Algebraically flat or projective algebras

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Abstract

We define and study algebraically flat algebras in order to have a better understanding of algebraically projective algebras of finite type (the projective algebras of literature). A close examination of the differential properties of these algebras leads to our main structure theorem. As a corollary, we get that an algebraically projective algebra of finite type over a field is either a polynomial ring or the affine algebra of a complete intersection. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

This paper originates with the theory of projective algebras. We were motivated by an unsolved conjecture: a projective algebra of finite type over a field $A$ is a polynomial ring. An example by Costa shows that the statement is false if $A$ is not a field. As Costa noticed, the cancellation problem for polynomial rings over fields is solved if the conjecture is true [9].

Flatness is well known to be useful when studying projectivity. In Section 1, we are aiming to build a convenient theory of flatness for algebras. Roughly speaking, the flatness of an $A$-module $M$ is characterized by properties of linear relations in $M$. Replacing linear relations with polynomial relations gives the solution. We have chosen to follow Lazard’s treatment of flatness [17]. An $A$-algebra $B$ is called algebraically flat (a-flat) if every morphism of $A$-algebras $P \to B$ where $P$ is of finite presentation can be factored $P \to L \to B$ where $L$ is a polynomial algebra in finitely many indeterminates. When $A$ and $B$ are Noetherian, replacing polynomial algebras
with smooth algebras in the above definition gives the characterization of regular morphisms by Popescu–Spivakovski [31]. Our definition gives most of the usual flatness properties. In particular, an $A$-algebra $B$ is a-flat if and only if $B$ is a direct limit of polynomial algebras in finitely many indeterminates over $A$. Symmetric algebras of flat modules are a-flat algebras. D. Popescu defined algebraically pure morphisms (a-pure morphisms) [26]. These morphisms are closely related to a-flat morphisms, since an a-flat morphism is a-pure and faithfully flat. Under some finiteness conditions, a-pure morphisms descend factorization of morphisms. As a consequence, a-purity descends a-flatness and smoothness. Evidently, a-flatness localizes but we do not know whether it globalizes. Here are some concrete examples of a-flat morphisms. If $I$ is a flat ideal of linear type in a ring $A$, its Rees ring $A[IX]$ is a-flat over $A$. Then a Rees ring over a Prüfer domain is a-flat.

We define the flat rank $\text{f-rk}(B)$ of an a-flat algebra $B$. Then $\text{f-rk}(B) \leq r$ if and only if $B$ is a direct limit of polynomial algebras in $r$ indeterminates. If $B$ is of finite type, $\text{f-rk}(B) = \lambda(B)$, the least number of elements required to generate $B$.

In this paper, projectively trivial rings are a prominent tool because a connected ring $A$ is projectively trivial if and only if each of its finitely generated projective modules is free [22]. We say that a ring $A$ is PPF if finitely generated projective $A[X_1,\ldots,X_n]$-modules are free for each integer $n$. A principal domain is PPF by the Quillen–Suslin’s theorem. If $\mathbb{P}$ is a property of rings, $\mathbb{P}$-morphisms are well known. We have been led to introduce a variant: universal $\mathbb{P}$-morphisms. We show that a regular PPF integral domain is a UFD and that an a-flat morphism is a universal connected PPF morphism. Hence, if $A \to B$ is a-flat and $A$ is PPF, so is $B$. Moreover, an a-flat morphism between noetherian rings is a regular UFD morphism.

Section 2 contains the main results of this paper and is devoted to algebraically projective (a-projective) algebras. They are the projective objects in a category of algebras over a ring. An a-projective algebra is projective. These algebras have been studied by many authors as D. L. Costa, T. Asanuma, J. W. Brewer, A. R. Kustin, J. Yanik.

Our results show that a-projective algebras share many properties with polynomial rings. An a-projective algebra of finite type is of finite presentation and an a-projective algebra is a-flat. The converse is true if $B$ is of finite presentation. In this case, $B$ is the direct limit of polynomial algebras in $\text{f-rk}(B) = \lambda(B)$ indeterminates. This gives a partial answer to the conjecture evoked at the beginning.

The following is a key result. If $B$ is an a-projective algebra of finite type, $A \to B$ is a projective, smooth, universal regular morphism, its $B$-module of Kähler differentials $\Omega_A(B)$ is projective and $K \otimes_A B$ is a regular UFD for every ring morphism $A \to K$ where $K$ is a field. Moreover, if $R$ is a connected PPF ring, so is $R \otimes_A B$ for every ring morphism $A \to R$ and $\Omega_R(R \otimes_A B)$ is free with finite rank.

An a-projective $A$-algebra $B$ of finite type is a retract of a polynomial ring $L = A[X_1,\ldots,X_n]$. An idempotent endomorphism $u$ of the $A$-algebra $L$ is associated to $B$. The sequence $\{u(X_1) = f_1,\ldots,u(X_n) = f_n\}$ is called a representation of $B$ and $J = (X_1 - f_1,\ldots,X_n - f_n)$ a representation ideal, for $B = A[f_1,\ldots,f_n]$ and $B \simeq L/J$. With this notation, if $A$ is a connected PPF ring, then $J/J^2$ is a free $B$-module with finite rank,
\[ n = \text{rk}_B(\Omega_A(B)) + \text{rk}_B(J/J^2) \] and if in addition \( A \) is Noetherian, \( \dim(B) \leq \dim(A) + \text{rk}_B(\Omega_A(B)) \).

Our main result is as follows. Let \( K \) be a PPF affine regular integral domain and \( K \to B \) an \( a \)-projective morphism of finite type which is not a polynomial algebra, then a representation ideal \( J \) such that \( \text{ht}(J) > \dim(K) \) is a complete intersection and \( \dim(B) = \text{rk}_B(\Omega_K(B)) \). In particular, if \( K \) is a field, a representation ideal is a complete intersection.

We give some notation. All rings considered are unital commutative and ring morphisms are unital. Hence a commutative \( A \)-algebra \( B \) can be identified with its structural ring morphism \( A \to B \). The set of all units of a ring \( A \) is denoted by \( U(A) \), the set of all idempotents by \( \text{Bool}(A) \) and the nilradical by \( \text{Nil}(A) \). If \( P \) is a prime ideal of \( A \), the associated residual field is denoted by \( k(P) \). The symmetric algebra of an \( A \)-module \( M \) is denoted by \( S_A(M) \). Any unexplained notation is standard.

1. Definition and properties of algebraically flat morphisms

In the following, a polynomial \( A \)-algebra \( L \) over a ring \( A \) is an \( A \)-algebra \( A[X_i] \in I \) in a set of indeterminates \( \{X_i\}_{i \in I} \) (if \( I \) is empty, \( L = A \)). We denote by \( \mathcal{P}_A \) the class of all polynomial algebras of the form \( A[X_1, \ldots, X_n] \) where \( n \) is an integer.

**Definition 1.1.** An \( A \)-algebra \( B \) (or a ring morphism \( A \to B \)) is called algebraically flat (\( a \)-flat) if the following condition (\( aF \)) holds:

\( (aF) \) Every morphism of \( A \)-algebras \( P \to B \) where \( P \) is an \( A \)-algebra of finite presentation, can be factored \( P \to L \to B \) where \( L \) is a polynomial \( A \)-algebra.

In the above definition, the polynomial \( A \)-algebra \( L \) can be replaced with \( L \in \mathcal{P}_A \) or with an \( a \)-flat algebra \( L \). Clearly, a polynomial \( A \)-algebra is \( a \)-flat.

Our first result gives the structure of \( a \)-flat morphisms. Lazard gave a similar result for flat modules [17]. We mimic the proof given in [5]. The proof is detailed because some arguments are different in the category of algebras.

**Lemma 1.2.** Let \( A \to B \) be a ring morphism and assume that there exists a direct system \( \{B_*\}_{\lambda \in \Lambda} \) of \( A \)-algebras \( B_* \) such that \( B = \lim_{\to} B_* \). Let \( P \to B \) be a morphism of \( A \)-algebras where \( P \) is of finite presentation. There is some index \( \lambda \) such that \( P \to B \) can be factored \( P \to B_{\lambda} \to B \).

**Proof.** Consider a morphism \( f : P \to B \) of \( A \)-algebras where the algebra \( P = A[X_1, \ldots, X_n]/(p_1, \ldots, p_s) \) is defined by the polynomials \( p_1, \ldots, p_s \). Denote by \( x_i \) the class of \( X_i \) in \( P \) and set \( f(x_i) = b_i \). There are an index \( \lambda \) and some \( v_1, \ldots, v_n \) in \( B_* \) such that \( p_\lambda (v_1, \ldots, v_n) = 0 \) for \( k = 1, \ldots, s \) and \( v_i \mapsto b_i \) for \( i = 1, \ldots, n \). Then \( P \to B \) can be factored \( P \to B_{\lambda} \to B \).

**Theorem 1.3.** Let \( A \to B \) be a ring morphism. Then \( B \) is \( a \)-flat if and only if there exists a direct system \( \{L_*\}_{\lambda \in \Lambda} \) of \( A \)-algebras \( L_* \in \mathcal{P}_A \) such that \( B = \lim_{\to} L_* \). In this case, the canonical morphisms \( L_{\lambda} \to L_{\mu} \) are of finite presentation.
Proof. Assume that $B = \lim L_i$ where $L_i \in \mathcal{P}_A$. Then Lemma 1.2 shows that $B$ is a-flat. Conversely, assume that $B$ is a-flat. Then $B = \lim B_i$ where $\{B_i\}_{i \in A}$ is a direct system of $A$-algebras of finite presentation, indexed by a partially ordered directed set $A$ (the partial ordering hypothesis is essential) [11, O.6.3.10]. There is no harm to change $A$ into $A \times \mathbb{N}$ equipped with the lexicographic order provided we set $B_{i \cdot (n, 1)} = B_i$ for each $n \in \mathbb{N}$. Thus we can assume that $A$ has no maximum element. Denote the canonical morphisms by $g_\lambda : B_\lambda \to B$ and $g_{\mu, \lambda} : B_\mu \to B_\lambda$ for $\lambda \leq \mu$. Consider an element $\sigma \in A$.

