sin\big(\sqrt{2\lambda}\delta(x_0,x)\big) \quad \text{if} \quad \delta(x_0,x) \le \frac{\pi}{2\sqrt{2\lambda}}, \tag{2.1.5}$$

1 otherwise. Then f is C^1 on $V\setminus\{x_0\}$, is C^2 except at x_0 and on the geodesic sphere $\{\delta(x_0,x) = \pi/\sqrt{8\lambda}\}$. Let us estimate $f''(x)$ for x in the geodesic ball and different from x_0; from the estimates of Lemma 1.2.1, if u is a tangent vector with parallel and orthogonal components v and w,

$$\begin{aligned} f''(x)\langle u,u\rangle &= \sqrt{2\lambda}\cos\big(\sqrt{2\lambda}\delta(x_0,x)\big)\delta''_{11}(x_0,x)\langle u,u\rangle \\ &\quad - 2\lambda\sin\big(\sqrt{2\lambda}\delta(x_0,x)\big)\big|\delta'_1(x_0,x)\langle u\rangle\big|^2 \\ &\ge \sqrt{2\lambda K}\frac{\cos\Big(\frac{\sqrt{K}}{2}\delta(x_0,x)\Big)}{\sin\big(\sqrt{K}\delta(x_0,x)\big)}\Big(\cos\big(\sqrt{K}\delta(x_0,x)\big)\wedge 0\Big)|w|^2 \\ &\quad - 2\lambda f(x)|v|^2 \end{aligned} \tag{2.1.6}$$

because $\sqrt{2\lambda}\ge\sqrt{K}/2$. But if $0<z<\pi/2$, one has

$$\frac{\cos(2z)\cos z}{\sin(2z)} = \frac{1-2\sin^2 z}{2\sin z} \ge -\frac{1}{2}\sin z \tag{2.1.7}$$

so

$$\begin{aligned} f''(x)\langle u,u\rangle &\ge -\sqrt{\frac{\lambda K}{2}}\sin\Big(\frac{\sqrt{K}}{2}\delta(x_0,x)\Big)|w|^2 - 2\lambda f(x)|v|^2 \\ &\ge -2\lambda f(x)|u|^2. \end{aligned} \tag{2.1.8}$$

By regularizing f, we can apply Theorem 2.1.3 and deduce the corollary. □

The maximum principle can also be used for convex domains of V. Let Δ be an open subset of V and suppose that its boundary $\partial\Delta$ is a smooth submanifold of dimension $d-1$; consider the function

$$f_0(x) = -(-1)^{1_\Delta(x)}\delta(x,\partial\Delta). \tag{2.1.9}$$

Each point of $\partial\Delta$ admits on a neighbourhood a system of local coordinates $(y_1, \dots, y_d)$ such that $f_0(x) = y_d$; in particular f_0 is smooth on a neighbourhood of $\partial\Delta$. We will say that the boundary is convex if $f_0'' \geq 0$ on $\partial\Delta$ (f_0'' is the second fundamental form of $\partial\Delta$ relative to the normal vector field exiting Δ).

Corollary 2.1.5. *Let Δ be a relatively compact open subset of V with smooth convex boundary. There exists a positive constant λ such that any martingale of $\mathcal{E}_\lambda$ converging to a variable in Δ lives in Δ.*

Proof. We can construct a smooth function f on V such that $f = f_0$ on a neighbourhood of $\partial\Delta$, $f > 0$ on $V \setminus \overline{\Delta}$, $f < 0$ on Δ and $f = 1$ except on a compact subset. The domain Δ is convex so $f''(x)$ is non-negative on $\{f(x) = 0\}$. Since f'' is Lipschitz, we can deduce that $f''/|f|$ is lower bounded by some contant number; we conclude with Theorem 2.1.3. $\square$

§*2.2 The uniqueness results*

We can easily deduce from the maximum principle the following uniqueness result.

Theorem 2.2.1. *Consider a constant function $\lambda > 0$ such that $\lambda \geq \frac{K}{2} \vee \frac{1}{2}\left(\frac{\pi}{R}\right)^2$. Then for any measurable variable, there is at most one martingale of the class $\mathcal{E}_\lambda$ converging almost surely to this variable.*

Proof. Put $\rho = (2\lambda)^{-1/2}\pi$ and consider the function

$$f(x) = \sin\Big(\frac{\pi\delta(x)}{2\rho}\Big)1_{\{\delta(x)<\rho\}} + 1_{\{\delta(x)\geq\rho\}} \tag{2.2.1}$$

defined on $V \times V$. Then on $\{0 < \delta < \rho\}$, the function f is smooth and

$$f''(x)\langle u, u\rangle = \frac{\pi}{2\rho}\cos\Big(\frac{\pi\delta(x)}{2\rho}\Big)\delta''(x)\langle u, u\rangle - \Big(\frac{\pi}{2\rho}\Big)^2 f(x)\big|\delta'(x)\langle u\rangle\big|^2. \tag{2.2.2}$$

¿From Lemma 1.1.1 we deduce

$$\begin{aligned} f''(x)\langle u, u\rangle &\geq -\frac{\pi}{2\rho}\sqrt{K}\tan\Big(\frac{1}{2}\sqrt{K}\delta(x)\Big)\cos\Big(\frac{\pi\delta(x)}{2\rho}\Big)|w|^2 - \frac{1}{2}\Big(\frac{\pi}{\rho}\Big)^2 f(x)|v|^2 \\ &\geq -\frac{1}{2}\Big(\frac{\pi}{\rho}\Big)^2 f(x)|w|^2 - \frac{1}{2}\Big(\frac{\pi}{\rho}\Big)^2 f(x)|v|^2 \end{aligned} \tag{2.2.3}$$

where the last inequality follows from $K \le \left(\frac{\pi}{\rho}\right)^2$. Since $|u|^2 = |v|^2 + |w|^2$ and from the definition of ρ, we deduce that $f'' + \lambda f \ge 0$; moreover the set $\{f = 0\}$ is exactly the diagonal of $V \times V$. On the other hand, if M_t and $\overline{M}_t$ are two martingales of $\mathcal{E}_\lambda$ such that $M_\infty = \overline{M}_\infty$, then $(M_t, \overline{M}_t)$ is a $V \times V$-valued martingale converging to a variable in the diagonal; since

$$\langle\!\langle (M, \overline{M}) \rangle\!\rangle_t = \langle\!\langle M \rangle\!\rangle_t + \langle\!\langle \overline{M} \rangle\!\rangle_t, \tag{2.2.4}$$

we deduce from the Cauchy-Schwartz inequality that $(M, \overline{M})$ is in $\mathcal{E}_{\lambda/2}$. By regularizing f, we can apply Theorem 2.1.3 and deduce that $(M, \overline{M})$ lives in the diagonal. □

Remark. If M and $\overline{M}$ take their values in an open subset Δ of V such that any two points of Δ are linked by at least one minimizing Δ-valued geodesic segment, then in previous theorem, the constant numbers K and R refer to Δ rather than V. If moreover this geodesic segment is unique, one can take $R = \infty$.

Example. A geodesic ball $B(O, \rho)$ is said to be regular (see [9]) if $B \times B$ does not meet the cut locus and $\rho < \pi/(2\sqrt{K})$. By applying Proposition 2.1.2 to the function

$$x \mapsto \cos\bigl(\sqrt{K}\delta(O, x)\bigr),$$

we deduce from Theorem 2.2.1 that two martingales taking their values in a regular geodesic ball and converging to the same variable coincide: this is the uniqueness result of [9].

We now verify that the part depending on R in the constant λ of Theorem 2.2.1 cannot be improved; in particular for manifolds satisfying $K \le \pi^2/R^2$, this constant appears to be the best one.

Proposition 2.2.2. *Suppose that λ is a constant satisfying $0 < \lambda < \frac{1}{2}\left(\frac{\pi}{R}\right)^2$. If on our probability space, there exists a one-dimensional $\mathcal{F}_t$ Wiener process, then one can find two different martingales of $\mathcal{E}_\lambda$ converging to the same variable.*

Proof. From the definition of the injectivity radius, one can deduce that there exist two different geodesic lines $c(t)$ and $\overline{c}(t)$, $0 \le t \le 1$, of length less than $\pi/\sqrt{2\lambda}$ and satisfying $c(0) = \overline{c}(0)$, $c(1) = \overline{c}(1)$. Then if W_t is the Wiener process and if

$$\tau = \inf\bigl\{t \ge 0, \quad \bigl|W_t\bigr| \ge 1/2\bigr\}, \tag{2.2.5}$$

we consider the martingales

$$M_t = c\Big(\frac{1}{2} + W_{t\wedge\tau}\Big), \qquad \overline{M}_t = \overline{c}\Big(\frac{1}{2} + W_{t\wedge\tau}\Big). \tag{2.2.6}$$

Then these two martingales converge to the same variable and their quadratic variations are at most $\alpha\pi^2\tau/(2\lambda)$ for some $\alpha < 1$. But from Itô's formula, the process $\cos(\sqrt{\alpha}\pi W_t)\exp(\alpha\pi^2 t/2)$ is a local martingale; when stopped at τ, it is positive, so it is a supermartingale and in particular

$$\cos\big(\sqrt{\alpha}\pi/2\big)\mathbb{E}\exp\big(\alpha\pi^2\tau/2\big) \le 1. \tag{2.2.7}$$

Thus M and $\overline{M}$ are in $\mathcal{E}_\lambda$. $\square$

For martingales which are not in $\mathcal{E}_{\pi^2/2R^2}$, the uniqueness does not hold; however one can sometimes obtain a weaker result. To this end, we use the notion of Riemannian covering manifold (see [1]): it consists of a Riemannian manifold $\widetilde{V}$ and of a function of $\widetilde{V}$ onto V which is locally isometric; in particular, a manifold admits a unique (up to an isometry) simply connected covering manifold called its universal cover. Uniqueness for V-valued martingales of $\mathcal{E}_\lambda$ implies the uniqueness for $\widetilde{V}$-valued martingales of $\mathcal{E}_\lambda$ but the converse is not necessarily true, and we are going to prove two weaker forms of this converse statement. We now define a notion of homotopy; we will say that two adapted continuous processes Y_t and $\overline{Y}_t$ converging almost surely to the same variable L are homotopic if there exists a family (Y_t^α), $0 \le \alpha \le 1$ of adapted processes such that $(\alpha, t) \mapsto Y_t^\alpha$ is almost surely continuous on $[0,1]\times[0,\infty]$, $Y_t^0 = Y_t$, $Y_t^1 = \overline{Y}_t$ and $Y_\infty^\alpha = L$.

