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Algebraically flat or projective algebras

Gabriel Picavet

Laboratoire de Mathématiques Pures, Université de Clermont II, 63177 Aubière Cedex, France

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Abstract

We define and study algebraically flat algebras in order to have a better understanding of algebraically projective algebras of finite type (the projective algebras of literature). A close examination of the differential properties of these algebras leads to our main structure theorem. As a corollary, we get that an algebraically projective algebra of finite type over a field is either a polynomial ring or the affine algebra of a complete intersection. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

This paper originates with the theory of projective algebras. We were motivated by an unsolved conjecture: a projective algebra of finite type over a field A is a polynomial ring. An example by Costa shows that the statement is false if A is not a field. As Costa noticed, the cancellation problem for polynomial rings over fields is solved if the conjecture is true [9].

Flatness is well known to be useful when studying projectivity. In Section 1, we are aiming to build a convenient theory of flatness for algebras. Roughly speaking, the flatness of an A-module M is characterized by properties of linear relations in M. Replacing linear relations with polynomial relations gives the solution. We have chosen to follow Lazard's treatment of flatness [17]. An A-algebra B is called algebraically flat (a-flat) if every morphism of A-algebras $P \to B$ where P is of finite presentation can be factored $P \to L \to B$ where L is a polynomial algebra in finitely many indeterminates. When A and B are Noetherian, replacing polynomial algebras

E-mail address: gabriel.picavet@math.univ-bpclermont.fr (G. Picavet).

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with smooth algebras in the above definition gives the characterization of regular morphisms by Popescu–Spivakovski [31]. Our definition gives most of the usual flatness properties. In particular, an A-algebra B is a-flat if and only if B is a direct limit of polynomial algebras in finitely many indeterminates over A. Symmetric algebras of flat modules are a-flat algebras. D. Popescu defined algebraically pure morphisms (a-pure morphisms) [26]. These morphisms are closely related to a-flat morphisms, since an a-flat morphism is a-pure and faithfully flat. Under some finiteness conditions, a-pure morphisms descend factorization of morphisms. As a consequence, a-purity descends a-flatness and smoothness. Evidently, a-flatness localizes but we do not know whether it globalizes. Here are some concrete examples of a-flat morphisms. If I is a flat ideal of linear type in a ring A, its Rees ring $A[IX]$ is a-flat over A. Then a Rees ring over a Prüfer domain is a-flat.

We define the flat rank $\text{f-rk}(B)$ of an a-flat algebra B. Then $\text{f-rk}(B) \leqslant r$ if and only if B is a direct limit of polynomial algebras in r indeterminates. If B is of finite type, $\text{f-rk}(B) = \lambda(B)$, the least number of elements required to generate B.

In this paper, projectively trivial rings are a prominent tool because a connected ring A is projectively trivial if and only if each of its finitely generated projective modules is free [22]. We say that a ring A is PPF if finitely generated projective $A[X_1,\dots,X_n]$-modules are free for each integer n. A principal domain is PPF by the Quillen–Suslin's theorem. If $\mathbb{P}$ is a property of rings, $\mathbb{P}$-morphisms are well known. We have been led to introduce a variant: universal $\mathbb{P}$-morphisms. We show that a regular PPF integral domain is a UFD and that an a-flat morphism is a universal connected PPF morphism. Hence, if $A \to B$ is a-flat and A is PPF, so is B. Moreover, an a-flat morphism between noetherian rings is a regular UFD morphism.

Section 2 contains the main results of this paper and is devoted to algebraically projective (a-projective) algebras. They are the projective objects in a category of algebras over a ring. An a-projective algebra is projective. These algebras have been studied by many authors as D. L. Costa, T. Asanuma, J. W. Brewer, A. R. Kustin, J. Yanik.

Our results show that a-projective algebras share many properties with polynomial rings. An a-projective algebra of finite type is of finite presentation and an a-projective algebra is a-flat. The converse is true if B is of finite presentation. In this case, B is the direct limit of polynomial algebras in $\text{f-rk}(B) = \lambda(B)$ indeterminates. This gives a partial answer to the conjecture evoked at the beginning.