By a-flatness, there exist a polynomial ring $L_\sigma = A[X_1, \ldots, X_n]$ and some morphisms $u_\sigma, w_\sigma$ such that $B_\sigma \xrightarrow{u_\sigma} L_\sigma \xrightarrow{w_\sigma} B = B_\sigma \xrightarrow{g_\sigma} B$. Then set $w_\sigma(X_i) = b_i$. There exist some $\tau > \sigma$ and $x_1, \ldots, x_n \in B$ such that $b_i = g_\tau(x_i)$ for $i = 1, \ldots, n$, because there is no maximum element in $A$. Next define an $A$-algebra morphism $w'_\sigma : L_\sigma \to B_\tau$ by $w'_\sigma(X_i) = x_i$ for $i = 1, \ldots, n$. We get a morphism $g_\tau \circ w'_\sigma : L_\sigma \to B_\tau \to B$ such that $w_\sigma = g_\tau \circ w'_\sigma$ since $g_\tau \circ w'_\sigma(X_i) = g_\tau(x_i) = b_i = w_\sigma(X_i)$. Then the relation $g_\tau \circ w'_\sigma \circ u_\sigma = g_\tau \circ g_\tau, \sigma$ follows. Now we can use [11, O.6.3.11]. Since $A \to B_\sigma$ is of finite type, there is some $\nu \geq \tau$ such that $g_{\nu, \tau} \circ w'_\sigma \circ u_\sigma = g_{\nu, \tau} \circ g_{\nu, \sigma} = g_{\nu, \sigma}$. Define a map $f : A \to A$ by letting $f(\sigma) = \nu$. Set $v_\sigma = g_{\nu, \tau} \circ w'_\sigma$. Hence we have $v_\sigma \circ u_\sigma = g_{f(\sigma), \sigma}$ with $f(\sigma) > \sigma$ so that $B_\sigma \xrightarrow{u_\sigma} L_\sigma \xrightarrow{v_\sigma} B_{f(\sigma)} = B_{f(\sigma)} \xrightarrow{g_{f(\sigma), \sigma}} B$. We are now in position to apply [5, 1.6, Lemma 2], that is to say we can change the partial ordering on $A$ so that $B = \lim L_i$. To complete the proof, observe that a morphism of $A$-algebras $\alpha : A[Y_1, \ldots, Y_m] \to A[X_1, \ldots, X_n]$ is of finite presentation. Setting $\alpha(Y_j) = p_j(X_1, \ldots, X_n)$, it is easy to see that $\alpha$ can be identified to the canonical morphism $A[S_1, \ldots, S_m] \to A[S_1, \ldots, S_m; X_1, \ldots, X_n]/(S_1 - p_1, \ldots, S_m - p_m)$.

Corollary 1.4. The symmetric algebra $S_A(M)$ of an $A$-flat module $M$ is a-flat.

Proof. Observe that $M$ is a direct limit of free modules with finite rank [5]. Hence, $S_A(M)$ is a direct limit of polynomial algebras.

Now, we characterize a-flat morphisms in the same way as Lazard did for flat modules [5].

Theorem 1.5. Let $A \to B$ be a ring morphism. Then $A \to B$ is a-flat if and only if the following condition $(\mathcal{AF}^\prime)$ holds:

$(\mathcal{AF}^\prime)$ For every $A$-algebra $P$ of finite presentation and every surjective morphism of $A$-algebras $s : C \to B$, the natural map $\text{Hom}_{A\text{-alg}}(P, C) \to \text{Hom}_{A\text{-alg}}(P, B)$ is surjective.

Proof. Assume that $(\mathcal{AF})$ holds and let $L = A[X_i]_{i \in I} \to B$ be a surjective morphism. Then a morphism of $A$-algebras $P \to B$ can be factored $P \to L \to B$ and $(\mathcal{AF})$ is verified. Conversely, assume that $(\mathcal{AF})$ holds. Let $s : C \to B$ and $f : P \to B$ be morphisms of $A$-algebras where $P$ is of finite presentation and $s$ is surjective. Then $f$ can be factored $P \xrightarrow{g} A[X_1, \ldots, X_n] \xrightarrow{h} B$ so that $f = h \circ g$. If $n = 0$, using the structural morphism $k : A \to C$ and observing that $h$ is the structural morphism of $B$, we get $s \circ (k \circ g) = f$. If $n \neq 0$, letting $b_i = h(X_i)$ for $i = 1, \ldots, n$, we pick $c_i \in C$
such that $s(c_i) = b_i$. Hence a morphism of $A$-algebras $k : A[X_1, \ldots, X_n] \to C$ is defined by $k(X_i) = c_i$ so that $h = s \circ k$. It follows that $f = s \circ (k \circ g)$. Thus the proof is complete. 

\textbf{Definition 1.6.} Let $A \to B$ be a ring morphism and $n$ an integer.

1. A size $n$ (polynomial) relation in $B$ is a pair $(p, \beta) \in A[X_1, \ldots, X_n] \times B^n$ such that $p(\beta) = 0$.
2. A system of (polynomial) relations in $B$ is a set of finitely many size $n$ relations $(p_1, \beta_1), \ldots, (p_m, \beta_m)$ and $\sum_{j=1}^m A[X_1, \ldots, X_n] p_j$ is its associated ideal.
3. Let $s : C \to B$ be a morphism of $A$-algebras. We say that a system of relations $(p_1, \beta), \ldots, (p_m, \beta)$ in $B$ has a pullback in $C$ if there exists $\gamma \in C^n$ such that $s(\gamma) = \beta$ and $(p_1, \gamma), \ldots, (p_m, \gamma)$ is a system of relations in $C$.

\textbf{Theorem 1.7.} Let $B$ be an $A$-algebra, the following statements are equivalent:

1. $B$ is a-flat over $A$.
2. For every surjective morphism of $A$-algebras $s : C \to B$, each relation (respectively, each system of relations) in $B$ has a pullback in $C$ via $s$.
3. There is a surjective morphism $s : L \to B$ of $A$-algebras, where $L$ is a polynomial algebra such that each relation (respectively, each system of relations) in $B$ has a pullback in $L$ via $s$.
4. There is a surjective morphism $s : F \to A$ of $A$-algebras, where $F$ is an $a$-flat $A$-algebra such that each relation (respectively, each system of relations) in $B$ has a pullback in $F$ via $s$.
5. The following condition ($\mathcal{AF}'$) holds:

$(\mathcal{AF}'')$ If $b = (b_1, \ldots, b_n) \in B^n$ is a zero of $p \in A[X_1, \ldots, X_n]$, there exist $\beta \in B^n$ and $f_1, \ldots, f_n$ in a polynomial algebra $A[Y_1, \ldots, Y_m]$ such that $p(f_1, \ldots, f_n) = 0$ and $b_i = f_i(\beta)$ for $i = 1, \ldots, n$.

\textbf{Proof.} To see that (1) $\Rightarrow$ (2), observe that a system of relations in $B$ with associated ideal $I$ defines a morphism of $A$-algebras $A[X_1, \ldots, X_n]/I \to B$ and then use Theorem 1.5. Obviously, (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). We show that (4) $\Rightarrow$ (1), assuming only that each of the relations has a pullback in $F$. Consider a morphism $f : P \to B$ where $P = A[X_1, \ldots, X_n]/(p_1, \ldots, p_m)$. Set $f(x_i) = b_i$ where $x_i$ is the class of $X_i$ in $P$ and $\beta = (b_1, \ldots, b_n)$. We get a system of relations $(p_1, \beta), \ldots, (p_m, \beta)$. Each relation $(p_i, \beta)$ has a pullback $(p_i, \gamma_i)$ in $F$. We set $\gamma_i = (c_{i1}, \ldots, c_{in})$. Let $P'$ be $P \otimes \cdots \otimes P$ with $n$ factors and let $P' \to B$ be the canonical morphism. There is at least a factorization $P \to P' \to B$. Set $X_i = \{X_{i1}, \ldots, X_{in}\}$ where the $X_{ij}$ are indeterminates. Now $P'$ is isomorphic to $A[X_1, \ldots, X_n]/J$ where $J$ is the ideal generated by $\{p_i(X_j)\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Define a morphism $A[X_1, \ldots, X_n] \to F$ by $X_{ij} \mapsto c_{ij}$. We get a morphism $P' \to F$ such that $P' \to F \to B$ commutes. Thus we have a factorization $P \to F \to B$. Then use the remark in (1.1). Now, (5) is a translation of (3). \qed

Algebraically flat morphisms are closely related to algebraically pure morphisms (a-pure morphisms) considered by Popescu [26].
**Definition 1.8.** A morphism of $A$-algebras $f : B \to C$ is called $a$-pure if for every commutative diagram of $A$-algebras

\[
\begin{array}{ccc}
T & \xrightarrow{g} & P \\
\downarrow u & & \downarrow v \\
B & \xrightarrow{f} & C
\end{array}
\]

where $T$ is of finite type and $P$ of finite presentation, there exists a morphism of $A$-algebras $d : P \to B$ such that $u = d \circ g$.

Obviously, if $B \to C$ is $a$-pure as a morphism of $B$-algebras, then $B \to C$ is $a$-pure as a morphism of $A$-algebras.

Algebraically pure morphisms can be characterized by polynomial relations. They are stable under arbitrary base changes. An $a$-pure morphism of $A$-algebras is universally injective.

**Definition 1.9.** A morphism of $A$-algebras $f : B \to C$ defines $B$ as a retract of $C$ if there is some morphism of $A$-algebras $s : C \to B$ such that $s \circ f = \text{Id}_B$.

In this case, $C = f(B) \oplus J$ is a direct sum of $B$-modules where $J = \text{Ker}(s)$. If $u = f \circ s$, then $u : C \to C$ is an idempotent endomorphism of the $A$-algebra $C$ such that $\text{Im}(u) = f(B)$ and $\text{Ker}(u) = J$. Conversely, an idempotent endomorphism of $A$-algebras $u : C \to C$ gives an $A$-algebra $\text{Im}(u) = B$ which is a retract of $C$ [9].

An $A$-algebra $B$ is called retractable if $A$ is a retract of $B$ with respect to the structural morphism $A \to B$.

**Theorem 1.10** (Popescu [26]). Let $A \to B$ be a ring morphism.

1. $A \to B$ is $a$-pure if and only if there exists a direct system $\{P_\lambda\}_{\lambda \in \Lambda}$ of retractable $A$-algebras of finite presentation $P_\lambda$ such that $B = \varprojlim P_\lambda$.