Theorem 2.2.3. *Let λ be a non-negative function, let $\widetilde{V}$ be the universal cover of V and suppose that the uniqueness holds for $\widetilde{V}$-valued martingales of $\mathcal{E}_\lambda$. Let M_t and $\overline{M}_t$ be two V-valued martingales of $\mathcal{E}_\lambda$ converging to the same variable which are homotopic. Then $M = \overline{M}$.*

Remark. The manifold V and its universal cover $\widetilde{V}$ have the same sectional curvatures but the injectivity radius may be larger for the cover; if the sectional curvatures are non-positive, the injectivity radius of the cover is infinite, so we can take for λ any positive constant.

Proof. We will denote by Y_t^α the family of processes going from M_t to $\overline{M}_t$. Consider the cover $\widetilde{V}$, the projection $\Phi : \widetilde{V} \to V$ and choose a point X_0 in $\widetilde{V}$ such that $\Phi(X_0) = M_0$; then consider the lifting X_t^α of the process Y_t^α so that $X_0^0 = X_0$; then for any α, X_t^α is an adapted process and X_∞^α does not depend on α; moreover $\Phi(X_t^0) = M_t$ and $\Phi(X_t^1) = \overline{M}_t$ so X_t^0 and X_t^1 are martingales of $\widetilde{V}$ converging to the same variable; since Φ is locally isometric, they have the same quadratic variation as M_t and $\overline{M}_t$, so they are in $\mathcal{E}_\lambda$. From our assumption, $X^0 = X^1$, so by projection on V, $M = \overline{M}$. $\square$

We now prove a result where the homotopy condition is replaced by differentiability. More precisely, we will limit ourselves to the case of a Wiener probability space and will consider martingales which are smooth enough in the sense of the differential calculus on this Wiener space (see §1.3).

Theorem 2.2.4. *Let λ be a non-negative function, let $\widetilde{V}$ be the universal cover of V and suppose that the uniqueness holds for $\widetilde{V}$-valued martingales of $\mathcal{E}_\lambda$. Let M_t and $\overline{M}_t$ be two V-valued martingales of $\mathcal{E}_\lambda$ converging to the same variable; suppose that M and $\overline{M}$ are in $\mathcal{D}^\star(V)$. Then $M = \overline{M}$.*

The first step in the proof of this theorem is the

Lemma 2.2.5. *Let Ψ be a real-valued function defined on $C([0,\infty], V)$ which is bounded and uniformly Lipschitz for the sup norm. If X_t is a continuous process in $\mathcal{D}^\star(V)$, then $\Psi(X)$ is in $\mathcal{D}(\mathbb{R})$.*

Proof. If $x(t)$ is a step function such that $\delta(x(t-), x(t)) < R$, we can consider the continuous function $y(t)$ obtained by geodesic interpolation and put $\Psi_0(x) = \Psi(y)$; since the sectional curvatures are bounded above by K, we can check that Ψ_0 is uniformly Lipschitz on

$$\Big\{\sup_t \delta(x(t-), x(t)) < R \wedge \frac{\pi}{2\sqrt{K}}\Big\}.$$

Then we can construct a function Ψ_1 defined on the set of step functions, which is uniformly Lipschitz and such that $\Psi_1 = \Psi_0$ on

$$\Big\{\sup_t \delta(x(t-), x(t)) < \frac{R}{2} \wedge \frac{\pi}{4\sqrt{K}}\Big\}.$$

Now consider a sequence of step processes Y_t^n converging to X_t and such that (1.3.6) holds; then $\Psi_1(Y^n)$ converges in probability to $\Psi(X)$. Moreover, by embedding V in a Euclidean space, $\Psi_1(Y^n)$ is a Lipschitz function of a finite number of variables of $\mathcal{D}(\mathbb{R})$, so it is in $\mathcal{D}(\mathbb{R})$, and by applying (1.3.3) and (1.3.6), it appears that $D\Psi_1(Y^n)$ is bounded in $\mathbb{L}^2$, so from Lemma 1.3.2, $\Psi(X)$ is in $\mathcal{D}(\mathbb{R})$. □

Proof of Theorem 2.2.4. Let Φ be the projection of $\widetilde{V}$ on V; note that if $\Phi(x) = \Phi(\overline{x})$, then either $x = \overline{x}$ or $\delta(x, \overline{x}) \geq 2R$. Consider a point X_0 in $\widetilde{V}$ such that $\Phi(X_0) = M_0$ and lift the process M_t in order to obtain a $\widetilde{V}$-valued martingale X_t; similarly let $\overline{X}_0$ be a point in $\widetilde{V}$ such that $\Phi(\overline{X}_0) = \overline{M}_0$ and let $\overline{X}_t$ be the corresponding lifting of $\overline{M}_t$. We deduce from Lemma 2.2.5 applied to $(M, \overline{M})$ that for any bounded Lipschitz function ψ on $\widetilde{V} \times \widetilde{V}$, the variable $\psi(X_\infty, \overline{X}_\infty)$ is in $\mathcal{D}(\mathbb{R})$; in particular, the variable $\delta(X_\infty, \overline{X}_\infty) \wedge (2R)$ is in $\mathcal{D}(\mathbb{R})$; but since $\Phi(X_\infty) = \Phi(\overline{X}_\infty)$, this variable takes its values in $\{0, 2R\}$, so from Lemma 1.3.1 it is deterministic. If it is zero, then $X_\infty = \overline{X}_\infty$ so from our assumption, $X = \overline{X}$ and therefore $M = \overline{M}$. Thus if we assume that M and $\overline{M}$ are not almost surely equal, then $\delta(X_\infty, \overline{X}_\infty) \geq 2R$ for any choice of $(X_0, \overline{X}_0)$; in particular, if we denote by $\mathcal{L}(X_0, \overline{X}_0; X_\infty, \overline{X}_\infty)$ the law of $(X_\infty, \overline{X}_\infty)$ corresponding to an admissible choice of $(X_0, \overline{X}_0)$ and if we consider the measure

$$m = \sum_{(X_0, \overline{X}_0)} \mathcal{L}(X_0, \overline{X}_0; X_\infty, \overline{X}_\infty) \tag{2.2.8}$$

then for any measurable subset A of $\widetilde{V}$ of diameter less than $2R$, we have $m(A \times A) = 0$. On the other hand since the diameter of A is less than $2R$, we also have

$$m(A \times A) = \mathbb{P}\big[M_\infty \in \Phi(A)\big] \tag{2.2.9}$$

so it cannot be zero for any A of small diameter. Thus the assumption $M \neq \overline{M}$ implies a contradiction. □

§2.3 A stability result

We have seen in last subsection that under some conditions, a martingale M_t is determined by its limit M_∞. The aim of this subsection is to estimate the perturbation on M_t induced by a perturbation on M_∞.

Theorem 2.3.1. *For $x \in V$, let $\kappa(x)$ be the smallest non-negative number dominating the sectional curvatures at x. Let M_t^α, $\alpha \in \mathbb{R}$ be a measurable family of V-valued martingales converging to M_∞^α such that $\langle\langle M^\alpha \rangle\rangle_\infty$ is bounded by a constant number; put $M_t = M_t^0$. We suppose that the variable*

$$\sup_{\alpha<\beta} \frac{\delta(M_\infty^\alpha, M_\infty^\beta) \wedge 1}{\beta - \alpha}$$

is integrable. Define

$$Z_t = \limsup_{\alpha,\beta \to 0} \frac{\delta(M_t^\alpha, M_t^\beta)}{|\beta - \alpha|}. \tag{2.3.1}$$

Then the process

$$S_t = Z_t \exp \frac{1}{2} \int_0^t \kappa(M_s) d\langle\langle M \rangle\rangle_s \tag{2.3.2}$$

is a uniformly integrable submartingale.

Proof. Let ψ be a nondecreasing function defined on $[0, \infty)$ such that $\psi(z) = z$ if z is small enough, $\psi'(z) = 0$ is z is large enough and $f = \psi \circ \delta$ is smooth. We deduce from the calculation of Lemma 1.1.1 that $f'' + 2\lambda f \geq 0$ for some bounded continuous non-negative function λ such that $\lambda(x, x) = \kappa(x)/4$. Thus for any $\alpha < \beta$, the process

$$S_t^{\alpha,\beta} = \frac{f(M_t^\alpha, M_t^\beta)}{\beta - \alpha} \exp \int_0^t \lambda(M_s^\alpha, M_s^\beta)\big(d\langle\langle M^\alpha \rangle\rangle_s + d\langle\langle M^\beta \rangle\rangle_s\big) \tag{2.3.3}$$

is a bounded submartingale. We deduce that the family of variables $\sup_t S_t^{\alpha,\beta}$ is bounded in probability so in particular, we obtain the uniform convergence in probability of M_t^α to M_t as $\alpha \to 0$. Consider V as a submanifold of a Euclidean space; then the family M_t^α becomes a family of Euclidean semimartingales; if we decompose it, its quadratic variation is uniformly bounded and the variation of its bounded variation part is bounded in probability; thus (apply for instance Corollary 2.2.5 of [13]) the uniform convergence in probability of M_t^α to M_t implies the uniform convergence in probability of $\langle\langle M^\alpha \rangle\rangle_t$ to $\langle\langle M \rangle\rangle_t$. Since $\lambda(M_t^\alpha, M_t^\beta)$ converges to $\kappa(M_t)/4$, we can also get (apply Theorem 2.4.4 of [13]) the convergence in probability of the integral in (2.3.3) to the integral in (2.3.2) as $(\alpha, \beta) \to (0, 0)$. There exists a sequence (α_k, β_k) such that the convergence holds almost surely, so

$$S_t = \limsup_k S_t^{\alpha_k,\beta_k}. \tag{2.3.4}$$

The processes $S_t^{\alpha,\beta}$ are submartingales which are dominated by a uniformly integrable process; we can conclude by means of Fatou's lemma. $\square$

In this theorem, we have let α, β tend to 0, but of course we can let them tend to another value; thus we obtain the almost sure Lipschitz continuity of M^α with respect to α and estimate $\partial M_t^\alpha/\partial\alpha$; this process can be viewed as a stochastic Jacobi field along M_t^α.