The following is a key result. If B is an a-projective algebra of finite type, $A \to B$ is a projective, smooth, universal regular morphism, its B-module of Kähler differentials $\Omega_A(B)$ is projective and $K \otimes_A B$ is a regular UFD for every ring morphism $A \to K$ where K is a field. Moreover, if R is a connected PPF ring, so is $R \otimes_A B$ for every ring morphism $A \to R$ and $\Omega_R(R \otimes_A B)$ is free with finite rank.

An a-projective A-algebra B of finite type is a retract of a polynomial ring $L = A[X_1,\dots,X_n]$. An idempotent endomorphism u of the A-algebra L is associated to B. The sequence $\{u(X_1) = f_1,\dots,u(X_n) = f_n\}$ is called a representation of B and $J = (X_1 - f_1,\dots,X_n - f_n)$ a representation ideal, for $B = A[f_1,\dots,f_n]$ and $B \simeq L/J$. With this notation, if A is a connected PPF ring, then J/J^2 is a free B-module with finite rank,

$n = \mathrm{rk}_B(\Omega_A(B)) + \mathrm{rk}_B(J/J^2)$ and if in addition A is Noetherian, $\dim(B) \leqslant \dim(A) + \mathrm{rk}_B(\Omega_A(B))$.

Our main result is as follows. Let K be a PPF affine regular integral domain and $K \to B$ an a-projective morphism of finite type which is not a polynomial algebra, then a representation ideal J such that $\mathrm{ht}(J) > \dim(K)$ is a complete intersection and $\dim(B) = \mathrm{rk}_B(\Omega_K(B))$. In particular, if K is a field, a representation ideal is a complete intersection.

We give some notation. All rings considered are unital commutative and ring morphisms are unital. Hence a commutative A-algebra B can be identified with its structural ring morphism $A \to B$. The set of all units of a ring A is denoted by $\mathrm{U}(A)$, the set of all idempotents by $\mathrm{Bool}(A)$ and the nilradical by $\mathrm{Nil}(A)$. If P is a prime ideal of A, the associated residual field is denoted by $\mathrm{k}(P)$. The symmetric algebra of an A-module M is denoted by $\mathrm{S}_A(M)$. Any unexplained notation is standard.

1. Definition and properties of algebraically flat morphisms

In the following, a polynomial A-algebra L over a ring A is an A-algebra $A[X_i]_{i\in I}$ in a set of indeterminates $\{X_i\}_{i\in I}$ (if I is empty, $L = A$). We denote by $\mathscr{P}_A$ the class of all polynomial algebras of the form $A[X_1,\dots,X_n]$ where n is an integer.

Definition 1.1. An A-algebra B (or a ring morphism $A \to B$) is called algebraically flat (a-flat) if the following condition $(\mathscr{AF})$ holds:

$(\mathscr{AF})$ Every morphism of A-algebras $P \to B$ where P is an A-algebra of finite presentation, can be factored $P \to L \to B$ where L is a polynomial A-algebra.

In the above definition, the polynomial A-algebra L can be replaced with $L \in \mathscr{P}_A$ or with an a-flat algebra L. Clearly, a polynomial A-algebra is a-flat.

Our first result gives the structure of a-flat morphisms. Lazard gave a similar result for flat modules [17]. We mimic the proof given in [5]. The proof is detailed because some arguments are different in the category of algebras.

Lemma 1.2. *Let $A \to B$ be a ring morphism and assume that there exists a direct system $\{B_\lambda\}_{\lambda\in\Lambda}$ of A-algebras B_λ such that $B = \varinjlim B_\lambda$. Let $P \to B$ be a morphism of A-algebras where P is of finite presentation. There is some index λ such that $P \to B$ can be factored $P \to B_\lambda \to B$.*

Proof. Consider a morphism $f: P \to B$ of A-algebras where the algebra $P = A[X_1,\dots,X_n]/(p_1,\dots,p_s)$ is defined by the polynomials $p_1,\dots,p_s$. Denote by x_i the class of X_i in P and set $f(x_i) = b_i$. There are an index λ and some $v_1,\dots,v_n$ in B_λ such that $p_k(v_1,\dots,v_n) = 0$ for $k = 1,\dots,s$ and $v_i \mapsto b_i$ for $i = 1,\dots,n$. Then $P \to B$ can be factored $P \to B_\lambda \to B$. $\square$