2. If $A \to B$ is of finite presentation, then $A \to B$ is $a$-pure if and only if $B$ is retractable.

**Corollary 1.11.** An $a$-flat morphism is $a$-pure and faithfully flat.

**Proposition 1.12.** Let $f : A \to B$ be an $a$-flat morphism, then

\[
\text{U}(B) = f(\text{U}(A)) + \text{Nil}(B) \quad \text{and} \quad \text{Bool}(B) = f(\text{Bool}(A)).
\]

**Proof.** Let $b, b' \in B$ be such that $bb' = 1$. Let $g : L \to B$ be a surjective morphism where $L$ is a polynomial ring. The relation $(XY - 1, (b, b'))$ has a pullback in $L$ via $g$. Therefore, there is some polynomial $p = u + n$ where $u \in \text{U}(A)$ and $n \in \text{Nil}(A[X])$ are
such that \( g(p) = b \). For \( e \in \text{Bool}(B) \), the relation \((X^2 - X, (e, e))\) has a pullback in \( L \) via \( g \). There is an \( e \in \text{Bool}(A) \) such that \( g(e) = e \). \( \square \)

**Remark 1.13.** An a-pure morphism need not be flat. It is enough to consider a non-noetherian ring \( A \) such that \( A \to A[[X]] \) is not flat. Moreover, a faithfully flat a-pure morphism need not be a-flat. To see this, let \( K \) be an algebraically closed field. Then by [26, 1.8], a ring morphism \( K \to B \) is a-pure. Choose \( B = K[X] \)). In view of Proposition 1.12, we have \( U(B) = f(U(K)) = K \setminus \{0\} \) if \( B \) is a-flat which is absurd.

Now we study the stability of the class of a-flat morphisms with respect to the usual constructions of algebra. Clearly, an isomorphism is a-flat.

**Proposition 1.14.** Let \((A;F)\) be the class of a-flat morphisms. 
(1) If \( f : A \to B \) and \( g : B \to C \) are in \((A;F)\), then \( g \circ f \) lies in \((A;F)\). In particular, \( A \to B[X_1, \ldots, X_n] \) is a-flat when \( A \to B \) is a-flat.
(2) If \( A \to B \) lies in \((A;F)\), then \( A' \to B \otimes_A A' \) lies in \((A;F)\) for every ring morphism \( A \to A' \).
(3) If \( \{B_\lambda\}_{\lambda \in \Lambda} \) is a direct system of a-flat \( A \)-algebras with direct limit \( B \), then \( B \) is an a-flat \( A \)-algebra.
(4) Let \( f : A \to B \) be a ring morphism and \( g : B \to C \) an a-pure morphism of \( A \)-algebras such that \( g \circ f \) lies in \((A;F)\), then \( f : A \to B \) lies in \((A;F)\). The same conclusion is valid if \( B \to C \) is an a-pure morphism of \( B \)-algebras.
(5) If \( B \) is a retract of \( C \) and \( C \) is in \((A;F)\), so is \( B \).

**Proof.** Thanks to Theorem 1.3, (2) is obvious. We show (3). Let \( P \) be an \( A \)-algebra of finite presentation and \( P \to B \) a morphism. According to Lemma 1.2, there is some index \( \lambda \) such that \( P \to B \) can be factored \( P \to B_\lambda \to B \). Since \( A \to B_\lambda \) is a-flat, there is some polynomial \( A \)-algebra \( L \) such that \( P \to B_\lambda = P \to L \to B_\lambda \) whence a factorization \( P \to L \to B \). Therefore, \( A \to B \) is a-flat. Now, if \( A \to B \) is a-flat, so is \( A \to B \to B[X_1, \ldots, X_n] \) (write \( B \) as a direct limit of polynomial algebras \( B_\lambda \)). Then \( B[X_1, \ldots, X_n] \) is the direct limit of the polynomial \( A \)-algebras \( B_\lambda[X_1, \ldots, X_n] \) so that \( A \to B[X_1, \ldots, X_n] \) is a-flat. Next, we show (1). Assume that \( f : A \to B \) and \( g : B \to C \) are a-flat and consider a morphism of \( A \)-algebras \( h : P \to C \) where \( P \) is of finite presentation. Suppose that \( P = A[Y_1, \ldots, Y_n]/I \) where \( I = (p_1, \ldots, p_s) \) in \( A[Y_1, \ldots, Y_n] \). Set \( Q = B[Y_1, \ldots, Y_n]/J \) where \( J = IB[Y_1, \ldots, Y_n] \). Then \( Q \) is a \( B \)-algebra of finite presentation such that there is a factorization \( P \to Q \to C \) where \( Q \to C \) is a morphism of \( B \)-algebras. Therefore, \( Q \to C \) can be factored \( Q \to K \to C \) where \( K \) is a polynomial \( B \)-algebra. According to the beginning of the proof, \( A \to K \) is a-flat. Since \( P \to Q \to K \) is a morphism of \( A \)-algebras, there is a factorization \( P \to L \to K \) where \( L \) is a polynomial \( A \)-algebra. In short, we get a factorization \( P \to L \to C \) and \( A \to C \) is a-flat. Now, we prove (4). Assume that \( g \circ f \) is a-flat and that \( g \) is an a-pure morphism of \( A \)-algebras. Consider a morphism \( h : P \to B \) of \( A \)-algebras where \( P \) is an \( A \)-algebra of finite presentation. Then \( P \to B \to C \) is a morphism of \( A \)-algebras. By
a-flatness of $C$, there are a polynomial $A$-algebra $L$ and a commutative diagram

$$
\begin{array}{ccc}
P & \longrightarrow & L \\
\downarrow & & \downarrow \\
B & \longrightarrow & C
\end{array}
$$

By the definition of a-purity, we get a factorization $P \to L \to B = P \to B$. Hence, $B$ is a-flat. The last statement of (4) follows from Definition 1.8. The proof of (5) uses Definition 1.1. $\square$

**Lemma 1.15.** Let $A \to B$, $A \to C$ and $A \to A'$ be ring morphisms where $A'$ is a direct limit of $A$-algebras $\langle A_\lambda \rangle$.

1. Let $f : B \otimes_A A' \to C \otimes_A A'$ be a morphism of $A'$-algebras. If $A \to B$ is of finite presentation, there is some index $\mu$ and a direct system of morphisms of $A_\lambda$-algebras $\{f_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}_{\lambda \geq \mu}$ such that $f = \lim_{\lambda \to \mu} f_\lambda$.

2. Let $\{f_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}$ and $\{g_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}$ be direct systems of morphisms of $A_\lambda$-algebras with limits $f$ and $g$. If $f = g$ and $A \to B$ is of finite type, there is some index $\lambda$ such that $f_\lambda = g_\lambda$.

**Proof.** Use [11, O.6.3.10]. $\square$

**Theorem 1.16.** Let $A \to A'$ be an a-pure ring morphism and $P$ an $A$-algebra of finite presentation.

1. For every pair of morphisms $u : P \to C$, $v : B \to C$ of $A$-algebras, $v$ factorizes $u$ if and only if $v \otimes A'$ factorizes $u \otimes A'$.

2. For every pair of morphisms $v : B \to P$ and $u : B \to C$ of $A$-algebras where $A \to B$ is of finite type, $v$ factorizes $u$ if and only if $v \otimes A'$ factorizes $u \otimes A'$.

It follows that a-pure morphisms descend universally a-flatness and smoothness.

**Proof.** We show (1). Let $P \otimes_A A' \xrightarrow{f} B \otimes_A A' \xrightarrow{\otimes_A A'} C \otimes_A A'$ be a factorization in the category of $A'$-algebras such that $u \otimes A' = (v \otimes A') \circ f$. If $A \to A'$ has a retraction $A' \to A$, tensor with $\otimes_A A$ to get a factorization $P \to B \to C$. Now assume that $A \to A'$ is an arbitrary a-pure morphism. We reduce the proof to the previous case. We know that $A' = \varprojlim A_\lambda$ where $A \to A_\lambda$ is retractable (see (1.10)). In view of Lemma 1.15(1) $f = \lim_{\lambda \to \mu} f_\lambda$ (where $\lambda \geq \mu$). Then we have $v \otimes A' = \lim_{\lambda \to \mu} v \otimes A_\lambda$ and $u \otimes A' = \lim_{\lambda \to \mu} u \otimes A_\lambda$.

Set $k_\lambda = v \otimes A_\lambda \circ f_\lambda$. We get $\lim_{\lambda \to \mu} k_\lambda = u \otimes A'$. It follows from Lemma 1.15(2) that there is a factorization $P \otimes_A A_\lambda \to B \otimes_A A_\lambda \to C \otimes_A A_\lambda$ in the category of $A_\lambda$-algebras for some index $\lambda$. A similar proof gives (2). We examine the descent properties of an a-pure morphism $A \to A'$. Let $A \to B$ be a ring morphism such that $A' \to B \otimes_A A'$ is a-flat. Use the criterion of Theorem 1.5 and (1) to show that $A \to B$ is a-flat. Next assume that $A' \to B \otimes_A A'$ is smooth. Since a-purity implies purity, $A \to B$ is of finite presentation [25, 5.3]. Then it is enough to show that $\text{Hom}_{A_{\text{alg}}}(B, C) \to \text{Hom}_{A_{\text{alg}}}(B, C/I)$ is surjective for each $A$-algebra $C$ equipped with an ideal $I$ such that
$I^2 = 0$. This is true after tensoring with $A'$ and the result follows from (1) since $B$ is of finite presentation.

**Proposition 1.17.** Let $A \rightarrow B$ and $A \rightarrow C$ be ring morphisms. The $A$-algebra $B \otimes_A C$ is a-flat if and only if $A \rightarrow B$ and $A \rightarrow C$ are a-flat.

**Proof.** If $A \rightarrow B$ and $A \rightarrow C$ are a-flat, so is $A \rightarrow B \otimes_A C$ by Proposition 1.14 (1), (2). Now, the a-flatness of $A \rightarrow B \otimes_A C$ implies its a-purity by Corollary 1.11 so that $A \rightarrow B$ is a-pure by Popescu [26]. Then Theorem 1.16 shows that $A \rightarrow C$ is a-flat and so is $A \rightarrow B$. □

**Proposition 1.18.** Let $s_1, \ldots, s_n$ in a ring $A$ be such that $(s_1, \ldots, s_n) = A$. Then $A \rightarrow \prod_{i=1}^n A_{s_i} = A'$ is of finite presentation, faithfully flat and locally retractable. It follows that if $A \rightarrow B$ is a ring morphism such that $A' \rightarrow B \otimes_A A'$ is a-flat, then $A \rightarrow B$ is locally a-flat.

**Proof.** It is well known that $A \rightarrow A'$ is of finite presentation and faithfully flat. Now, let $P$ be a prime ideal of $A$. There is some $s_i$ such that $s_i \notin P$ so that $(A_{s_i})_P \simeq A_P$. It follows that $A_P$ is a retract of $A'_P$. Now, if $A' \rightarrow B \otimes_A A'$ is a-flat, so is $A'_P \rightarrow B'_P$. □

**Proposition 1.19.** Let $A \rightarrow B$ and $A' \rightarrow B'$ be two a-flat ring morphisms. Then $R = A \times A' \rightarrow B \times B' = S$ is a-flat.