3. The existence in the differentiable case

In this section we are given a random variable L and we want to construct a V-valued martingale converging to this variable. Henceforth we will limit ourselves to the case of a Wiener probability space and in this section, we will consider the case of a differentiable variable L such that DL satisfies some estimate.

§3.1 Statement of the result

The aim of this section is to prove the

Theorem 3.1.1. *Suppose that Ω is the m-dimensional Wiener space. Let ρ_t be a deterministic non-negative function satisfying*

$$\int_0^\infty \rho_t^2 dt < 1/K \tag{3.1.1}$$

and let L be a variable of $\mathcal{D}(V)$ satisfying

$$\Big(\sum_{j=1}^m \mathbb{E}\big[|D_t^j L| \mid \mathcal{F}_t\big]^2\Big)^{1/2} \le \rho_t. \tag{3.1.2}$$

Then there exists a martingale M_t converging to L and such that

$$\frac{d}{dt}\langle\!\langle M\rangle\!\rangle_t \le \frac{\rho_t^2}{1 - K\int_t^\infty \rho_s^2 ds}. \tag{3.1.3}$$

Moreover M is in $\mathcal{D}^\star(V)$.

Remark 1. If the sectional curvatures are non-positive ($K = 0$), condition (3.1.1) simply means that ρ_t is square integrable. In the general case, if ρ_t is square integrable but

(3.1.1) does not hold, if t_0 is such that $\int_{t_0}^{\infty}\rho_t^2 dt = 1/K$, we deduce from Theorem 3.1.1 the existence of a martingale M_t, $t > t_0$ converging to L, but do not know whether it can be extended at and before t_0.

Remark 2. We deduce from (3.1.3) that $\langle\!\langle M\rangle\!\rangle_\infty$ is uniformly bounded so M_t is in the uniqueness class of Theorem 2.2.1.

Remark 3. When one is given a random variable L, in order to apply Theorem 3.1.1, one has to find a Wiener process with respect to which L is measurable; to this end, one has sometimes to enlarge the probability space. The choice of the Wiener process is not unique and different choices may have different behaviours for the application of the theorem.

Remark 4. The above results can also be applied when the time interval is the whole real line $[-\infty,+\infty]$, so that we obtain martingales converging as $t \to -\infty$ as soon as $\int_{-\infty}^{+\infty}\rho_t^2 dt < 1/K$. On the other hand, consider the case of the sphere S^d and let M_t be the Brownian motion on S^d stopped at some fixed deterministic time T; then M_t has no limit as $t \to -\infty$. Use the canonical isometric embedding of S^d in $\mathbb{R}^{d+1}$; then M_t can be defined on a Wiener space of dimension $d+1$ by the stochastic differential equation

$$dM_t = \big(I - M_t M_t^\star\big)dW_t - \frac{d}{2}M_t dt \tag{3.1.4}$$

for $-\infty < t < \infty$. One can check that M_T is in $\mathcal{D}(\mathbb{R}^{d+1})$ for $d \geq 3$ and that

$$D_t M_T\big(D_t M_T\big)^\star = \big(I - M_T M_T^\star\big)\exp\Big(-2\int_t^T M_s^\star dW_s - d(T-t)\Big) \tag{3.1.5}$$

for $t \leq T$ (compare the equations satisfied by both sides as T varies); by taking the trace and by integrating, we obtain

$$\sum_j \mathbb{E}\big[|D_t^j M_T|^2 \;\big|\; \mathcal{F}_t\big] = d\exp\big\{-(d-2)(T-t)\big\}. \tag{3.1.6}$$

Thus we can take for ρ_t the square root of the right-hand side and $\int\rho_t^2 dt$ is $d/(d-2)$; it is finite but is more than $1/K = 1$; we also see that the constant $1/K$ in Theorem 3.1.1 cannot be improved (at least without taking into account the dimension of the manifold).

The theorem will be proved in several steps. First note that by a deterministic change of time, we can always assume that L is $\mathcal{F}_1$ measurable and that ρ_t, $0 \le t \le 1$ is a constant $\rho < 1/\sqrt{K}$; then we have to find a martingale satisfying

$$\frac{d}{dt}\langle\!\langle M\rangle\!\rangle_t \le \frac{\rho^2}{1 - K\rho^2(1-t)}. \tag{3.1.7}$$

In the proof we will suppose that we are in this simplified framework.

The basic idea is to construct a sequence of step processes which will be proved to converge to our martingale. Let us now define this sequence. Let $\mu_0 < R \wedge \frac{\pi}{\sqrt{K}}$. Then there exists a constant $0 < \mu < \mu_0$ and a nondecreasing smooth function ψ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that

$$\psi'(z) \le \frac{\sin\bigl(\sqrt{K}z\bigr)}{\sqrt{K}}, \qquad \bigl|\psi''(z)\bigr| \le 1 \tag{3.1.8}$$

for any z, $\psi'(z) = 0$ for $z \ge \mu_0$ and

$$\psi(z) = \frac{1 - \cos\bigl(\sqrt{K}z\bigr)}{K} \tag{3.1.9}$$

if $z \le \mu$, where this expression is by convention $z^2/2$ if $K = 0$. Define also $g = \psi \circ \delta$; we have constructed a smooth function on $V \times V$ which is equivalent to $\delta^2(x)/2$ as x tends to the diagonal. Let $(t_i^n, 1 \le i \le n)$ be a sequence of subdivisions of $[0,1]$ such that the step size $\varepsilon_n = \sup|t_{i+1}^n - t_i^n|$ converges to 0 as $n \to \infty$. Put also $X_n^n = L$; assuming that X_{i+1}^n is given and $\mathcal{F}_{t_{i+1}^n}$ measurable, let us construct the variable X_i^n. This construction will be based on an approximation of the construction of the Riemannian centre of mass described in [8]. The random function

$$x \mapsto \mathbb{E}\bigl[g(x, X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\bigr]$$

is non-negative, smooth, bounded by $\psi(\infty)$; moreover, if V is not compact, this function converges almost surely to $\psi(\infty)$ as $x \to \infty$; thus it takes its minimal value at at least one point and we can find a $\mathcal{F}_{t_i^n}$ measurable variable X_i^n satisfying

$$\mathbb{E}\bigl[g(X_i^n, X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\bigr] = \inf_x \mathbb{E}\bigl[g(x, X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\bigr]. \tag{3.1.10}$$

Then we can consider the sequence of step processes M_t^n taking the value X_i^n for $t_i^n \le t < t_{i+1}^n$. The martingale M_t will be the uniform limit in probability of the processes M_t^n.

§3.2 Preliminary properties

We first give a geometric result about the derivatives of the function g.

Lemma 3.2.1. *Let $x = (x_0, x_1)$ be a point of $V \times V$, $u_0 \in T_{x_0}V$ and $u_1 \in T_{x_1}V$. Then*

$$g''_{00}(x)\langle u_0, u_0\rangle \geq \bigl(1 - Kg(x) - Cg^2(x)\bigr)|u_0|^2 \tag{3.2.1}$$

for some $C > 0$ and

$$\bigl|g''_{01}(x)\langle u_0, u_1\rangle\bigr| \leq |u_0|\,|u_1|. \tag{3.2.2}$$

Proof. We only need to prove the result on $\{0 < \delta(x) < \mu_0\}$. Let us first prove (3.2.1). Use the decomposition $u_0 = v_0 + w_0$ into parallel and orthogonal vectors; we deduce from Lemma 1.2.1 that

$$g''_{00}(x)\langle u_0, u_0\rangle \geq \bigl(\psi'' \circ \delta\bigr)(x)|v_0|^2 + \bigl(\psi' \circ \delta\bigr)(x)\sqrt{K}\cot\bigl(\sqrt{K}\delta(x)\bigr)|w_0|^2. \tag{3.2.3}$$

In particular this expression is bounded below by some $-C_0|u_0|^2$ so there exists a $C > 0$ such that

$$g''_{00}(x)\langle u_0, u_0\rangle \geq (1 - Cg^2(x))|u_0|^2 \tag{3.2.4}$$

if $\delta(x) \geq \mu$; on the other hand, if $0 < \delta(x) < \mu$, (3.2.3) can be written as

$$g''_{00}(x)\langle u_0, u_0\rangle \geq (1 - Kg(x))|u_0|^2. \tag{3.2.5}$$

The estimate (3.2.1) is then deduced from (3.2.4) and (3.2.5). Then, in order to prove (3.2.2), we apply

$$g''_{01}(x)\langle u_0, u_1\rangle = \bigl(\psi'' \circ \delta\bigr)(x)\delta'_0(x)\langle u_0\rangle\delta'_1(x)\langle u_1\rangle + \bigl(\psi' \circ \delta\bigr)(x)\delta''_{01}(x)\langle u_0, u_1\rangle \tag{3.2.6}$$

and we deduce from Lemma 1.2.1 and (3.1.8) that

$$\begin{aligned}\bigl|g''_{01}(x)\langle u_0, u_1\rangle\bigr| &\leq |v_0|\,|v_1| + |w_0|\,|w_1| \\ &\leq |u_0|\,|u_1|. \end{aligned} \tag{3.2.7}$$

□

Lemma 3.2.2. *Let Q be a probability measure on V; then the function*

$$Q \mapsto \inf_x \int g(x,z)Q(dz)$$

is continuous (for the weak convergence on probability measures). When this function is less than some constant positive number, the infimum is obtained at exactly one point which depends continuously on Q.