Theorem 1.3. *Let $A \to B$ be a ring morphism. Then B is a-flat if and only if there exists a direct system $\{L_\lambda\}_{\lambda\in\Lambda}$ of A-algebras $L_\lambda \in \mathscr{P}_A$ such that $B = \varinjlim L_\lambda$. In this case, the canonical morphisms $L_\lambda \to L_\mu$ are of finite presentation.*

Proof. Assume that $B = \varinjlim L_\lambda$ where $L_\lambda \in \mathscr{P}_A$. Then Lemma 1.2 shows that B is a-flat. Conversely, assume that B is a-flat. Then $B = \varinjlim B_\lambda$ where $\{B_\lambda\}_{\lambda\in\Lambda}$ is a direct system of A-algebras of finite presentation, indexed by a partially ordered directed set Λ (the partial ordering hypothesis is essential) [11, O.6.3.10]. There is no harm to change Λ into $\Lambda \times \mathbb{N}$ equipped with the lexicographic order provided we set $B_{(\lambda,n)} = B_\lambda$ for each $n \in \mathbb{N}$. Thus we can assume that Λ has no maximum element. Denote the canonical morphisms by $g_\lambda : B_\lambda \to B$ and $g_{\mu,\lambda} : B_\lambda \to B_\mu$ for $\lambda \leqslant \mu$. Consider an element $\sigma \in \Lambda$. By a-flatness, there exist a polynomial ring $L_\sigma = A[X_1,\dots,X_n]$ and some morphisms u_σ, w_σ such that $B_\sigma \xrightarrow{u_\sigma} L_\sigma \xrightarrow{w_\sigma} B = B_\sigma \xrightarrow{g_\sigma} B$. Then set $w_\sigma\ (X_i) = b_i$. There exist some $\tau > \sigma$ and $x_1,\dots,x_n \in B_\tau$ such that $b_i = g_\tau(x_i)$ for $i = 1,\dots,n$, because there is no maximum element in Λ. Next define an A-algebra morphism $w'_\sigma : L_\sigma \to B_\tau$ by $w'_\sigma(X_i) = x_i$ for $i = 1,\dots,n$. We get a morphism $g_\tau \circ w'_\sigma : L_\sigma \to B_\tau \to B$ such that $w_\sigma = g_\tau \circ w'_\sigma$ since $g_\tau \circ w'_\sigma(X_i) = g_\tau(x_i) = b_i = w_\sigma(X_i)$. Then the relation $g_\tau \circ w'_\sigma \circ u_\sigma = g_\sigma = g_\tau \circ g_{\tau,\sigma}$ follows. Now we can use [11, O.6.3.11]. Since $A \to B_\sigma$ is of finite type, there is some $v \geqslant \tau$ such that $g_{v,\tau} \circ w'_\sigma \circ u_\sigma = g_{v,\tau} \circ g_{\tau,\sigma} = g_{v,\sigma}$. Define a map $f : \Lambda \to \Lambda$ by letting $f(\sigma) = v$. Set $v_\sigma = g_{v,\tau} \circ w'_\sigma$. Hence we have $v_\sigma \circ u_\sigma = g_{f(\sigma),\sigma}$ with $f(\sigma) > \sigma$ so that $B_\sigma \xrightarrow{u_\sigma} L_\sigma \xrightarrow{v_\sigma} B_{f(\sigma)} = B_\sigma \xrightarrow{g_{f(\sigma),\sigma}} B_{f(\sigma)}$. We are now in position to apply [5, 1.6, Lemma 2], that is to say we can change the partial ordering on Λ so that $B = \varinjlim L_\lambda$. To complete the proof, observe that a morphism of A-algebras $\alpha : A[Y_1,\dots,Y_m] \to A[X_1,\dots,X_n]$ is of finite presentation. Setting $\alpha(Y_j) = p_j(X_1,\dots,X_n)$, it is easy to see that α can be identified to the canonical morphism $A[S_1,\dots,S_m] \to A[S_1,\dots,S_m;\ X_1,\dots,X_n]/(S_1 - p_1,\dots,S_m - p_m)$. □

Corollary 1.4. *The symmetric algebra* $\mathrm{S}_A(M)$ *of an* A*-flat module* M *is a-flat.*

Proof. Observe that M is a direct limit of free modules with finite rank [5]. Hence, $\mathrm{S}_A(M)$ is a direct limit of polynomial algebras. □

Now, we characterize a-flat morphisms in the same way as Lazard did for flat modules [5].