**Proof.** There is an isomorphism of $R$-algebras for each integer $n$

\[ f : R[X_1, \ldots, X_n] \rightarrow A[X_1, \ldots, X_n] \times A'[X_1, \ldots, X_n] \]

where $f$ is defined by $f(\sum (a_i, a'_i)x^i) = (\sum a_iX^i, \sum a'_iX^i)$. Then use the criterion (\(\mathcal{AF}_n\)) of (1.7). □

**Remark 1.20.** If $A \rightarrow B$ is an a-flat morphism, then so is $A_S \rightarrow B_S$ for each multiplicative subset $S$ of $A$ (see Proposition 1.14 (2)). In particular, $A_P \rightarrow B_P$ is a-flat for each prime ideal $P$ of $A$.

1. We do not know whether a-flatness globalizes or not, although we suspect that the answer is negative. The following remarks show that a-flatness globalizes in some cases.

2. Let $A \rightarrow B$ be a ring morphism of finite presentation. If $A \rightarrow B$ is locally polynomial (for every prime ideal $P$ of $A$, there is some integer $n$ such that $B_P \simeq A_P[X_1, \ldots, X_n]$), then a result of Bass, Connell and Wright says that $B \simeq S_A(M)$ where $M$ is a finitely generated projective module [3, 4.4]. It follows that such an algebra is a-flat. Actually, $A \rightarrow B$ is algebraically projective (see the next section).

3. Let $P$ be the set of all prime integers. Let $S$ be a subset of $P$ and set $B(S) = \mathbb{Z}[p^{-1}X; p \in S]$. Then $B(P)$ is the direct limit of the $\mathbb{Z}$-algebras $B(S)$ where $S$ varies in the set of all finite subsets of $P$. It is straightforward to check that $\mathbb{Z} \rightarrow B(P)$ and $\mathbb{Z} \rightarrow B(S)$ are locally polynomial (for instance, see [10]). In view of (2), $B(S)$ is a-flat so that $B(P)$ is a-flat by Proposition 1.14 (3).
(4) Let \( A \) be a ring, \( I \) an ideal of \( A \) and \( R_A(I) = A[IX] \) its Rees algebra. The ideal \( I \) is of linear type if the canonical surjective map \( S_A(I) \to R_A(I) \) is an isomorphism \cite{13}. If \( I \) is of linear type and flat, then \( R_A(I) \) is an a-flat algebra. Notice that \( R_A(I) \) is a-flat only if \( I \) is flat. Indeed, \( I \) is a direct summand of \( R_A(I) \) and an a-flat algebra is flat.

(5) In particular, assume that \( A \) is an integral domain and \( I \) is an invertible ideal whence projective, then \( I \) is of linear type \cite[IV, 2, Theorem 2']{23}. Moreover, \( A \to R_A(I) \) is a-flat and locally polynomial since \( I \) is locally principal. Now, if \( I \) is a directed union of invertible ideals \( I_\lambda \), then \( R_A(I) = \lim\limits_{\longrightarrow} R_A(I_\lambda) \) shows that \( R_A(I) \) is a-flat. If \( A \) is a Prüfer domain, each of its nonzero finitely generated ideals is invertible. Thus a Rees algebra over \( A \) is a-flat.

(6) Let \( A \) be a noetherian ring and \( I \) an ideal of \( A \). Set \( \text{gr}_I(A) = \bigoplus_a I^a/I^{a+1} \). Then \( I \) is of linear type if and only if \( S_A(I/I^2) \cong \text{gr}_I(A) \) \cite{13, 3.1}. Therefore, if \( I \) is of linear type and \( I/I^2 \) is \((A/I)\)-flat, \( \text{gr}_I(A) \) is an a-flat \((A/I)\)-algebra. Hence, if the ideal \( I \) is completely secant (see \cite[5.2, Theorem 1]{8}), \( \text{gr}_I(A) \) is an a-flat \((A/I)\)-algebra (actually, an a-projective algebra since \( I/I^2 \) is \((A/I)\)-projective).

**Proposition 1.21.** Let \( A \to B \) be a flat ring morphism. If \( B \) is a direct limit of a system of \( A \)-algebras \( \{B_i\} \) such that each \( B_i \cong A[X_1, \ldots, X_n]/(f_1, \ldots, f_p) \) and each \( f_i \) is a linear homogeneous polynomial, then \( A \to B \) is a-flat.

**Proof.** Denote by \( \rho_i : B_i \to B \) the canonical morphisms. We can consider that each \( B_i = S_A(M_i) \) where \( M_i \) is an \( A \)-module of finite presentation. By flatness of \( A \to B \) and Lazard’s theorem, we get a factorization \( M_i \to F_i \to B \) where \( F_i \) is free with finite rank. Taking symmetric algebras, we get a factorization \( B_i \to L_i \to B \) where \( L_i \) is a-free. An appeal to Lemma 1.2 shows that \( A \to B \) is a-flat. \( \square \)

An a-flat \( A \)-algebra \( B \) is a direct limit of polynomial algebras \( L_\lambda \) with finite transcendence degree over \( A \). We examine the situation when the set of integers \( \text{tr.deg}_A(L_\lambda) \) has an upper bound.

**Definition 1.22.** Let \( A \to B \) be an a-flat morphism. We say that the flat rank \( f\text{-rk}(B) \) of \( B \) over \( A \) is \( r \in \mathbb{N} \) if \( r \) is the least integer such that \( B = \lim_{\longrightarrow} L_\lambda, \ L_\lambda \in \mathcal{P}_A \) and \( \text{tr.deg}_A(L_\lambda) \leq r \) for each \( \lambda \).

**Proposition 1.23.** Let \( A \to B \) be an a-flat morphism. The following statements are equivalent:

1. \( f\text{-rk}(B) \leq r \).
2. \( B \) is a direct limit of polynomial \( A \)-algebras with transcendence degree \( r \).
3. Each morphism of \( A \)-algebras \( P \to B \) where \( P \) is of finite presentation can be factored \( P \to L \to B \) where \( \text{tr.deg}_A(L) = r \) and \( L \in \mathcal{P}_A \).
4. Each finitely generated \( A \)-subalgebra of \( B \) is contained in an \( A \)-subalgebra of \( B \) generated by \( r \) elements.
Proof. Assume that (1) holds and consider $C = \langle b_1, \ldots, b_n \rangle \subset B$. Denote the canonical morphisms by $\rho_\mu : L_\mu \to B$. There is an index $\mu$ such that $C \subset \rho_\mu(L_\mu)$ and $\rho_\mu(L_\mu)$ can be generated by $r$ elements. Hence, (1) implies (4). Assuming that (4) is verified, we show (3). Consider a morphism of $A$-algebras $P \to B$ where $P$ is of finite presentation. It can be factored $P \to L \to B$ where $L \in \mathcal{P}_A$. Let $C$ be the image of $L$ in $B$, there is an $A$-subalgebra $C' = \langle b_1, \ldots, b_r \rangle$ of $B$ which contains $C$. Set $L' = A[X_1, \ldots, X_r]$, there is a surjective morphism $L \to L' \to C'$. Since $L$ is a free object, $L \to C \to C'$ can be factored $L \to L' \to C'$. Therefore, we get a factorization $P \to L' \to B$ of $P \to B$. Now (3) implies (2); it is enough to mimic the proof of Theorem 1.3. Obviously, (2) implies (1). □

Remark 1.24. Assume that $A \to B$ is a flat of finite type. The flat rank of $B$ is the least number $\lambda(B)$ of elements required to generate $B$ over $A$.

If $A$ is a field, $B$ is an integral domain since a direct limit of integral domains. Then $\text{tr.deg}_A(B)$ is defined and is $\leq \lambda(B)$. Therefore, the flat rank and the transcendence degree of $B$ are equal if and only if $B \in \mathcal{P}_A$.

We intend to give some homological properties of a-flat morphisms. The following definition may be found in McDonald’s book [22, p. 328]. For the definition and properties of stably free modules, see for instance Lam’s book [16, I.4].

Definition 1.25. A ring $A$ is called projectively trivial if each idempotent matrix over $A$ is diagonalizable under a similarity transform.

According to [22, IV.49], if $A$ is a connected projectively trivial ring, each of its finitely generated projective modules is free. The converse can be easily shown.

Definition 1.26. A ring $A$ is called PPF if for each integer $n$, every finitely generated projective $A[X_1, \ldots, X_n]$-module is free.

Hence, if $A$ is connected and PPF, $A[X_1, \ldots, X_n]$ is projectively trivial. If $A$ is a principal ideal domain (or a Bézout domain such that prime ideals have finite heights), the Quillen–Suslin’s theorem states that $A$ is a PPF ring ([16,19]).

Definition 1.27. Let $A \to B$ be a ring morphism and $P$ a property of rings. We say that $A \to B$ is a universal $P$-morphism if $B \otimes_A A'$ has $P$ for any base change $A \to A'$ where $A'$ has $P$.

An a-flat morphism $A \to B$ is a universal $P$-morphism for many properties $P$ like reduced, (integral) domain since $B$ is a direct limit of polynomial algebras over $A$. However, this definition is not identical to the following usual definition.

Definition 1.28. Let $P$ be a property of rings. Then $A \to B$ is called a $P$-morphism if $A \to B$ is flat and for each prime ideal $P$ of $A$, the ring $B \otimes_A K$ has $P$ for every finite field extension $k(P) \to K$. 

Therefore, if $A \to B$ is a flat universal $P$-morphism, $A \to B$ is a $P$-morphism. Actually, the flatness condition is verified in many cases by universal $P$-morphisms. Recall that a universal reduced morphism $A \to B$ is flat if $A$ is reduced [18, II, Proposition 2].

**Theorem 1.29.** Let $A \to B$ be an $a$-flat morphism.

(1) $A \to B$ is a universal connected PPF morphism.

(2) If $A$ is a connected PPF ring, so is $B$. Hence, every stably free projective $B$-module is free so that $B$ is a Hermite ring.