Remark 1. In particular, this result can be applied to the definition (3.1.10) of X_i^n by taking for Q the conditional law of X_{i+1}^n given $\mathcal{F}_{t_i^n}$.

Remark 2. From the Bienaymé-Chebychev inequality, the second part of the lemma holds in particular when $\int\int g(x,z)Q(dx)Q(dz)$ is less than some constant number.

Proof. Since g is uniformly continuous, the map $Q \mapsto \int g(x,z)Q(dz)$ is continuous uniformly in x; thus the map $Q \mapsto \inf_x \int g(x,z)Q(dz)$ is continuous. Then fix Q and note that if V is not compact the function $\gamma : x \mapsto \int g(x,z)Q(dz)$ tends to its supremum as $x \to \infty$ so its infimum is obtained at at least one point. From (3.2.1), γ is strictly convex on the set where it is less than some constant number; thus, since γ is uniformly Lipschitz, there exists some constant numbers C_1 and C_2 such that γ is strictly convex on the ball $B(x, C_1)$ as soon as $\gamma(x) < C_2$. Moreover, from the Bienaymé-Chebychev inequality, there exists some $C_3 < C_2$ such that $\gamma(x) < C_3$ implies

$$Q\big[B(x, C_1/2)\big] > 1/2. \tag{3.2.8}$$

Now if x_0 and x_1 are two points where the function γ gets its minimal value and if this value is less than C_3, then from (3.2.8), the two balls $B(x_0, C_1/2)$ and $B(x_1, C_1/2)$ must intersect, so $\delta(x_0, x_1)$ is less than C_1; but since γ is strictly convex on $B(x_0, C_1)$, we necessarily have $x_0 = x_1$ so the infimum is obtained at exactly one point. The continuous dependence of this point follows easily. $\square$

Now, assuming that X_{i+1}^n is in $\mathcal{D}(V)$ and satisfies some estimate, we are going to estimate the distance between X_i^n and X_{i+1}^n so that one can apply Lemma 3.2.2.

Lemma 3.2.3. *Suppose that X_{i+1}^n is in $\mathcal{D}(V)$ and that*

$$\sum_{j=1}^{m} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\Big[|D_t^j X_{i+1}^n| \;\Big|\; \mathcal{F}_t\Big]^2 dt \le \nu_i^n \tag{3.2.9}$$

almost surely for some deterministic number ν_i^n. There exists a constant number $C > 0$ such that

$$\mathbb{E}\big[g(X_i^n, X_{i+1}^n) \;\big|\; \mathcal{F}_{t_i^n}\big] \le \frac{\nu_i^n}{2} + C\big(\nu_i^n\big)^2 \tag{3.2.10}$$

and

$$\mathbb{E}\big[g^2(X_i^n, X_{i+1}^n) \;\big|\; \mathcal{F}_{t_i^n}\big] \le C\big(\nu_i^n\big)^2. \tag{3.2.11}$$

Proof. The basic tool in the proof of this result is the following formula: if $h(x_0, x_1)$ is a smooth real Lipschitz function on $V \times V$, if $\tau \le t_{i+1}^n$ is a stopping time and if X is a $\mathcal{F}_\tau$ measurable V-valued variable, then

$$h(X, X_{i+1}^n) = \mathbb{E}\big[h(X, X_{i+1}^n) \;\big|\; \mathcal{F}_\tau\big] + \sum_{j=1}^{m} \int_{\tau}^{t_{i+1}^n} \mathbb{E}\Big[h_1'(X, X_{i+1}^n)\langle D_t^j X_{i+1}^n\rangle \;\Big|\; \mathcal{F}_t\Big] dW_t^j. \tag{3.2.12}$$

This formula is a consequence of (1.3.4) (first consider the case $h(x_0, x_1) = h_0(x_0)h_1(x_1)$ and then use classical approximation techniques). In particular, (3.2.12) enables to estimate the conditional variance of $h(X, X_{i+1}^n)$ given $\mathcal{F}_\tau$. Since we are on a Wiener space, one can find a time-continuous version for the conditional law of X_{i+1}^n given $\mathcal{F}_t$; on the other hand we have seen in Lemma 3.2.2 that the map $Q \mapsto \inf_x \int g(x, z)Q(dz)$ is continuous; from these two remarks, the process $\inf_x \mathbb{E}[g(x, X_{i+1}^n)|\mathcal{F}_t]$ is continuous. Note that it takes the value $\mathbb{E}[g(X_i^n, X_{i+1}^n)|\mathcal{F}_{t_i^n}]$ at $t = t_i^n$ and the value 0 at time $t = t_{i+1}^n$. Now consider the event

$$A = \Big\{\mathbb{E}\big[g(X_i^n, X_{i+1}^n) \;\big|\; \mathcal{F}_{t_i^n}\big] \ge \psi(\mu)/2\Big\}. \tag{3.2.13}$$

¿From the above continuity, on A there exist a stopping time $t_i^n \le \tau \le t_{i+1}^n$ and a $\mathcal{F}_\tau$ measurable variable X such that

$$\mathbb{E}\big[g(X, X_{i+1}^n) \;\big|\; \mathcal{F}_\tau\big] = \psi(\mu)/2 \tag{3.2.14}$$

and X is a variable which minimizes this expression among $\mathcal{F}_\tau$ measurable variables. By applying (3.2.12) to the variable $g(X, X_{i+1}^n)$, we deduce from (3.2.9) that on A,

$$g(X, X_{i+1}^n) = \psi(\mu)/2 + O(\sqrt{\nu_i^n}) \tag{3.2.15}$$

in the spaces $\mathbb{L}^q$ for the probability $\mathbb{P}$ conditioned on $\mathcal{F}_{t_i^n}$. On the other hand apply (3.2.12) to $g_0'(X, X_{i+1}^n)$; its conditional mean is zero (since X is solution of a variational problem), so we deduce that

$$g_0'(X, X_{i+1}^n) = O(\sqrt{\nu_i^n}). \tag{3.2.16}$$

The sets $\{g(x) = \psi(\mu)/2\}$ and $\{g_0'(x) = 0\}$ are disjoint, so if ν_i^n is less than some constant number, there is a contradiction between (3.2.15) and (3.2.16), so $\mathbb{P}[A] = 0$ and therefore

$$\mathbb{E}\big[g(X_i^n, X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\big] < \psi(\mu)/2 \tag{3.2.17}$$

almost surely. By applying (3.2.12) to $g(X_i^n, X_{i+1}^n)$ with $\tau = t_i^n$, we obtain

$$\begin{aligned}
\mathbb{P}\big[\delta(X_i^n, X_{i+1}^n) \geq \mu \bigm| \mathcal{F}_{t_i^n}\big] &= \mathbb{P}\big[g(X_i^n, X_{i+1}^n) \geq \psi(\mu) \bigm| \mathcal{F}_{t_i^n}\big] \\
&\leq \mathbb{P}\Big[g(X_i^n, X_{i+1}^n) - \mathbb{E}[g(X_i^n, X_{i+1}^n)|\mathcal{F}_{t_i^n}] \geq \psi(\mu)/2 \Bigm| \mathcal{F}_{t_i^n}\Big] \\
&\leq C\big(\nu_i^n\big)^2.
\end{aligned} \tag{3.2.18}$$

Apply also (3.2.12) to $g_0'(X_i^n, X_{i+1}^n)$; from (3.2.2) we get

$$\mathbb{E}\Big[\big|g_0'(X_i^n, X_{i+1}^n)\big|^2 \Bigm| \mathcal{F}_{t_i^n}\Big] \leq \nu_i^n \tag{3.2.19}$$

and

$$\mathbb{E}\Big[\big|g_0'(X_i^n, X_{i+1}^n)\big|^4 \Bigm| \mathcal{F}_{t_i^n}\Big] \leq C\big(\nu_i^n\big)^2. \tag{3.2.20}$$

On $\{\delta(x) < \mu\}$, we have

$$\big|g_0'(x)\big|^2 = \big|\psi'\big(\delta(x)\big)\big|^2 = \frac{\sin^2\big(\sqrt{K}\delta(x)\big)}{K} \tag{3.2.21}$$

and

$$g(x) = \frac{2}{K}\sin^2\Big(\frac{\sqrt{K}\delta(x)}{2}\Big) \leq \frac{1}{2}\big|g_0'(x)\big|^2 + C\big|g_0'(x)\big|^4. \tag{3.2.22}$$

Thus

$$\mathbb{E}\big[g(X_i^n, X_{i+1}^n) \mid \mathcal{F}_{t_i^n}\big] \le \frac{1}{2}\mathbb{E}\Big[\big|g_0'(X_i^n, X_{i+1}^n)\big|^2 \Bigm| \mathcal{F}_{t_i^n}\Big] + C\mathbb{E}\Big[\big|g_0'(X_i^n, X_{i+1}^n)\big|^4 \Bigm| \mathcal{F}_{t_i^n}\Big]$$
$$+ C\mathbb{P}\Big[\delta(X_i^n, X_{i+1}^n) \ge \mu \Bigm| \mathcal{F}_{t_i^n}\Big]$$
$$\le \frac{\nu_i^n}{2} + C\big(\nu_i^n\big)^2 \tag{3.2.23}$$

from (3.2.18), (3.2.19) and (3.2.20). The estimate (3.2.11) is proved similarly by noticing that $g(x) \le C|g_0'(x)|^2$ on $\{\delta(x) < \mu\}$. $\square$

§*3.3 A stochastic implicit function theorem*

¿From Lemmas 3.2.2 and 3.2.3, we know that if X_{i+1}^n is in $\mathcal{D}(V)$ and DX_{i+1}^n is not too large, then X_i^n is uniquely defined by (3.1.10) and the distance between X_i^n and X_{i+1}^n is estimated by (3.2.10) and (3.2.11); moreover

$$\mathbb{E}\big[g_0'(X_i^n, X_{i+1}^n) \mid \mathcal{F}_{t_i^n}\big] = 0 \tag{3.3.1}$$

and X_i^n depends continuously on the conditional law of X_{i+1}^n given $\mathcal{F}_{t_i^n}$. We still have to prove that X_i^n is in $\mathcal{D}(V)$ and to estimate DX_i^n; to this end, we need a stochastic implicit function theorem.