Theorem 1.5. *Let* $A \to B$ *be a ring morphism. Then* $A \to B$ *is a-flat if and only if the following condition* $(\mathscr{AF}')$ *holds*:

$(\mathscr{AF}')$ *For every* A*-algebra* P *of finite presentation and every surjective morphism of* A*-algebras* $s : C \to B$, *the natural map* $\mathrm{Hom}_{A\text{-alg}}(P,C) \to \mathrm{Hom}_{A\text{-alg}}(P,B)$ *is surjective.*

Proof. Assume that $(\mathscr{AF}')$ holds and let $L = A[X_i]_{i\in I} \to B$ be a surjective morphism. Then a morphism of A-algebras $P \to B$ can be factored $P \to L \to B$ and $(\mathscr{AF})$ is verified. Conversely, assume that $(\mathscr{AF})$ holds. Let $s : C \to B$ and $f : P \to B$ be morphisms of A-algebras where P is of finite presentation and s is surjective. Then f can be factored $P \xrightarrow{g} A[X_1,\dots,X_n] \xrightarrow{h} B$ so that $f = h \circ g$. If $n = 0$, using the structural morphism $k : A \to C$ and observing that h is the structural morphism of B, we get $s \circ (k \circ g) = f$. If $n \neq 0$, letting $b_i = h(X_i)$ for $i = 1,\dots,n$, we pick $c_i \in C$

such that $s(c_i) = b_i$. Hence a morphism of A-algebras $k : A[X_1,\dots,X_n] \to C$ is defined by $k(X_i) = c_i$ so that $h = s \circ k$. It follows that $f = s \circ (k \circ g)$. Thus the proof is complete. □

Definition 1.6. Let $A \to B$ be a ring morphism and n an integer.

(1) A size n (polynomial) relation in B is a pair $(p,\beta) \in A[X_1,\dots,X_n] \times B^n$ such that $p(\beta) = 0$.

(2) A system of (polynomial) relations in B is a set of finitely many size n relations $(p_1,\beta),\dots,(p_m,\beta)$ and $\sum_{j=1}^{m} A[X_1,\dots,X_n]p_j$ is its associated ideal.

(3) Let $s : C \to B$ be a morphism of A-algebras. We say that a system of relations $(p_1,\beta),\dots,(p_m,\beta)$ in B has a pullback in C via s, if there exists $\gamma \in C^n$ such that $s(\gamma) = \beta$ and $(p_1,\gamma),\dots,(p_m,\gamma)$ is a system of relations in C.

Theorem 1.7. *Let B be an A-algebra, the following statements are equivalent*:

(1) *B is a-flat over A.*

(2) *For every surjective morphism of A-algebras $s : C \to B$, each relation (respectively, each system of relations) in B has a pullback in C via s.*

(3) *There is a surjective morphism $s : L \to B$ of A-algebras, where L is a polynomial algebra such that each relation (respectively, each system of relations) in B has a pullback in L via s.*

(4) *There is a surjective morphism $s : F \to B$ of A-algebras, where F is an a-flat A-algebra such that each relation (respectively, each system of relations) in B has a pullback in F via s.*

(5) *The following condition $(\mathscr{AF}'')$ holds*:
$(\mathscr{AF}'')$ If $b = (b_1,\dots,b_n) \in B^n$ is a zero of $p \in A[X_1,\dots,X_n]$, there exist $\beta \in B^m$ and $f_1,\dots,f_n$ in a polynomial algebra $A[Y_1,\dots,Y_m]$ such that $p(f_1,\dots,f_n) = 0$ and $b_i = f_i(\beta)$ for $i = 1,\dots,n$.