**Proof.** We can assume that $A$ is a connected PPF ring. First observe that $B$ is connected by Proposition 1.12. According to Proposition 1.14 (1), $A \to B[X_1, \ldots, X_n]$ is $a$-flat. Thus it is enough to show that a finitely generated projective $B$-module is free. By virtue of [22, IV.G.1], $B$ is projectively trivial because $B$ is a direct limit of projectively trivial rings. Use Definition 1.25 to complete. The statement (2) follows since a non finitely generated stably free module is free. \(\square\)

**Lemma 1.30.** Let $A$ be a PPF regular integral domain. Then $A$ is a unique factorization domain.

**Proof.** Consider a nonzero divisorial ideal $I$ of $A$. Since $I$ is finitely generated over $A$ and $A$ is regular, its projective dimension is finite. Thus, according to [5, X.8.1, Proposition 2], $I$ has a finite free resolution of finite length by Definition 1.26. It follows from [4, 4.7, Corollary 3] that $A$ is a unique factorization domain. \(\square\)

For simplicity’s sake, we give [32, 1.1] as a reference for the following Popescu–Spivakovsky’s theorem or Spivakovsky’s paper for a more recent treatment [31].

**Theorem 1.31.** Let $A \to B$ be a ring morphism between Noetherian rings. The following statements are equivalent:

(1) $A \to B$ is a regular morphism.

(2) $B$ is a direct limit of smooth $A$-algebras (of finite type).

(3) Every morphism of $A$-algebras $P \to B$ where $P$ is an $A$-algebra of finite presentation can be factored $P \to S \to B$ where $S$ is a smooth $A$-algebra.

**Corollary 1.32.** An $a$-flat morphism $A \to B$ between Noetherian rings is a regular morphism.

Artamonov showed the following result for algebraically projective algebras of finite type over a field [1, Proposition 7]

**Theorem 1.33.** Let $A \to B$ be an $a$-flat morphism between Noetherian rings. Then $A \to B$ is a (regular) factorial morphism. If in addition, $A \to B$ is essentially of finite type, then $K \otimes_A B$ is a regular UFD for every ring morphism $A \to K$ where $K$ is a field.
Proof. Let \( P \) be a prime ideal of \( A \) and \( k(P) \to K \) a finite extension of fields, then \( K \otimes_A B = F \) is a regular integral domain since \( K \to F \) is a universal integral domain morphism (see Definition 1.27). Then use Lemma 1.30. Now, if \( A \to B \) is essentially of finite type, \( A \to B \) is a universal Noetherian morphism and the proof is complete.

We have just seen that differential properties of a-flat morphisms are involved. If \( A \to B \) is a ring morphism, we denote by \( \Omega_A(B) \) the \( B \)-module of Kähler differentials of \( B \) over \( A \).

**Proposition 1.34.** Let \( A \to B \) be an a-flat morphism. Then \( \Omega_A(B) \) is a flat \( B \)-module.

**Proof.** Since \( B \) is the direct limit of polynomial \( A \)-algebras \( L_i \) in finitely many indeterminates, the conclusion follows from \( \Omega_A(B) = \lim_{\longrightarrow} (\Omega_A(L_i) \otimes_{L_i} B) \) and \( \Omega_A(L_i) \) is a free \( L_i \)-module with finite rank.

2. Algebraically projective morphisms

**Definition 2.1.** An \( A \)-algebra \( B \) is called algebraically projective (a-projective), if the natural map \( \text{Hom}_{A \text{-alg}}(B, C) \to \text{Hom}_{A \text{-alg}}(B, D) \) is surjective for every surjective morphism of \( A \)-algebras \( C \to D \).

The symmetric \( A \)-algebra \( S_A(P) \) of a projective \( A \)-module \( P \) is a-projective [9].

In the literature, a-projective algebras are called projective algebras or weakly projective algebras. The word projective has many meanings. So we have preferred to introduce another name. In this paper, a projective algebra is an \( A \)-algebra \( B \) such that the \( A \)-module \( B \) is projective. Retracts of algebras are defined in Definition 1.9.

**Proposition 2.2.** Let \( B \) be an \( A \)-algebra. The following statements are equivalent:

1. \( B \) is a-projective.
2. \( B \) is a retract of a polynomial algebra.
3. For every \( A \)-algebra \( R \) and every surjective morphism \( C \to B \), the natural map \( \text{Hom}_{A \text{-alg}}(R, C) \to \text{Hom}_{A \text{-alg}}(R, B) \) is surjective.

Therefore, an a-projective morphism is universally a-projective, projective and faithfully flat.

**Proof.** (1) \( \iff \) (2) is well known (for instance, see [9]). Assume that (1) holds and consider morphisms \( R \to B \) and \( C \to B \) where \( C \to B \) is surjective. Then \( \text{Id}_B \) can be factored \( B \to C \to B \) and (3) is proved. Clearly, (3) implies that \( B \) is a retract of a polynomial ring and (2) is shown. Assume that \( f : B \to L \) defines \( B \) as a retract of a polynomial ring \( L \). Then by (1.9), there is a direct sum \( L = f(B) \oplus J \) of \( B \)-modules whence a direct sum of \( A \)-modules. Since \( L \) is free over \( A \), \( B \) is projective over \( A \).

**Lemma 2.3.** Let \( C \) be an \( A \)-algebra of finite presentation and \( B \) a retract of \( C \). Then \( B \) is of finite presentation. Hence, an a-projective algebra of finite type is of finite presentation.
Proof. Let \( f : B \rightarrow C \) and \( s : C \rightarrow B \) be the morphisms defining \( B \) as a retract. We need only to show that \( J = \ker(s) \) is of finite type since finite presentation is stable under composition. Let \( \{c_1, \ldots, c_n\} \) be a system of generators of the algebra \( C \). From \( C = f(B) \oplus J \) we deduce \( c_i = b_i + x_i \) where \( b_i \in B \) and \( x_i \in J \) so that \( \sum C x_i \subset J \). Now let \( x = p(x_1, \ldots, x_n) \in J \) where \( p(X_1, \ldots, X_n) \in B[X_1, \ldots, X_n] \). Observe that \( p(0, \ldots, 0) = 0 \) because \( J \cap B = \{0\} \). It follows that \( J \subset \sum C x_i \). Now, an a-projective \( A \)-algebra \( B \) of finite type is a retract of a polynomial ring \( L \in \mathcal{P}_A \) over \( A \). 

The following result gives a partial answer to the question: is a projective algebra of finite type a polynomial algebra?

**Theorem 2.4.** Let \( A \rightarrow B \) be a ring morphism.

1. If \( B \) is a-projective, then \( B \) is a-flat.
2. If \( A \rightarrow B \) is of finite presentation and a-flat, \( B \) is a-projective. In particular, if \( A \) is Noetherian or an integral domain and \( A \rightarrow B \) is of finite type and a-flat, then \( B \) is a-projective.

Hence, if \( A \rightarrow B \) is of finite presentation, \( B \) is a-projective if and only if \( B \) is a-flat. In this case, \( B \) is a direct limit of polynomial algebras over \( A \) with transcendence degree \( \text{f-rk}(B) = \lambda(B) \).

**Proof.** To show (1), use Proposition 2.2 (3) and the a-flatness definition. If \( A \rightarrow B \) is of finite presentation and a-flat, \( \text{Id}_B \) can be factored \( B \rightarrow L \rightarrow B \) where \( L \) is a polynomial algebra. Hence \( B \) is a-projective. Now, if \( A \rightarrow B \) is of finite type and flat and \( A \) is Noetherian or an integral domain, \( A \rightarrow B \) is of finite presentation [12, 1.3.4.7]. For the last statement, see Proposition 1.23 and Remark 1.24.

Ohm and Rush defined content modules [24]. A projective module is a content module. Moreover, Rush introduced weak content algebras [28]. We will use the following characterization. If \( B \) is an \( A \)-algebra such that \( B \) is a content module, \( B \) is weak content if and only if \( PB = B \) or \( PB \) is a prime ideal for each prime ideal \( P \) of \( A \) [28, 1.2]. For instance, a polynomial algebra is weak content.

**Proposition 2.5.** Let \( A \rightarrow B \) be a ring morphism.

1. If \( A \rightarrow B \) is a weak content injective morphism, every finitely generated flat module over \( A \) is projective if and only if every finitely generated flat module over \( B \) is projective.
2. If \( A \rightarrow B \) is a-projective, then \( A \rightarrow B \) is weak content and injective.

**Proof.** The first result is quoted in [28, Note, p. 333] while the lacking proof is a consequence of [27]. Assume that \( B \) is a-projective. Then \( B \) is a content module over \( A \) because \( B \) is projective over \( A \) (see Proposition 2.2). Since \( B \) is a-flat, \( A \rightarrow B \) is a universal domain morphism. It follows that \( PB \) is a prime ideal for each prime ideal \( P \) of \( A \). Therefore, \( A \rightarrow B \) is weak content.

A ring \( A \) is called FGFP if each of its finitely generated flat modules is projective. Jöndrup showed that the FGFP property is stable under flat and finite morphisms [14].
An integral domain or a semilocal ring is FGFP. Moreover, $A$ is FGFP if and only if $A[\mathfrak{X}]$ is FGFP [14].

**Theorem 2.6.** Let $A \rightarrow B$ be an a-projective morphism. If $A$ is a PPF connected FGFP ring (for instance a PID), then every finitely generated flat module over $B[\mathfrak{X}_1, \ldots, \mathfrak{X}_n]$ is free.

**Proof.** Use Proposition 2.5 and Theorem 1.29 since a-projective implies a-flat. □

We look at the differential properties of a-projective morphisms. In order to avoid many references, we use the definitions and results of [8] although they may be found elsewhere.

**Proposition 2.7.** Let $A \rightarrow B$ be an a-projective morphism which is a retract of a polynomial algebra $L$. Denote by $J$ the kernel of $L \rightarrow B$.

1. $A \rightarrow B$ is formally smooth.
2. There is a left-invertible morphism of $B$-modules $\Omega_A(B) \rightarrow \Omega_A(L) \otimes_L B$. Hence, $\Omega_A(B)$ is a projective $B$-module.
3. There is an isomorphism of $B$-modules $J/ J^2 \simeq \Omega_B(L) \otimes_L B$.

**Proof.** Let $C$ be an $A$-algebra and $I$ an ideal of $C$ such that $I^2 = 0$. The natural map $\text{Hom}_{A\text{-alg}}(B, C) \rightarrow \text{Hom}_{A\text{-alg}}(B, C/I)$ is surjective. Hence, $A \rightarrow B$ is formally smooth [8, X.7.2, Definition 1]. Then (2) can be shown in the same way as in [2, 6.5]. Consider the factorization $B \rightarrow L \rightarrow B$ of $\text{Id}_B$. Since $\text{Id}_B$ is formally smooth, there is an exact sequence of $B$-modules $0 \rightarrow J/J^2 \rightarrow \Omega_B(L) \otimes_L B \rightarrow \Omega_B(B) \rightarrow 0$ [8, X.7.2, Remarques]. Since $\Omega_B(B)$ is zero, (3) follows. □

In the following, we consider only a-projective morphisms of finite type, hence of finite presentation by Lemma 2.3.