Lemma 3.3.1. *Suppose that X_{i+1}^n is in $\mathcal{D}(V)$. If $\mathbb{E}[g(X_i^n, X_{i+1}^n)|\mathcal{F}_{t_i^n}]$ is uniformly less than some constant number, then X_i^n is in $\mathcal{D}(V)$.*

Proof. Let k be an integer; from Lemma 3.2.2, we know that there exists a constant $C_0 > 0$ (not depending on k) such that on the subset

$$A_k^{C_0} = \Big\{(y_1, \ldots, y_k);\ \frac{1}{k^2}\sum_{j,l} g(y_j, y_l) < C_0\Big\} \tag{3.3.2}$$

of V^k, the relation

$$\sum_j g\big(\beta_0(y_1, \ldots, y_k), y_j\big) = \inf_{x\in V}\sum_j g(x, y_j) \tag{3.3.3}$$

defines a unique point $\beta_0(y_1, \ldots, y_k)$. Moreover from the classical implicit function theorem, and using the estimates of Lemma 3.2.1, this function is smooth and locally

$$\delta\big(\beta_0(y), \beta_0(\overline{y})\big) \le \frac{C}{k}\sum_{j=1}^k \delta(y_j, \overline{y}_j) \tag{3.3.4}$$

where C is a constant number which does not depend on k. Now let ϕ be a smooth real function defined on $[0,\infty)$ such that $\phi(z)=1$ if $z\le C_0/2$, $\phi(z)=0$ if $z\ge C_0$; consider V as a submanifold of some Euclidean space $\mathbb{R}^r$ and define on V^k the $\mathbb{R}^r$-valued function

$$\beta(y)=\beta_0(y)\phi\Big(\frac{1}{k^2}\sum_{j,l}g(y_j,y_l)\Big)+\frac{1}{k}\sum_j y_j\bigg(1-\phi\Big(\frac{1}{k^2}\sum_{j,l}g(y_j,y_l)\Big)\bigg). \tag{3.3.5}$$

Then

$$\big|\beta(y)-\beta(\overline{y})\big|\le\frac{C}{k}\sum_{j=1}^k\delta(y_j,\overline{y}_j) \tag{3.3.6}$$

for some C not depending on k; moreover on $A_k^{C_0/2}$, β is V-valued and satisfies (3.3.3). Now enlarge the probability space into $\Omega\times\Omega^{\mathbb{N}}$ and let $(W_t,W_t^1,\dots)$ be the canonical process, the components of which are independent standard Wiener processes. If $(\omega,\omega^1,\dots)$ is a generic element of this space, denote

$$\overline{\omega}^j(t)=\omega(t\wedge t_i^n)+\omega^j(t\vee t_i^n)-\omega^j(t_i^n) \tag{3.3.7}$$

and

$$Y^k=\beta\big(X_{i+1}^n(\overline{\omega}^1),\dots,X_{i+1}^n(\overline{\omega}^k)\big). \tag{3.3.8}$$

For almost each fixed $(\omega^1,\dots)$, the variables $X_{i+1}^n(\overline{\omega}^l)$ are in $\mathcal{D}(V)$ so, since β satisfies (3.3.6), the variables Y^k are in $\mathcal{D}(\mathbb{R}^r)$ and

$$\big|D_t^jY^k\big|\le\frac{C}{k}\sum_{l=1}^k\big|D_t^jX_{i+1}^n(\overline{\omega}^l)\big| \tag{3.3.9}$$

for $t<t_i^n$, so that

$$\mathbb{E}\Big[\int_0^{t_i^n}\big|D_t^jY^k\big|^2dt\;\Big|\;W^1,\dots\Big]\le\frac{C}{k}\sum_{l=1}^k\mathbb{E}\Big[\int_0^{t_i^n}\big|D_t^jX_{i+1}^n(\overline{\omega}^l)\big|^2dt\;\Big|\;W^1,\dots\Big]. \tag{3.3.10}$$

Moreover from the law of large numbers and the continuity property of Lemma 3.2.2, Y^k converges almost surely on $\Omega\times\Omega^{\mathbb{N}}$ to X_i^n as $k\to\infty$ and

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^k\mathbb{E}\Big[\int_0^{t_i^n}\big|D_t^jX_{i+1}^n(\overline{\omega}^l)\big|^2dt\;\Big|\;W^1,\dots\Big]=\mathbb{E}\Big[\int_0^{t_i^n}\big|D_t^jX_{i+1}^n\big|^2dt\;\Big|\;\mathcal{F}_{t_i^n}\Big] \tag{3.3.11}$$

almost surely; so let us fix $(\omega_1,\dots)$ at a point at which these two convergences hold almost surely on Ω; from (3.3.10) and (3.3.11), DY^k is bounded in $\mathbb{L}^2$ so we deduce from Lemma 1.3.2 that X_i^n is in $\mathcal{D}(V)$. □

Lemma 3.3.2. *Under the assumptions of Lemma 3.3.1,*

$$|D_t^j X_i^n| \le \Big(1 + K\mathbb{E}\big[g(X_i^n, X_{i+1}^n) \mid \mathcal{F}_{t_i^n}\big] + C\mathbb{E}\big[g^2(X_i^n, X_{i+1}^n) \mid \mathcal{F}_{t_i^n}\big]\Big)\mathbb{E}\big[|D_t^j X_{i+1}^n| \mid \mathcal{F}_{t_i^n}\big] \tag{3.3.12}$$

for $t \le t_i^n$.

Proof. When we consider V as a submanifold of $\mathbb{R}^r$, each tangent space T_xV becomes canonically isomorphic to a vectorial subspace of $\mathbb{R}^r$; in particular the function $h = g_0'$ can be considered as $\mathbb{R}^r$-valued and (3.3.1) can be viewed as an equality in $\mathbb{R}^r$; since X_i^n and X_{i+1}^n are in $\mathcal{D}(V)$ we can differentiate this relation (we apply the properties of the Wiener derivative applied to a conditional expectation) and obtain

$$\mathbb{E}\Big[h_0'(X_i^n, X_{i+1}^n)\langle D_t^j X_i^n\rangle + h_1'(X_i^n, X_{i+1}^n)\langle D_t^j X_{i+1}^n\rangle \Bigm| \mathcal{F}_{t_i^n}\Big] = 0 \tag{3.3.13}$$

for $t \le t_i^n$; this relation is again in $\mathbb{R}^r$. Now take the scalar product of (3.3.13) with $D_t^j X_i^n$ which is $\mathcal{F}_{t_i^n}$ measurable; by replacing h by g_0', we get

$$\mathbb{E}\big[g_{00}''(X_i^n, X_{i+1}^n)\langle D_t^j X_i^n, D_t^j X_i^n\rangle \mid \mathcal{F}_{t_i^n}\big] = -\mathbb{E}\big[g_{01}''(X_i^n, X_{i+1}^n)\langle D_t^j X_i^n, D_t^j X_{i+1}^n\rangle \mid \mathcal{F}_{t_i^n}\big]. \tag{3.3.14}$$

¿From the estimates of Lemma 3.2.1, we easily deduce (3.3.12). □

§*3.4 End of the proof*

We will first prove the weak existence of the martingale; more precisely, let $C(V)$ be the space of V-valued continuous functions defined on $[0, 1]$, and consider the product space $\Omega \times C(V)$ and its canonical process (W_t, M_t); variables defined on Ω such as L can also be considered as variables defined on $\Omega \times C(V)$.

Lemma 3.4.1. *On $\Omega \times C(V)$ there exists a probability $\mathbb{P}_0$ such that if $\mathcal{G}_t$ is the completed right-continuous filtration generated by (W_t, M_t), then W_t is a $\mathcal{G}_t$ Wiener process and M_t is a $\mathcal{G}_t$ martingale satisfying $M_1 = L$ and (3.1.7).*

Proof. Suppose that X_{i+1}^n is in $\mathcal{D}(V)$ and

$$\Big(\sum_{j=1}^m \mathbb{E}\big[|D_t X_{i+1}^n| \mid \mathcal{F}_t\big]^2\Big)^{1/2} \le \rho_{i+1}^n \tag{3.4.1}$$

(we have assumed that this condition is satisfied for $i+1=n$ with $\rho_n^n=\rho$). Then (3.2.9) is satisfied with

$$\nu_i^n=\big(\rho_{i+1}^n\big)^2\big(t_{i+1}^n-t_i^n\big). \tag{3.4.2}$$

In particular, from Lemma 3.2.3,

$$\mathbb{E}\big[g(X_i^n,X_{i+1}^n)\bigm|\mathcal{F}_{t_i^n}\big]\le\frac{\big(\rho_{i+1}^n\big)^2}{2}\big(t_{i+1}^n-t_i^n\big)+C\big(\rho_{i+1}^n\big)^4\big(t_{i+1}^n-t_i^n\big)^2. \tag{3.4.3}$$