Proof. To see that (1) ⇒ (2), observe that a system of relations in B with associated ideal I defines a morphism of A-algebras $A[X_1,\dots,X_n]/I \to B$ and then use Theorem 1.5. Obviously, (2) ⇒ (3) and (3) ⇒ (4). We show that (4) ⇒ (1), assuming only that each of the relations has a pullback in F. Consider a morphism $f : P \to B$ where $P = A[X_1,\dots,X_n]/(p_1,\dots,p_m)$. Set $f(x_i) = b_i$ where x_i is the class of X_i in P and $\beta = (b_1,\dots,b_n)$. We get a system of relations $(p_1,\beta),\dots,(p_m,\beta)$. Each relation (p_i,β) has a pullback (p_i,γ_i) in F. We set $\gamma_i = (c_{i,1},\dots,c_{i,n})$. Let P' be $P \otimes \cdots \otimes P$ with n factors and let $P' \to B$ be the canonical morphism. There is at least a factorization $P \to P' \to B$. Set $\mathbf{X}_i = \{X_{i,1},\dots,X_{i,n}\}$ where the $X_{i,j}$ are indeterminates. Now P' is isomorphic to $A[\mathbf{X}_1,\dots\mathbf{X}_n]/J$ where J is the ideal generated by $\{p_i(\mathbf{X}_j)\}$ for $i = 1,\dots,m$ and $j = 1,\dots,n$. Define a morphism $A[\mathbf{X}_1,\dots,\mathbf{X}_n] \to F$ by $X_{i,j} \mapsto c_{i,j}$. We get a morphism $P' \to F$ such that $P' \to F \to B$ commutes. Thus we have a factorization $P \to F \to B$. Then use the remark in (1.1). Now, (5) is a translation of (3). □

Algebraically flat morphisms are closely related to algebraically pure morphisms (a-pure morphisms) considered by Popescu [26].

Definition 1.8. A morphism of A-algebras $f:B \to C$ is called a-pure if for every commutative diagram of A-algebras

$$\begin{array}{ccc} T & \xrightarrow{g} & P \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow{f} & C \end{array}$$

where T is of finite type and P of finite presentation, there exists a morphism of A-algebras $d:P \to B$ such that $u = d \circ g$.

Obviously, if $B \to C$ is a-pure as a morphism of B-algebras, then $B \to C$ is a-pure as a morphism of A-algebras.

Algebraically pure morphisms can be characterized by polynomial relations. They are stable under arbitrary base changes. An a-pure morphism of A-algebras is universally injective.

Definition 1.9. A morphism of A-algebras $f:B \to C$ defines B as a retract of C if there is some morphism of A-algebras $s:C \to B$ such that $s \circ f = \mathrm{Id}_B$.

In this case, $C = f(B) \oplus J$ is a direct sum of B-modules where $J = \mathrm{Ker}(s)$. If $u = f \circ s$, then $u:C \to C$ is an idempotent endomorphism of the A-algebra C such that $\mathrm{Im}(u) = f(B)$ and $\mathrm{Ker}(u) = J$. Conversely, an idempotent endomorphism of A-algebras $u:C \to C$ gives an A-algebra $\mathrm{Im}(u) = B$ which is a retract of C [9].

An A-algebra B is called retractable if A is a retract of B with respect to the structural morphism $A \to B$.

Theorem 1.10 (Popescu [26]). *Let $A \to B$ be a ring morphism.*

(1) *$A \to B$ is a-pure if and only if there exists a direct system $\{P_\lambda\}_{\lambda \in \Lambda}$ of retractable A-algebras of finite presentation P_λ such that $B = \varinjlim P_\lambda$.*

(2) *If $A \to B$ is of finite presentation, then $A \to B$ is a-pure if and only if B is retractable.*

Corollary 1.11. *An a-flat morphism is a-pure and faithfully flat.*

Proposition 1.12. *Let $f:A \to B$ be an a-flat morphism, then*

$$\mathrm{U}(B) = f(\mathrm{U}(A)) + \mathrm{Nil}(B) \quad \textit{and} \quad \mathrm{Bool}(B) = f(\mathrm{Bool}(A)).$$