**Theorem 2.8.** Let $A \rightarrow B$ be an a-projective morphism of finite type.

1. $A \rightarrow B$ is a projective smooth morphism.
2. $A \rightarrow B$ is a universal regular morphism.
3. $K \otimes_A B$ is a regular unique factorization domain for every ring morphism $A \rightarrow K$ where $K$ is a field.
4. $R \otimes_A B$ is a connected PPF ring for every ring morphism $A \rightarrow R$ where $R$ is a connected PPF ring. In this case, $\Omega_R(R \otimes_A B)$ is a free $R \otimes_A B$-module with finite rank.

**Proof.** In view of Proposition 2.7, $A \rightarrow B$ is formally smooth and is of finite presentation. Thus, $A \rightarrow B$ is smooth. Now, if $A$ is Noetherian, $A \rightarrow B$ is a universal regular morphism by [8, X.7.10, Theorem 4]. We can reduce to the Noetherian case by virtue of Proposition 2.9 (4). Hence (1) and (2) are proved. Now (3) follows from Theorem 1.33 since an a-projective morphism is a-flat. The first part of (4) is a consequence of Theorem 1.29. Set $S = R \otimes_A B$, then $R \rightarrow S$ is of finite type so that $\Omega_R(S)$ is a finitely
generated S-module. By Proposition 2.7, \( \Omega_R(S) \) is a projective S-module and hence is free with finite rank according to the first part of (4). □

The following result is well known (except (4)) and defines representations of a-projective algebras of finite type [9].

**Proposition 2.9.** Let \( A \to B \) be an a-projective morphism of finite type and \( L = A[X_1, \ldots, X_n] \to B \) defining \( B \) as a retract of \( L \). Let \( J \) be the kernel of \( L \to B \) and \( u : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n] \) the associated idempotent endomorphism of \( A \)-algebras. Then \( \{ f_i = u(X_i) \mid i = 1, \ldots, n \} \) verifies:

1. \( f_i(f_1, \ldots, f_n) = f_i \) for \( i = 1, \ldots, n \).
2. \( J = \ker(u) = (X_1 - f_1, \ldots, X_n - f_n) \).
3. \( B \simeq \operatorname{Im}(u) = A[f_1, \ldots, f_n] \simeq A[X_1, \ldots, X_n]/J. \)

Conversely, a sequence of polynomials \( f_1, \ldots, f_n \in A[X_1, \ldots, X_n] \) verifying (1) defines an a-projective algebra \( A \to A[X_1, \ldots, X_n]/(X_1 - f_1, \ldots, X_n - f_n) \).

4. There exist a Noetherian ring \( R \), an a-projective ring morphism of finite type \( R \to S \) and a ring morphism \( R \to A \) such that \( B = A \otimes_R S \).

**Proof.** To show (4), consider the set \( G \) of all the coefficients of \( f_i \). It is enough to take \( \mathbb{Z}[G] \) that \( R \simeq A \) and \( S = R[X_1, \ldots, X_n]/(X_1 - f_1, \ldots, X_n - f_n) \). □

A sequence \( \{ f_1, \ldots, f_n \} \) is called a representation of \( B \) and the ideal of the representation is \( J = (X_1 - f_1, \ldots, X_n - f_n) \). A representation \( \{ f_1, \ldots, f_n \} \) is called standard if \( f_i(0, \ldots, 0) = 0 \) for each \( i \).

Thanks to the following results, we can get more interesting representations.

**Lemma 2.10.** Let \( A \to B \) be an a-projective morphism of finite type, \( u \) an associated idempotent endomorphism defining a representation \( \{ f_1, \ldots, f_n \} \) of \( B \). Let \( \phi \) be an \( A \)-automorphism of the algebra \( A[X_1, \ldots, X_n] \) and set \( v = \phi \circ u \circ \phi^{-1} \).

1. \( v \) is an idempotent endomorphism of the \( A \)-algebra \( A[X_1, \ldots, X_n] \) defining a representation \( \{ g_1, \ldots, g_n \} \) of \( B \).
2. \( (\phi(X_1 - f_1), \ldots, \phi(X_n - f_n)) = (X_1 - g_1, \ldots, X_n - g_n) \).

**Proof.** Obviously, \( \phi \) induces an isomorphism of \( A \)-algebras \( \operatorname{Im}(u) \to \operatorname{Im}(v) \) and we have \( \phi(\ker(u)) = \ker(v) \). □

**Proposition 2.11.** Let \( A \to B \) be an a-projective morphism of finite type which is not a polynomial algebra.

1. \( B \) has a standard representation \( \{ g_1, \ldots, g_n \} \subset A[X_1, \ldots, X_n] \) such that its ideal contains a polynomial of the form \( aX_n^s + p_{s-1}X_n^{s-1} + \cdots + p_1X_n \) where \( a \in A \) is nonzero, \( s \neq 0 \) is an integer and \( p_i \in A[X_1, \ldots, X_{n-1}] \).
2. Moreover, if \( A \) is Noetherian and \( B \) has a representation ideal \( J \) such that \( \operatorname{ht}(J) > \dim(A) \), we can assume that \( a = 1 \).
Proof. Let \( \{f_1, \ldots, f_n\} \) be a representation of \( B \). We can assume that \( f_n \notin A \). Let \( a_i \) be the constant term of \( f_i \) and define \( \varphi \) by \( \varphi(x_i) = x_i + a_i \). We get \( g_i = v(x_i) = \varphi(u(x_1 - a_1)) = f_i(x_1 + a_1, \ldots, x_n + a_n) - a_i \). Arguing as in [9, 3.2], we find that the constant term of \( g_i \) is zero. Thus we can assume that the representation is standard. Now, define \( \psi \) by \( \psi(x_1) = x_1 \) and \( \psi(x_i) = x_i + X_n^i \). The constant term of each polynomial \( v(x_i) \) is still zero. Following Nagata’s proof of the Noether normalization Lemma, we can choose integers \( n_i \) such that \( \psi(x_i - f_n) \) has the required form. Thus, (1) is shown. Now, (2) is an immediate consequence of a Suslin’s result involving the same automorphism [20, 6.1.5].

Proposition 2.12. Let \( A \) be a UFD and \( Q \) a prime ideal of \( A[T] \) such that \( Q \cap A = P \) and \( P[T] \neq Q \). There is some irreducible polynomial \( f(T) \in A[T] \) such that \( Q = P[T] + A[T]f(T) \).

Proof. Set \( B = A/P \) and consider the prime ideal \( Q' \) of \( B[T] \) lying over \( Q \) so that \( Q' \neq 0 \) and \( Q' \cap B = 0 \). Let \( g(T) \) be a polynomial of least positive degree in \( Q' \) (hence, \( g(T) \in Q' \setminus P[T] \)). Pick an irreducible polynomial \( f(T) \) in \( Q' \setminus P[T] \) dividing \( g(T) \). The content ideal of \( f(T) \) is \( A \) and thus the content ideal of \( f(T) \) is \( B \). Then \( f(T) \) cannot lie in \( B \) and we can write \( g(T) = f(T)h(T) \). In this case, the degree of \( h(T) \) is zero (if not, we get \( 0 < d'(f(T)) < d'(g(T)) \), contradicting the definition of \( g(T) \) since \( f(T) \notin Q' \)). It follows that \( h(T) = \bar{a} \in B \). Therefore, \( f(T) \) is a polynomial of least positive degree in \( Q' \) with content ideal \( B \). A result of Sharma shows that \( Q' = (\bar{f}(T)) \) [30, Corollary 3] and the proof is complete.

When \( M \) is a finitely generated \( A \)-module, we denote by \( \mu(M) \) the minimal number of generators of \( M \).

Proposition 2.13. Let \( A \) be a Noetherian UFD and \( Q \) a prime ideal of \( R = A[T] \) lying over \( P \) in \( A \). Assume that \( Q \) contains a monic polynomial, \( Q/Q^2 \) is \( R/Q \)-free and that stably free \( R/Q \)-modules are free, then \( \mu(Q) = \mu(Q/Q^2) \).

Proof. By Proposition 2.12, we have \( Q = (P, f(T)) \) since \( Q \) contains a monic polynomial. A result by Mandal and Roy gives the conclusion [21, 3.6].

Theorem 2.14. Let \( A \to B \) be an \( a \)-projective morphism of finite type with representation ideal \( J \) in \( L = A[X_1, \ldots, X_n] \).

1. If \( A \) is a connected PPF ring, then \( J/J^2 \) and \( \Omega_A(B) \) are free \( B \)-modules such that \( n = \text{rk}_B(\Omega_A(B)) + \text{rk}_B(J/J^2) \).

2. If \( A \) is a field, \( J \) is a completely secant prime ideal so that \( S_A(J/J^2) \simeq \text{gr}_J(L) \). Moreover, \( J/J^2 \) and \( \Omega_A(B) \) are free \( B \)-modules such that \( \text{rk}_B(\Omega_A(B)) = \dim(B) \) and \( \text{ht}(J) = \text{rk}_B(J/J^2) = n - \dim(B) \).

3. If \( A \) is a connected Noetherian PPF ring, \( \dim(B) \leq \dim(A) + \text{rk}_B(\Omega_A(B)) \).

4. If \( A \) is an affine PPF integral domain over a field \( K \), so is the ring \( B \) and \( \text{rk}_B(J/J^2) \leq \text{ht}(J) \leq \mu(J) \) holds for the prime ideal \( J \).
An example of affine PPF integral domain $A$ over a field is given by an a-projective algebra of finite type over a field.