For any constant number C_0, if $\rho_{i+1}^n\le C_0$ and if n is large enough, then the expression of (3.4.3) is small enough and we can apply the previous lemmas; from Lemmas 3.3.1 and 3.3.2, X_i^n is in $\mathcal{D}(V)$ and satisfies an estimate similar to (3.4.1) with

$$\rho_i^n=\rho_{i+1}^n\Big[1+\frac{K}{2}\big(\rho_{i+1}^n\big)^2\big(t_{i+1}^n-t_i^n\big)+C\big(\rho_{i+1}^n\big)^4\big(t_{i+1}^n-t_i^n\big)^2\Big]. \tag{3.4.4}$$

By comparing with the ordinary backward equation

$$\dot{\phi}_s=-\frac{K}{2}\phi_s^3,\ s\le 1\qquad \phi_1=\rho, \tag{3.4.5}$$

which explodes at time $1-1/(K\rho^2)$ if $K>0$, it appears that ρ_i^n remains asymptotically bounded if the assumption $\rho<1/\sqrt{K}$ holds. More precisely, we have

$$\rho_i^n\le\frac{\rho}{\sqrt{1-K\rho^2(1-t_i^n)}}+o(1) \tag{3.4.6}$$

where the '$o(1)$' term tends to 0 as $n\to\infty$ uniformly in i. In particular,

$$\mathbb{E}\big[g(X_i^n,X_{i+1}^n)\bigm|\mathcal{F}_{t_i^n}\big]\le\frac{\rho^2\big(t_{i+1}^n-t_i^n\big)}{2\big(1-K(1-t_i^n)\rho^2\big)}+o\big(t_{i+1}^n-t_i^n\big) \tag{3.4.7}$$

and

$$\mathbb{E}\big[g^2(X_i^n,X_{i+1}^n)\bigm|\mathcal{F}_{t_i^n}\big]\le C\big(t_{i+1}^n-t_i^n\big)^2. \tag{3.4.8}$$

Now let h be a bounded smooth function with bounded derivatives defined on V; by applying the Taylor formula to h at X_i^n and by using the inverse exponential function at X_i^n which is defined on a neighbourhood of X_i^n, we obtain

$$\begin{aligned}h(X_{i+1}^n)=&h(X_i^n)+h'(X_i^n)\langle\exp_{X_i^n}^{-1}X_{i+1}^n\rangle\\&+\frac{1}{2}h''(X_i^n)\langle\exp_{X_i^n}^{-1}X_{i+1}^n,\exp_{X_i^n}^{-1}X_{i+1}^n\rangle+O\big(|\exp_{X_i^n}^{-1}X_{i+1}^n|^3\big).\end{aligned} \tag{3.4.9}$$

Since

$$\exp^{-1}_{X_i^n} X_{i+1}^n = g_0'(X_i^n, X_{i+1}^n) + O\big(g^{3/2}(X_i^n, X_{i+1}^n)\big), \tag{3.4.10}$$

by taking the conditional expectation in (3.4.9), by using (3.3.1) and (3.4.8), we deduce

$$\begin{aligned} \mathbb{E}\big[h(X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\big] =& h(X_i^n) + \frac{1}{2}\mathbb{E}\Big[h''(X_i^n)\langle g_0'(X_i^n, X_{i+1}^n), g_0'(X_i^n, X_{i+1}^n)\rangle \Bigm| \mathcal{F}_{t_i^n}\Big] \\ &+ O\Big(\big(t_{i+1}^n - t_i^n\big)^{3/2}\Big). \end{aligned} \tag{3.4.11}$$

If $\lambda_h(x)$ is the smallest eigenvalue of $h''(x)$ and if $\lambda_h^- = (-\lambda_h) \vee 0$, we deduce

$$\begin{aligned} \mathbb{E}\big[h(X_{i+1}^n) \bigm| \mathcal{F}_{t_i^n}\big] &\geq h(X_i^n) + \frac{1}{2}\lambda_h(X_i^n)\mathbb{E}\big[\big|g_0'(X_i^n, X_{i+1}^n)\big|^2 \bigm| \mathcal{F}_{t_i^n}\big] - C\big(t_{i+1}^n - t_i^n\big)^{3/2} \\ &\geq h(X_i^n) - C\lambda_h^-(X_i^n)\big(t_{i+1}^n - t_i^n\big) - C\big(t_{i+1}^n - t_i^n\big)^{3/2}. \end{aligned} \tag{3.4.12}$$

Now consider the step process M_t^n defined in §3.1, which is X_i^n on the time interval $[t_i^n, t_{i+1}^n)$; we want to prove that its law is C-tight. We deduce from (3.4.12) that

$$\mathbb{E}\big[h(M_t^n) \bigm| \mathcal{F}_s\big] \geq h(M_s^n) - C\int_s^t \lambda_h^-(M_u^n)du - C\sqrt{\varepsilon_n}(t-s) \tag{3.4.13}$$

for $s \leq t$ in the subdivision $(t_\cdot^n)$ and where we recall that ε_n is the step size of the subdivision. On the other hand if in Lemma 3.2.3 we replace t_i^n by another time s, we see that there exist $\mathcal{F}_s$ measurable variables $X_{s,t}$ such that

$$\mathbb{E}\big[g^2(X_{s,t}, M_t^n) \bigm| \mathcal{F}_s\big] \leq C(t-s)^2, \qquad \mathbb{E}\big[g(X_{s,t}, M_t^n) \bigm| \mathcal{F}_s\big] \leq C(t-s). \tag{3.4.14}$$

By applying (3.4.13) to the function $x_1 \mapsto g(x_0, x_1)$, we obtain

$$\mathbb{E}\big[g(X_{s,t}, M_t^n) - g(X_{s,t}, M_s^n) \bigm| \mathcal{F}_s\big] \geq -C(t-s) \tag{3.4.15}$$

so by substracting these two estimates,

$$g(X_{s,t}, M_s^n) \leq C(t-s). \tag{3.4.16}$$

Since $\delta \wedge 1 \leq C\sqrt{g}$, we deduce from (3.4.14) and (3.4.16) estimates on $\delta(X_{s,t}, M_s^n)$ and $\delta(X_{s,t}, M_t^n)$; from the triangle inequality, we get for s and t in the subdivision

$$\mathbb{E}\big[\delta^4(M_s^n, M_t^n) \wedge 1 \bigm| \mathcal{F}_s\big] \leq C(t-s)^2. \tag{3.4.17}$$

Moreover the law of $M_1^n = L$ is fixed and therefore tight, so the C-tightness of the process M^n is proved. This implies that the law of (W, M^n) is also C-tight, so let us consider the limit $\mathbb{P}_0$ of some subsequence; for any real bounded measurable function ϕ defined on $\Omega \times C(V)$ which is continuous with respect to the second variable, the mean of $\phi(W, M^n)$ converges (for the subsequence) to the $\mathbb{P}_0$-expectation of $\phi(W, M)$ (apply [7]). In particular, since $M_1^n = L$, we also have $M_1 = L$; it is also easy to check that W_t is a $\mathcal{G}_t$ Wiener process; on the other hand, by multiplying both sides of (3.4.13) by any continuous function of $(W_u, M_u; u \le s)$, by taking the expectation and by taking the limit as $n \to \infty$, we prove that

$$h(M_t) - C \int_0^t \lambda_h^-(M_s) ds$$

is a $\mathcal{G}_t$ submartingale. Thus, from the Darling definition [2] of martingales, the process M_t is a $\mathcal{G}_t$ martingale. Since $g \sim \delta^2/2$ in the neighbourhood of the diagonal, the quadratic variation of M_t^n is estimated from (3.4.7) and by taking the limit, we obtain (3.1.7). $\square$

We finish the proof of Theorem 3.1.1 in two steps; we first prove that the process M_t obtained in Lemma 3.4.1 is adapted to the filtration $\mathcal{F}_t$ of W_t so that it can be realized on Ω; then we check that M_t is in $\mathcal{D}^\star(V)$.

Proof of the strong existence. This will follow from the uniqueness theorem of §1; the Watanabe-Yamada method for proving strong existence for stochastic differential equations (see Chapter 8 of [15]) can also be applied in our context. On the space $\Omega \times C(V) \times C(V)$ with canonical process (W, M^1, M^2), consider the probability $\mathbb{P}_1$ defined by

$$\mathbb{E}_{\mathbb{P}_1}\left[f(W)\phi_1(M^1)\phi_2(M^2)\right] = \mathbb{E}_{\mathbb{P}}\Big[f(W)\mathbb{E}_{\mathbb{P}_0}\left[\phi_1(M) \mid W\right]\mathbb{E}_{\mathbb{P}_0}\left[\phi_2(M) \mid W\right]\Big] \quad (3.4.18)$$

for bounded measurable functionals f, ϕ_1, ϕ_2. Let $\mathcal{G}_t^1$ be the filtration generated by the process (W, M^1, M^2); one verifies that under $\mathbb{P}_1$, (W, M^1) and (W, M^2) have law $\mathbb{P}_0$, that W_t is a $\mathcal{G}_t^1$ Wiener process, that M^1 and M^2 are conditionally independent given W, so that M_t^1 and M_t^2 are $\mathcal{G}_t^1$ martingales with value L at final time; since they satisfy (3.1.7), we deduce from Theorem 2.2.1 that $M^1 = M^2$. Thus conditionally on $W = \omega$, M^1 and M^2 are independent and almost surely equal, so are equal to some $M(\omega)$. Now M is measurable with respect to the σ-field of W and M_t is independent from future increments of W, so M_t is $\mathcal{F}_t$ adapted; thus it is a $\mathcal{F}_t$ martingale defined on Ω and converging to L. $\square$

Proof of the regularity. By using elementary results concerning weak and stable convergences (see [7]), we can deduce from the previous proof that M^n converges uniformly in probability to M. Thus in order to prove that M_t is in $\mathcal{D}^\star(V)$ it is sufficient to prove (1.3.6) for $Y^n = M^n$, that is

$$\limsup_{n\to\infty} \sum_j \mathbb{E} \int_0^1 \sup_i \left| D_t^j X_i^n \right|^2 dt < \infty. \tag{3.4.19}$$

But we deduce from Lemma 3.3.2 that there exists a constant $C > 0$ such that for t fixed, the process $\exp(C t_i^n) |D_t^j X_i^n|$ is a discrete-time submartingale; from Doob's inequality, the left-hand side of (3.4.19) is estimated by $4e^{2C} \|L\|_{\mathcal{D}}^2$. $\square$

4. The existence on small convex domains

We now prove that if Δ is an open subset of V satisfying some assumption, then one can construct Δ-valued martingales in the non differentiable case; this can in particular be applied to manifolds with non-positive curvature. As in previous section, we suppose that the probability space is a Wiener space. If V is embedded in a Euclidean space $\mathbb{R}^r$, the spaces $\mathbb{L}^q(\Delta)$ are defined to be the subsets of the spaces $\mathbb{L}^q(\mathbb{R}^r)$ consisting of Δ-valued variables, with the induced topology (this does not depend on the embedding).