Proof. Let $b, b' \in B$ be such that $bb' = 1$. Let $g:L \to B$ be a surjective morphism where L is a polynomial ring. The relation $(XY - 1, (b, b'))$ has a pullback in L via g. Therefore, there is some polynomial $p = u + n$ where $u \in \mathrm{U}(A)$ and $n \in \mathrm{Nil}(A[X])$ are such that $g(p) = b$. For $e \in \mathrm{Bool}(B)$, the relation $(X^2 - X, (e, e))$ has a pullback in L via g. There is an $\varepsilon \in \mathrm{Bool}(A)$ such that $g(\varepsilon) = e$. □

Remark 1.13. An a-pure morphism need not be flat. It is enough to consider a non-noetherian ring A such that $A \to A[[X]]$ is not flat. Moreover, a faithfully flat a-pure morphism need not be a-flat. To see this, let K be an algebraically closed field. Then by [26, 1.8], a ring morphism $K \to B$ is a-pure. Choose $B = K[X]_X$. In view of Proposition 1.12, we have $\mathrm{U}(B) = f(\mathrm{U}(K)) = K \setminus \{0\}$ if B is a-flat which is absurd.

Now we study the stability of the class of a-flat morphisms with respect to the usual constructions of algebra. Clearly, an isomorphism is a-flat.

Proposition 1.14. *Let* (**AF**) *be the class of a-flat morphisms.*

(1) *If $f : A \to B$ and $g : B \to C$ are in* (**AF**), *then $g \circ f$ lies in* (**AF**). *In particular, $A \to B[X_1, \dots, X_n]$ is a-flat when $A \to B$ is a-flat.*
(2) *If $A \to B$ lies in* (**AF**), *then $A' \to B \otimes_A A'$ lies in* (**AF**) *for every ring morphism $A \to A'$.*
(3) *If $\{B_\lambda\}_{\lambda \in \Lambda}$ is a direct system of a-flat A-algebras with direct limit B, then B is an a-flat A-algebra.*
(4) *Let $f : A \to B$ be a ring morphism and $g : B \to C$ an a-pure morphism of A-algebras such that $g \circ f$ lies in* (**AF**), *then $f : A \to B$ lies in* (**AF**). *The same conclusion is valid if $B \to C$ is an a-pure morphism of B-algebras.*
(5) *If B is a retract of C and C is in* (**AF**), *so is B.*

Proof. Thanks to Theorem 1.3, (2) is obvious. We show (3). Let P be an A-algebra of finite presentation and $P \to B$ a morphism. According to Lemma 1.2, there is some index λ such that $P \to B$ can be factored $P \to B_\lambda \to B$. Since $A \to B_\lambda$ is a-flat, there is some polynomial A-algebra L such that $P \to B_\lambda = P \to L \to B_\lambda$ whence a factorization $P \to L \to B$. Therefore, $A \to B$ is a-flat. Now, if $A \to B$ is a-flat, so is $A \to B \to B[X_1, \dots, X_n]$ (write B as a direct limit of polynomial algebras B_λ). Then $B[X_1, \dots, X_n]$ is the direct limit of the polynomial A-algebras $B_\lambda[X_1, \dots, X_n]$ so that $A \to B[X_1, \dots, X_n]$ is a-flat. Next, we show (1). Assume that $f : A \to B$ and $g : B \to C$ are a-flat and consider a morphism of A-algebras $h : P \to C$ where P is of finite presentation. Suppose that $P = A[Y_1, \dots, Y_r]/I$ where $I = (p_1, \dots, p_s)$ in $A[Y_1, \dots, Y_r]$. Set $Q = B[Y_1, \dots, Y_r]/J$ where $J = IB[Y_1, \dots, Y_r]$. Then Q is a B-algebra of finite presentation such that there is a factorization $P \to Q \to C$ where $Q \to C$ is a morphism of B-algebras. Therefore, $Q \to C$ can be factored $Q \to K \to C$ where K is a polynomial B-algebra. According to the beginning of the proof, $A \to K$ is a-flat. Since $P \to Q \to K$ is a morphism of A-algebras, there is a factorization $P \to L \to K$ where L is a polynomial A-algebra. In short, we get a factorization $P \to L \to C$ and $A \to C$ is a-flat. Now, we prove (4). Assume that $g \circ f$ is a-flat and that g is an a-pure morphism of A-algebras. Consider a morphism $h : P \to B$ of A-algebras where P is an A-algebra of finite presentation. Then $P \to B \to C$ is a morphism of A-algebras. By a-flatness of C, there are a polynomial A-algebra L and a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & L \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