**Proof.** Let $A$ be a connected PPF ring. Since $A \to B$ is smooth by Theorem 2.8, there is an isomorphism of $B$-modules $\Omega_A(L) \otimes_L B \simeq J/J^2 \oplus \Omega_A(B)$ induced by the split exact sequence $0 \to J/J^2 \to \Omega_A(L) \otimes_L B \to \Omega_A(B) \to 0$ [8, X.7.2, Remark 1]. Observe that $\Omega_A(L) \otimes_L B$ is a free $B$-module with rank $n$. Therefore, $J/J^2$ and $\Omega_A(B)$ are finitely generated projective $B$-modules. These $B$-modules are free with finite rank by Theorem 2.15. Moreover, $\text{rk}(\Omega_A(B)) = \text{rk}_B(\Omega_A(B)) + \text{rk}_B(J/J^2)$. If $A$ is a field, $B$ is an integral domain so that $J$ is a prime ideal. From Theorem 2.8 (2), we deduce that $B$ is a regular ring. Now $L$ is a regular ring as well as $B$. It follows that $J$ is completely secant by [8, X.5.3, Proposition 2] and $S_d(J/J^2) \simeq \text{gr}_J(L)$ is a consequence of [8, X.5.2, Theorem 1]. To complete the proof of (2), it is enough to show that $\text{rk}_B(\Omega_A(B)) = \dim(B)$. If $M$ is a maximal ideal of the affine integral domain $B$ with quotient field $K$, then $\dim(B_M) = \dim(B) = \text{tr.deg}_K(M)$ (the quotient field of $B_M$ is $K$) [7, VIII.2.4, Theorem 3]. From $\Omega_A(B_M) \cong \Omega_A(B)_M$ and [8, X.6.5, Theorem 1], we deduce that $\text{rk}_B(\Omega_A(B)) = \text{rk}_{B_M}(\Omega_A(B)_M) = \text{tr.deg}_K(M)$ because $A \to B_M$ is a regular morphism [8, X.6.4, Proposition 6]. Now assume that $A$ is a connected PPF Noetherian ring. In view of (1), the $B$-module $\Omega_A(B)$ is free with finite rank. Let $P$ be a prime ideal of $A$ and set $F(P) = B \otimes_A k(P)$. Then $\text{rk}_{F(P)}(\Omega_{k(P)}(F(P))) = \text{rk}_B(\Omega_A(B))$ follows from $\Omega_A(B) \otimes_B F(P) \cong \Omega_{k(P)}(F(P))$. According to (2), we get $\text{rk}_{F(P)}(\Omega_{k(P)}(F(P))) = \dim(F(P))$ since $k(P) \to F(P)$ is a-projective so that $\dim(F(P)) = \text{rk}_B(\Omega_A(B))$. It follows that $\dim(B) \leq \dim(A) + \text{rk}_B(\Omega_A(B))$ by [7, VIII.3.4, Corollary 2]. Thus (3) is shown. If $A$ is an affine PPF integral domain, so are $L$ and $B$ because $K \to L$ is of finite type as well as $K \to B$. Since $L$ is an affine integral domain, we get from (3) that $\dim(B) - \dim(A) = n - \text{ht}(J) \leq \text{rk}_B(\Omega_A(B)) = n - \text{rk}_B(J/J^2)$. Therefore, (4) is proved since $\text{ht}(J) \leq \mu(J)$ holds for an arbitrary Noetherian ring. \[\square\]

**Theorem 2.15.** Let $K$ be a PPF affine regular integral domain (for instance, an a-projective algebra of finite type over a field) and $K \to B$ an a-projective morphism of finite type which is not a polynomial algebra. Then each representation ideal $J$ of $B$ such that $\text{ht}(J) > \dim(K)$ is a complete intersection and $\dim(B) = \text{rk}_B(\Omega_K(B))$.

In particular, if $K$ is a field then each representation ideal of $B$ is a complete intersection ideal.

**Proof.** Let $\{f_1, \ldots, f_n\} \subset K[X_1, \ldots, X_n]$ be a representation of $B$ and denote by $J$ the associated representation ideal. First assume that $n = 1$. In this case $f_1(X_1) = a \in K$ or $f_1(X_1) = X$ [9, 3.4] which yields $J = (X - a)$ or $J = 0$. Now assume that $n > 1$. We set $A = K[X_1, \ldots, X_{n-1}]$, $X_n = T$ so that $B = A[T]/J$ where $A$ is a Noetherian UFD since $K$ is a UFD by (1.30). According to (2.11)(2), we can assume that $J$ contains a monic polynomial of $A[T]$. Hence, $\mu(J) = \mu(J/J^2)$ follows from (2.13). Now, $\text{rk}_B(J/J^2) \leq \text{ht}(J) \leq \mu(J)$ is a consequence of (2.14)(4) and then $\mu(J/J^2) = \text{rk}_B(J/J^2)$ implies $\text{ht}(J) = \mu(J)$. It follows that $J$ is a complete intersection ideal. Moreover, $B$ is an affine integral domain and we have $n = \text{ht}(J) + \text{rk}_B(\Omega_K(B))$ so that $\dim(B) = \text{rk}_B(\Omega_K(B))$. \[\square\]
Definition 2.16. We call a ring \( B \) a global complete intersection ring if \( B \cong A[X_1, \ldots, X_n]/J \) where \( A \) is a regular ring and \( J \) is a complete intersection ideal (generated by a regular sequence).

It follows that \( A[X_1, \ldots, X_n] \) is a global complete intersection ring when \( A \) is a regular ring. We do not know whether the previous definition is independent of the presentation of the ring \( B \) although this is known for local rings.

The adjective global is added because of possible confusions with complete intersection rings (rings which are locally complete intersection).

Corollary 2.17. Let \( A \to B \) be an a-projective morphism of finite type. Then \( A \to B \) is a global complete intersection morphism.

Let \( A \to B \) be an a-projective morphism of finite type. In view of Proposition 2.11 (1), \( A \to B \) has a retract \( B \to A \) with kernel \( I = (f_1, \ldots, f_n) \). Tronin used this fact to exhibit some morphisms [33]. Consider the ideal \( M = (X_1, \ldots, X_n) \) of \( L = A[X_1, \ldots, X_n] \). There is a factorization \( B = A \oplus I \to L = A \oplus M \to B = A \oplus I \) of \( \text{Id}_B \) where \( \phi : B \to L \) is the canonical injection and \( \sigma \) is defined by \( \sigma(X_i) = f_i \). This factorization induces injective morphisms of \( A \)-algebras

\[
\Phi : B \to L \cong S_A(M/M^2) \to S_A(I/I^2) = B',
\]

\[
\Psi : B' = S_A(I/I^2) \to S_A(M/M^2) \to L \to B.
\]

Now, observe that \( B' \cong S_A(\Omega_A(B) \otimes_B A) \) since the exact sequence

\[
0 \to I/I^2 \to \Omega_A(B) \otimes_B A \to \Omega_A(A) \to 0,
\]

ensures us that \( I/I^2 \cong \Omega_A(B) \otimes_B A \) and \( \text{rk}_B(\Omega_A(B)) = \text{rk}_A(I/I^2) \).

Using our previous results, we can improve a result by Tronin [33].

Proposition 2.18. Let \( A \to B \) be an a-projective morphism of finite type.

(1) The following sequences are exact

\[
0 \to \Omega_A(B) \otimes_B B' \to \Omega_A(B') \to \Omega_B(B') \to 0,
\]

\[
0 \to \Omega_A(B') \otimes_B' B \to \Omega_A(B) \to \Omega_B'(B) \to 0.
\]

(2) If \( A \) is a connected PPF ring and \( \text{rk}_B(\Omega_A(B)) = r \), then \( \text{rk}_A(I/I^2) = r \) and there are two injective morphisms of \( A \)-algebras

\[
B \xrightarrow{\alpha} A[X_1, \ldots, X_r] = B' \text{ and } B' = A[X_1, \ldots, X_r] \xrightarrow{\beta} B,
\]

where \( r = \dim(B) \) when \( A \) is a field.

(3) If \( A \) is a PPF integral domain, \( \alpha \) and \( \beta \) induce separable algebraic extensions of the quotient fields.

Proof. See [33] for a proof of (1). To show (2), observe that \( \Omega_A(B) \) is a free \( B \)-module of rank \( r \) by (2.14) while \( I/I^2 \cong \Omega_A(B) \otimes_B A \) and \( B' = S_A(I/I^2) \cong A[X_1, \ldots, X_r] \). Next
notice that \( \dim(B) = \text{rk}_B(\mathcal{O}_A(B)) \) when \( A \) is a field by Theorem 2.14. We prove (3). Let \( K \) and \( K' \) be the respective quotient fields of \( B \) and \( B' \). Tensoring the first exact sequence with \( \otimes_{B'} K' \) gives an exact sequence of \( K' \)-vector spaces since \( B' \to K' \) is flat. The first two \( K' \)-vector spaces have the same rank \( r \) so that \( \mathcal{O}_K(K') \cong \mathcal{O}_B(B') \otimes_{B'} K' = 0 \). The conclusion follows from [6, V.16.6, Corollary 2].

**Remark 2.19.** Costa proved that when \( A \) is a field and \( A \to B \) is a-projective of finite type with representation \( \{ f_1, f_2 \} \subset A[X_1, X_2] \) or such that \( \dim(B) = 2 \), then \( B = A[X_1, X_2] \) (see [9, 3.5]). We can recover this result thanks to Proposition 2.18. Let \( A \) be a perfect field and \( A \to B \) an a-projective algebra of finite type with \( \dim(B) = 2 \). The \( A \)-algebra \( B \) is isomorphic to \( A[X_1, X_2] \). Indeed, the hypotheses of Castelnuovo’s affine theorem are fulfilled [29, Theorem 3] since in this case \( B' = A[X_1, X_2] \), \( B \) is regular, \( K \otimes_A B \) is a UFD for every morphism \( A \to K \) where \( K \) is a field and the quotient fields extension is separable by Proposition 2.18. If \( A \) is not perfect, let \( A \to C \) where \( C \) is an algebraic closure of \( A \). Then \( A \to C \) is faithfully flat and we can use the descent result of Proposition 2.23.

The previous proposition cannot be used to prove that \( B \) is isomorphic to a polynomial algebra when \( \dim(B) > 2 \) since Castelnuovo’s Theorem is no longer true when \( d > 2 \) [15, p. 297].

We give here some descent results.

**Proposition 2.20.** Algebraically pure morphisms descend a-projective algebras of finite presentation.

**Proof.** Observe that a pure morphism descends algebras of finite presentation [25, 5.3]. To conclude use Theorems 2.4 and 1.16. □

A ring morphism \( A \to A' \) is called strongly Nakayama if for every \( A \)-module \( M \), the equation \( M \otimes_A A' = 0 \) implies \( M = 0 \). A strongly Nakayama morphism \( A \to A' \) descends the surjectivity of \( A \)-module morphisms [25].

**Lemma 2.21.** Let \( A \to B \) be a ring morphism and \( A \to A' \) a strongly Nakayama morphism. If \( \{ b_2 \} \) is a family of elements in \( B \) such that \( \{ b_2 \otimes 1 \} \) generates the \( A' \)-algebra \( B \otimes_A A' \), then so does \( \{ b_2 \} \) in \( B \).