Theorem 4.1. *Suppose that Ω is a Wiener probability space. For $x \in V$, let $\kappa(x)$ be the smallest non-negative number dominating the sectional curvatures at x. Let $1 < p \le \infty$ and $1 \le q < \infty$ be such that $1/p + 1/q = 1$. Let Δ be an open connected and simply connected subset of V and suppose that there exists a smooth positive function f such that $0 < c_0 < f < c_1$ on Δ, $0 < f < c_0$ on $V \setminus \overline{\Delta}$ and $f'' + p\kappa f \le 0$ on Δ (the case $p = \infty$ is possible only when $\kappa \equiv 0$ on Δ). Then there exists a unique continuous function $L \mapsto (M_t)$ defined on $\mathbb{L}^q(\Delta)$, with values in the space of Δ-valued continuous processes endowed with the uniform convergence in probability, and such that*
(a) for any L, M_t is a martingale converging to L;
(b) if L is in $\mathcal{S}(\Delta)$, then $\langle\!\langle M \rangle\!\rangle_\infty$ is bounded.

Example. The regular geodesic balls of [9] satisfy the assumptions of the theorem for some p: if O is the centre, use the function $x \mapsto \cos(\sqrt{Kp}\delta(O,x))$. Thus the existence theorem of [9] can be deduced from Theorem 4.1.

Remark. We have stated the uniqueness of a continuous function $L \mapsto (M_t)$ satisfying (a) and (b), but we do not know whether M_t is the unique martingale converging to L.

Proof. First note that one can apply Theorem 2.1.3 to $\Delta^\beta = \{f > \beta\}$ for any $\beta > c_0$, so that any martingale with bounded quadratic variation converging to a variable in Δ^β lives in Δ^β; moreover from Proposition 2.1.2, any Δ-valued martingale is in $\mathcal{E}_{p\kappa/2}$; more precisely, the variables

$$\mathbb{E}\Big[\exp \frac{p}{2} \int_t^\infty \kappa(M_s) d\langle\!\langle M \rangle\!\rangle_s \;\Big|\; \mathcal{F}_t\Big]$$

are uniformly bounded by some constant number. Note also that two points of Δ are linked by at least one Δ-valued geodesic (one can prove that if x_0 is a point of Δ, the set of points which are linked to x_0 by a Δ-valued geodesic segment is both open and closed in Δ; to this end we use the fact that if the end points of a $\overline{\Delta}$-valued geodesic are in Δ, then the geodesic is Δ-valued); one can also prove from comparison theorems that there do not exist conjugate points on a Δ-valued geodesic; since Δ is simply connected, one can deduce from the homotopy lemma (see Lemma 2.6.4 of [10]) that two points of Δ are linked by exactly one geodesic; moreover this geodesic depends smoothly on its end points; in particular, Δ is diffeomorphic to a star-shaped open subset of $\mathbb{R}^d$ (use the exponential function at some point), so $\mathcal{S}(\Delta)$ is dense in $\mathbb{L}^q(\Delta)$. After these geometrical preliminaries, let us prove the theorem. First note that the uniqueness is a consequence of Theorem 2.2.1 and of the density of $\mathcal{S}(\Delta)$ in $\mathbb{L}^q(\Delta)$. For the existence, let L be a variable in $\mathbb{L}^q(\Delta)$; one can find a sequence L^n of functionals of $\mathcal{S}(\Delta)$ such that $\delta\big(L^n, L\big)$ converges to 0 in $\mathbb{L}^q$ and $f(L^n) > \beta^n$ for some $\beta^n > c_0$; for each n there exists a Δ-valued martingale M_t^n, $t \geq t_n$ with bounded quadratic variation and converging to L^n; the process M_t^n is in $\mathcal{D}^\star(V)$ and by applying Theorem 2.3.1 to perturbations on the Wiener process, we can check that

$$S_{s,t}^n = \big|D_s M_t^n\big| \exp \frac{1}{2} \int_s^t \kappa(M_u^n) d\langle\!\langle M^n \rangle\!\rangle_u, \tag{4.1}$$

$t \geq s$, is a submartingale; moreover for any fixed n, the derivatives $D_s M_\infty^n$ are bounded and the variables

$$\mathbb{E}\Big[\exp\frac{1}{2}\int_t^{\infty}\kappa(M_u^n)d\langle\!\langle M^n\rangle\!\rangle_u\;\Big|\;\mathcal{F}_t\Big]$$

are bounded so $D_sM_t^n$ is bounded. Thus M_t^n can be defined on the whole time interval ($t_n = 0$); if indeed $t_n > 0$, since $D_sM_{t_n}^n$ is bounded, we deduce that M_t^n can always be extended to an interval $[t_n - C, \infty]$. For each m and n, let $L^{m,n,\alpha}$, $0 \le \alpha \le 1$ be the geodesic interpolation between L^m and L^n; then $\alpha \mapsto L^{m,n,\alpha}$ is continuous into $\mathcal{S}(\Delta)$, we can construct the martingales $M_t^{m,n,\alpha}$ converging to $L^{m,n,\alpha}$, apply Theorem 2.3.1 to them and estimate the Jacobi field $\partial M_t^{m,n,\alpha}/\partial\alpha$ for any $0 \le \alpha \le 1$. Since these martingales are bounded in $\mathcal{E}_{p\kappa/2}$, we deduce that

$$\mathbb{P}\Big[\sup_t \delta(M_t^m, M_t^n) > u\Big] \le \frac{C}{u}\mathbb{E}\Big[\delta(L^m, L^n)^q\Big]^{1/q}. \tag{4.2}$$

Thus M_t^n has in $\overline{\Delta}$ a uniform limit in probability and this limit is a martingale converging to L; from Theorem 2.1.3, M_t is Δ-valued. The continuity of the map $L \mapsto (M_t)$ also follows from (4.2) which can be extended to any sequence of Δ-valued variables L^n. $\square$

The continuity property of Theorem 4.1 can be improved by the

Proposition 4.2. *Under the assumptions of Theorem 4.1, let (A, d) be a metric space and let L^α, $\alpha \in A$, be a family of Δ-valued variables such that almost surely, $\alpha \mapsto L^\alpha$ is uniformly continuous; suppose also that $\sup_\alpha \delta(O, L^\alpha)$ is in $\mathbb{L}^q$ for some (or any) O. For any α, let M_t^α be the martingale converging to L^α constructed in Theorem 4.1; then there is a version of (M_t^α) which is almost surely continuous on $A \times [0, \infty]$.*

Proof. We deduce from (4.2) that

$$\delta(M_0^\alpha, M_0^\beta) \le C\mathbb{E}\big[\delta(L^\alpha, L^\beta)^q\big]^{1/q}. \tag{4.3}$$

Similarly, by working conditionally on $\mathcal{F}_t$, we can check that

$$\delta(M_t^\alpha, M_t^\beta) \le C\mathbb{E}\big[\delta(L^\alpha, L^\beta)^q \;\big|\; \mathcal{F}_t\big]^{1/q}. \tag{4.4}$$

Thus

$$\operatorname*{ess\,sup}_{d(\alpha,\beta)\le\varepsilon}\sup_t \delta(M_t^\alpha, M_t^\beta) \le C\sup_t \mathbb{E}\Big[\sup_{d(\alpha,\beta)\le\varepsilon}\delta(L^\alpha, L^\beta)^q \;\Big|\; \mathcal{F}_t\Big]^{1/q} \tag{4.5}$$

and therefore, by using an inequality for submartingales, for $\eta > 0$,

$$\mathbb{P}\Big[\operatorname*{ess\,sup}_{d(\alpha,\beta)\le\varepsilon}\sup_t \delta(M_t^\alpha, M_t^\beta) > \eta\Big] \le \mathbb{P}\Big[\sup_t \mathbb{E}\big[\sup_{d(\alpha,\beta)\le\varepsilon}\delta(L^\alpha, L^\beta)^q \mid \mathcal{F}_t\big] > \frac{\eta^q}{C^q}\Big]$$
$$\le \frac{C^q}{\eta^q}\mathbb{E}\Big[\sup_{d(\alpha,\beta)\le\varepsilon}\delta(L^\alpha, L^\beta)^q\Big]. \tag{4.6}$$

Take the limit as $\varepsilon \downarrow 0$ on both sides of this inequality; the right-hand side converges to 0 for any $\eta > 0$, so we deduce

$$\lim_{\varepsilon\to 0}\operatorname*{ess\,sup}_{d(\alpha,\beta)\le\varepsilon}\sup_t \delta(M_t^\alpha, M_t^\beta) = 0 \tag{4.7}$$

almost surely. If we restrict ourselves to rational values of α, M_t^α is almost surely continuous with respect to α uniformly in (t,α), so it is continuous with respect to (t,α); for real values of α, if we choose for $(M_t^\alpha, 0 \le t \le \infty)$ a measurable accumulation point of (M_t^β), β rational tending to α, we obtain a version satisfying the condition of the proposition. $\square$

If Δ is not simply connected, we can consider its universal cover; for instance, if V is connected and its sectional curvatures are non-positive, then its universal cover $\widetilde{V}$ is a Cartan-Hadamard manifold, so we can apply Theorem 4.1 to $\Delta = \widetilde{V}$ with $p = \infty$; if L is a variable of $\mathbb{L}^1(V)$, we can lift it into a variable of $\mathbb{L}^1(\widetilde{V})$, construct a $\widetilde{V}$-valued martingale and by projection, prove the existence of a V-valued martingale converging to L. If V is not simply connected, the lifting is not unique and consequently, the martingale is not unique. Actually, we have a more precise existence theorem; by using the notion of homotopy defined in §2.2, we can prove the

Proposition 4.3. *Let Δ be a connected subset of V and suppose that its universal cover satisfies the conditions of Theorem 4.1. Let X_t be a Δ-valued $\mathcal{F}_t$ adapted continuous process converging almost surely to L. Suppose that $\sup_t \delta(O, X_t)$ is in $\mathbb{L}^q$. Then there exists a Δ-valued martingale converging to L and homotopic to X.*

Proof. Consider a continuous lifting $\widetilde{X}_t$ of X_t in $\widetilde{\Delta}$; we can construct a family of $\widetilde{\Delta}$-valued martingales $\widetilde{Y}_t^\alpha$, $0 \le \alpha \le \infty$, converging to $\widetilde{X}_\alpha$, and from Proposition 4.2, we can choose a version which is almost surely continuous with respect to (t,α); note that $\widetilde{Y}_t^\alpha = \widetilde{X}_\alpha$ for

$t \geq \alpha$. Let Y_t^α be the projection of $\widetilde{Y}_t^\alpha$ on Δ if $t \leq \alpha$, and put $Y_t^\alpha = X_t$ for $t \geq \alpha$; put also $M_t = Y_t^\infty$; then M_t is martingale converging to L and Y^α is a homotopy path from X to M. $\square$

5. Application to partial differential equations

We want to apply previous results to a smooth Markovian case. If one is given a Markov transition kernel, one has to realize it on a Wiener space; when this is possible, such a realization is not unique. In this section, we suppose that we have constructed a realization satisfying some assumptions and we deduce the existence of a probabilistic solution to the heat equation. Then we will consider the Dirichlet problem.

Theorem 5.1. *On our Wiener space, consider the stochastic differential equation*

$$dX_t = \beta(t, X_t)dt + \sigma(t, X_t) \circ dW_t \tag{5.1}$$

on a smooth submanifold U of some Euclidean space, for smooth coefficients β and σ; we suppose that when (5.1) is written in Itô's form in the Euclidean space, the coefficients are uniformly Lipschitz and have at most linear growth. We denote by $X_t^{s,x}$, $s \leq t \leq 0$ the stochastic flow associated to (5.1). Let ϕ be a C^1 Lipschitz function from U into V. We suppose that

$$\sup_x \mathbb{E}\left|\phi'(X_0^{t,x})\frac{\partial X_0^{t,x}}{\partial x}\sigma(t,x)\right| \leq \rho_t \tag{5.2}$$

for some function ρ_t such that $\int_{-\infty}^0 \rho_t^2 dt$ is finite and less than $1/K$. Then there exists a unique measurable V-valued function f defined on $(-\infty, 0] \times U$ such that $f(0,.) = \phi$ and for any (s,x), $f(t, X_t^{s,x})$, $s \leq t \leq 0$ is a V-valued martingale with bounded quadratic variation.

Proof. It follows from the study of (5.1) (see [12]) that $\phi(X_0^{s,x})$ is in $\mathcal{D}(V)$ and that

$$D_t\phi(X_0^{s,x}) = \phi'(X_0^{t,y})\frac{\partial X_0^{t,y}}{\partial y}\sigma(t,y)\bigg|_{y=X_t^{s,x}} \tag{5.3}$$

for $s \leq t \leq 0$. Thus from (5.2), we can apply Theorem 3.1.1 and $f(s,x)$ is necessarily the value at time s of the martingale constructed in this theorem. More precisely, if $S = (t_i)$

is a subdivision of $(-\infty, 0]$, for $t \geq s$ and $t_i \leq t < t_{i+1}$, we denote by $M_t^{S,s,x}$ the point at which the function $\mathbb{E}[g(x, M_{t_{i+1}}^{S,s,x}) \mid \mathcal{F}_t]$ takes its minimum: from the study of §3, this point is unique when

$$\varepsilon(S) = \sup_i \int_{t_i^n}^{t_{i+1}^n} \rho_t^2 dt \tag{5.4}$$

is small enough. Then $M_t^{S,s,x}$ converges, as $\varepsilon(S)$ tends to 0, to the martingale $M_t^{s,x}$ with limit $\phi(X_0^{s,x})$ and $f(s,x)$ is defined to be $M_s^{s,x}$; it follows that f is measurable and $f(0,.) = \phi$. Moreover, using the Markov property,

$$M_t^{S,s,x} = M_t^{S,t,X_t^{s,x}} \tag{5.5}$$

so, by taking the limit,

$$M_t^{s,x} = f(t, X_t^{s,x}). \tag{5.6}$$

Thus $f(t, X_t^{s,x})$ is a martingale with bounded quadratic variation. □

In the previous theorem f is only measurable; however a better regularity result can be checked from Theorem 2.3.1.

Proposition 5.2. *Under the assumptions of Theorem 5.1, suppose moreover that*

$$\sup_t \mathbb{E} \sup_x \left| \phi'(X_0^{t,x}) \frac{\partial X_0^{t,x}}{\partial x} \right| < \infty. \tag{5.7}$$

Then the function f is Lipschitz with respect to x uniformly in (t,x) and is continuous with respect to (t,x).

Proof. Fix a time t and two points x_0, x_1 of U. Let $x(\alpha)$, $0 \leq \alpha \leq 1$ be a minimizing geodesic in U between these two points (for the Riemannian metric induced by the Euclidean metric). Then $\phi(X_0^{t,x(\alpha)})$ is smooth with respect to α and from (5.7), we have

$$\mathbb{E} \sup_\alpha \left| \frac{\partial \phi\big(X_0^{t,x(\alpha)}\big)}{\partial \alpha} \right| \leq C \delta_U(x_0, x_1). \tag{5.8}$$

We deduce that f is uniformly Lipschitz with respect to x by applying Theorem 2.3.1. We still have to prove that f is continuous with respect to t at any (t_0, x_0); using the

Lipschitz continuity which has already been proved, the set $C(t_0)$ of points x such that f is continuous at (t_0, x) is closed; for any $s < t_0$, the process $f(t, X_t^{s,x_0})$ is almost surely continuous at t_0 (it is a martingale); the process X_t^{s,x_0} is also continuous. Now use

$$\delta\Big(f\big(t, X_{t_0}^{s,x_0}\big), f\big(t_0, X_{t_0}^{s,x_0}\big)\Big) \leq \delta\Big(f\big(t, X_{t_0}^{s,x_0}\big), f\big(t, X_t^{s,x_0}\big)\Big) + \delta\Big(f\big((t, X_t^{s,x_0}\big), f\big(t_0, X_{t_0}^{s,x_0}\big)\Big). \tag{5.9}$$

We deduce from the previous remarks that the two terms of the right-hand side converge to 0 as $t \to t_0$, so $f(t, X_{t_0}^{s,x_0})$ converges almost surely to $f(t_0, X_{t_0}^{s,x_0})$; thus $C(t_0)$ contains the closed support of the law of $X_{t_0}^{s,x_0}$; by letting $s \uparrow t_0$, we deduce that it contains x_0. □

We can also give (as in [9] for regular geodesic balls) a probabilistic solution to the Dirichlet problem in a small convex domain. The framework which was set in the introduction (the exit problem) is a particular case of the following one.

Theorem 5.3. *On a state space U, consider a time-homogeneous Markov process which can be realized as a stochastic flow X_t^x defined on our Wiener space. Let Δ be a domain satisfying the assumptions of Theorem 4.1 and let $\phi(X^x)$ be a bounded Δ-valued variable which is measurable with respect to the asymptotic σ-field of the Markov process. Then there exists a measurable function f from U to Δ such that for any x, $f(X_t^x)$ is a martingale converging to $\phi(X^x)$.*

Proof. Theorem 4.1 enables the construction of a map Ψ defined on the set of Δ-valued Wiener functionals L such that $\Psi(L)$ is a martingale with final value L; denote by $\Psi_0(L)$ the value at time 0 of this martingale; if L is decomposed as

$$L = L\big(W_s, s \leq t; W_s - W_t, s \geq t\big) \tag{5.10}$$

then one can check that the value at time t of this martingale is almost surely

$$M_t = \Psi_0\big(L(W_s, s \leq t; .)\big). \tag{5.11}$$

In the framework of the theorem, define

$$f(x) = \Psi_0\big(\phi \circ X^x\big). \tag{5.12}$$

Then f is measurable and using the Markov property for the flow X_t^x, the martingale with final value $\phi(X^x)$ is from (5.11)

$$M_t^x = \Psi_0\big(\phi \circ X^{X_t^x}\big) = f(X_t^x). \tag{5.13}$$

□

The only regularity on f proclaimed in Theorem 5.3 is the measurability; analytical methods may provide better regularity results but we have to notice that our framework is more general; for the continuity, one can try to use coupling methods as in [9]. If Δ is not simply connected, we can use its universal cover as in §4; for instance, we can solve the Dirichlet problem for connected manifolds with non-positive sectional curvatures. Note also that the method of Proposition 4.3 can also be used to solve the Dirichlet problem; choose a V-valued function g such that $g(X_t^x)$ converges to $\phi(X^x)$ as $t \to \infty$; then consider the family of martingales with limit $g(X_t^x)$ and let t tend to infinity; one can try to prove that this procedure provides a solution in the homotopy class of g.

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