**Proof.** Consider the morphism of \( A \)-algebras \( A[X_1] \to B \) defined by \( X_1 \mapsto b_2 \). Then \( A[X_2] \otimes_A A' \to B \otimes_A A' \) is surjective and so is \( A[X_2] \to B \). □

**Proposition 2.22.** Let \( A \to B \) and \( A \to A' \) be ring morphisms. If \( \{ b_2 \} \subset B \) is a family such that \( B \otimes_A A' = A'[b_2 \otimes 1] \) is a polynomial \( A' \)-algebra with respect to the elements \( b_2 \otimes 1 \), then \( B = A[b_2] \) is a polynomial \( A \)-algebra with respect to the elements \( b_2 \) in the following cases:

1. \( A \to A' \) is faithfully flat.
2. The kernel of the morphism \( A[X_1] \to B \) defined by \( X_1 \mapsto b_2 \) is a pure \( A \)-submodule of \( A[X_1] \) and \( A \to A' \) is strongly Nakayama.
Lemma 2.21. Let $A[X_i] \otimes A' \to B \otimes A'$ be surjective with kernel $I$. Then tensor the exact sequence $0 \to I \to A[X_i] \to B \to 0$ by $\otimes A'$. The new sequence is exact and then $I \otimes_A A' = 0$ implies $I = 0$. □

Proposition 2.23. Let $A \to B$ and $A \to A'$ be ring morphisms such that the $A'$-algebra $B \otimes_A A'$ is isomorphic to $A'[X_1, \ldots, X_n]$. Then $A$ is strongly Nakayama.

Proof. Denote by $f$ the isomorphism $B \otimes A' \to A'[X_1, \ldots, X_n]$ and set $f^{-1}(X_i) = \sum b_i \otimes a_i'$. Then $\{b_i \otimes 1\}$ generates the $A'$-algebra $B \otimes A'$ and $\{b_i\}$ generates the $A$-algebra $B$ by Lemma 2.22. Let $u : A[X_1, \ldots, X_n] \to B$ be the surjective morphism defined by $X_i \mapsto b_i$ with kernel $I$. The composite morphism $f \circ (u \otimes \text{Id}_{A'})$ is a surjective endomorphism of the $A'$-algebra of finite type $A'[X_1, \ldots, X_n]$, whence an isomorphism. Thus, $u \otimes \text{Id}_{A'}$ is an isomorphism and so is $u$ thanks to (2.22) if $A \to A'$ is faithfully flat. If $A \to B$ is projective, $A[X_1, \ldots, X_n] = I \oplus B$ implies that $I$ is a pure $A$-submodule of $A[X_1, \ldots, X_n]$ and the proof can be completed as above. □

Next we give some informations on differential properties of a-projective algebras.

For each positive integer $m$, we denote by $\text{M}_m(R)$ the ring of all size $m$ squared matrices with entries in the ring $R$ and by $\text{LG}_m(R)$ the set of all units in $\text{M}_m(R)$. A ring morphism $\phi : R \to S$ induces a ring morphism $\phi_m : \text{M}_m(R) \to \text{M}_m(S)$ with kernel $\text{M}_m(\text{Ker}(\phi))$. Let $A$ be a ring and $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ defining an $A$-endomorphism $u : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n]$ by $u(X_i) = f_i$. We consider the jacobian matrix $J_u = (\frac{\partial f_j}{\partial X_i}) \in \text{M}_n(A[X_1, \ldots, X_n])$ where $i$ is the index of the row and $j$ the index of the column. Now let $u, v$ be two $A$-endomorphisms of $A[X_1, \ldots, X_n]$. The rule of chained derivations gives here $J_{uv} = J_u(J_v)$.

Let $A \to B$ be an a-projective morphism of finite type with representation $\{f_1, \ldots, f_n\} \subset A[X_1, \ldots, X_n]$ and $u : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n]$ the associated idempotent endomorphism defined by $u(X_i) = f_i$. We get $J_u = J_{uv} = J_{uv}(J_u)$ so that $u(J_u)$ is an idempotent matrix of $\text{M}_n(A[X_1, \ldots, X_n])$ and its determinant lies in $\text{Bool}(A)$. The ideal of $A[X_1, \ldots, X_n]$ generated by the entries of $u(J_u)$ is idempotent whence generated by an element of $\text{Bool}(A)$.

Now assume that $A$ is a connected PPF ring. Then $u(J_u)$ is diagonalizable under a similarity transform. Thus there is some $M \in \text{LG}_n(A[X_1, \ldots, X_n])$ such that $Mu(J_u)M^{-1} = \text{Diag}(1, \ldots, 1, 0, \ldots, 0)$ where the last matrix is diagonal with $r$ nonzero entries. The kernel of the canonical surjective morphism $p : A[X_1, \ldots, X_n] \to B$ is $(X_1 - f_1, \ldots, X_n - f_n)$ and $p(M)p(u(J_u))p(M)^{-1} = \text{Diag}(1, \ldots, 1, 0, \ldots, 0)$. As usual, set $p(\frac{\partial f_j}{\partial X_i}) = \frac{\partial f_j}{\partial X_i}$ where $x_i$ denotes the class of $X_i$ in $B$. Therefore, the relation $\text{Diag}(1, \ldots, 1, 0, \ldots, 0) = p(M)(\frac{\partial f_j}{\partial X_i})p(M)^{-1}$ where $p(M) \in \text{LG}_n(B)$ follows from $p(X_i) = p(f_i)$.

Proposition 2.24. Let $B$ be an a-projective algebra of finite type over a connected PPF ring $A$ and $u$ an associated idempotent endomorphism defining a representation $\{f_1, \ldots, f_n\}$. Then $u(J_u)$ is similar to the matrix $\text{Diag}(1, \ldots, 1, 0, \ldots, 0)$ with
rkₜ(Ωₜ(B)) nonzero entries. If the representation is standard, rkₜ(Ωₜ(B)) = rkₜ((f₁, ..., fₙ)/(f₁, ..., fₙ)²).

Proof. Let z be the B-module endomorphism of Bⁿ with matrix ((c(xⱼ − fⱼ))/cᵫxᵢ) = Iₙ − (c fⱼ/cᵫxᵢ) in the canonical basis of Bⁿ. Since z is idempotent, we get Bⁿ = Im(z) ⊕ Ker(z). Then observe that Ωₜ(B) ∼= Bⁿ/Im(z) ∼= Ker(z). The result follows immediately, the last statement being a consequence of (2.18)(2). □

We come back to Lemma 2.10, where an A-automorphism φ of A[X₁, ..., Xₙ] is considered as well as v = φ ◦ u ◦ ψ where ψ = φ⁻¹. Then φ ◦ u = v ◦ φ gives Jₜφ(Jₜu) = Jₜv(Jₜφ) while φ ◦ ψ = Id = ψ ◦ φ gives Jₜφ(Jₜψ) = Iₙ and Jₜψ(Jₜφ) = Iₙ so that φ(Jₜψ)Jₜφ = Iₙ. It follows that Jₜ = Jₜφ(Jₜu)v(Jₜψ)⁻¹ = Jₜφ(Jₜu)v(φ(Jₜψ)) = Jₜφ(Jₜu)φ(u(Jₜψ)).

Now consider a matrix M = (xᵢⱼ) ∈ Mₙₙ(A) and the associated A-endomorphism φ defined by φ(Xᵢ) = ∑ᵢ xᵢⱼ Xⱼ for j = 1, ..., n that is to say φ is defined by the matrix equation (φ(X₁), ..., φ(Xₙ)) = (X₁, ..., Xₙ)M. Obviously, we have M = Jₜφ. Now assume that M ∈ LGₙₙ(A). With the previous notation, we get that v(Jₜψ) = Jₜφ so that Jₜ = Jₜφ(Jₜu)Jₜ⁻¹ and v(Jₜψ) = Jₜφ(u(Jₜψ)).

Proposition 2.25. Let A → B be an a-projective morphism of finite type with a standard representation {f₁, ..., fₙ} associated to the idempotent endomorphism u. Let hᵢ be the degree one homogeneous component of fᵢ so that there is a matrix equation (h₁, ..., hₙ) = (X₁, ..., Xₙ)Jₜ(u, 0, ..., 0).

1. {h₁, ..., hₙ} defines a representation of an a-projective algebra B₁. Its associated idempotent endomorphism h is defined by Jₜ = Jₜ(u, 0, ..., 0).

2. If in addition A is a connected PPF ring, the A-algebra B₁ is isomorphic to A[X₁, ..., Xₙ] where r = rkₜ(Ωₜ(B))

Proof. (1) is obvious since hᵢ(h₁, ..., hₙ) = hᵢ. Assume that A is a connected PPF ring. Denote by s : A[X₁, ..., Xₙ] → A the substitution morphism defined by s(Xᵢ) = 0 and observe that Jₜ = s(Jₜ) = s(u(Jₜu)). There is an equation Mu(Jₜu)M⁻¹ = Diag(1, ..., 1, 0, ..., 0) where M ∈ LGₙₙ(A[X₁, ..., Xₙ]). Thus we get s(M)Jₜs(M)⁻¹ = Diag(1, ..., 1, 0, ..., 0) where the number of nonzero entries is r = rkₜ(Ωₜ(B)) and s(M) ∈ LGₙₙ(A). Now s(M) defines an A-automorphism φ of A[X₁, ..., Xₙ]. Then k = φ ◦ h ◦ φ⁻¹ is an A-endomorphism associated to the matrix Diag(1, ..., 1, 0, ..., 0) so that k(X₁) = X₁, ..., k(Xᵢ) = Xᵢ and k(Xᵢ) = 0 for i > r. Hence B₁ is isomorphic to A[X₁, ..., Xₙ]. □

Remark 2.26. If A is a PPF affine regular integral domain, dim(B) = dim(Bₜ).

Remark 2.27. Assume that A is a connected PPF ring. Consider the A-automorphism φ defined in Proposition 2.25, v = φ ◦ u ◦ φ⁻¹ and set v(Xᵢ) = gᵢ. From fᵢ = hᵢ + hᵢ where tᵢ ∈ (X₁, ..., Xₙ)², we get that Xᵢ - g₁, ..., Xᵢ - gᵢ ∈ (X₁, ..., Xₙ)² and gᵢ+₁, ..., gₙ ∈ (X₁, ..., Xₙ)². It follows that gᵢ+₁, ..., gₙ ∈ (g₁, ..., gₙ)². Hence the classes of g₁, ..., gₙ in (g₁, ..., gₙ)/(g₁, ..., gₙ)² give a basis of this A-module (see Proposition 2.18 (2)).
References