By the definition of a-purity, we get a factorization $P \to L \to B = P \to B$. Hence, B is a-flat. The last statement of (4) follows from Definition 1.8. The proof of (5) uses Definition 1.1. □

Lemma 1.15. *Let $A \to B$, $A \to C$ and $A \to A'$ be ring morphisms where A' is a direct limit of A-algebras $\{A_\lambda\}$.*

(1) *Let $f : B \otimes_A A' \to C \otimes_A A'$ be a morphism of A'-algebras. If $A \to B$ is of finite presentation, there is some index μ and a direct system of morphisms of A_λ-algebras $\{f_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}_{\lambda \geqslant \mu}$ such that $f = \varinjlim f_\lambda$.*

(2) *Let $\{f_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}$ and $\{g_\lambda : B \otimes_A A_\lambda \to C \otimes_A A_\lambda\}$ be direct systems of morphisms of A_λ-algebras with limits f and g. If $f = g$ and $A \to B$ is of finite type, there is some index λ such that $f_\lambda = g_\lambda$.*

Proof. Use [11, O.6.3.10]. □

Theorem 1.16. *Let $A \to A'$ be an a-pure ring morphism and P an A-algebra of finite presentation.*

(1) *For every pair of morphisms $u : P \to C$, $v : B \to C$ of A-algebras, v factorizes u if and only if $v \otimes A'$ factorizes $u \otimes A'$.*

(2) *For every pair of morphisms $v : B \to P$ and $u : B \to C$ of A-algebras where $A \to B$ is of finite type, v factorizes u if and only if $v \otimes A'$ factorizes $u \otimes A'$.*

It follows that a-pure morphisms descend universally a-flatness and smoothness.

Proof. We show (1). Let $P \otimes_A A' \xrightarrow{f} B \otimes_A A' \xrightarrow{v \otimes A'} C \otimes_A A'$ be a factorization in the category of A'-algebras such that $u \otimes A' = (v \otimes A') \circ f$. If $A \to A'$ has a retraction $A' \to A$, tensor with $\otimes_{A'} A$ to get a factorization $P \to B \to C$. Now assume that $A \to A'$ is an arbitrary a-pure morphism. We reduce the proof to the previous case. We know that $A' = \varinjlim A_\lambda$ where $A \to A_\lambda$ is retractable (see (1.10)). In view of Lemma 1.15(1) $f = \varinjlim f_\lambda$ (where $\lambda \geqslant \mu$). Then we have $v \otimes A' = \varinjlim v \otimes A_\lambda$ and $u \otimes A' = \varinjlim u \otimes A_\lambda$. Set $k_\lambda = v \otimes A_\lambda \circ f_\lambda$. We get $\varinjlim k_\lambda = u \otimes A'$. It follows from Lemma 1.15(2) that there is a factorization $P \otimes_A A_\lambda \to B \otimes_A A_\lambda \to C \otimes_A A_\lambda$ in the category of A_λ-algebras for some index λ. A similar proof gives (2). We examine the descent properties of an a-pure morphism $A \to A'$. Let $A \to B$ be a ring morphism such that $A' \to B \otimes_A A'$ is a-flat. Use the criterion of Theorem 1.5 and (1) to show that $A \to B$ is a-flat. Next assume that $A' \to B \otimes_A A'$ is smooth. Since a-purity implies purity, $A \to B$ is of finite presentation [25, 5.3]. Then it is enough to show that $\mathrm{Hom}_{A\text{-alg}}(B, C) \to \mathrm{Hom}_{A\text{-alg}}(B, C/I)$ is surjective for each A-algebra C equipped with an ideal I such that

$I^2 = 0$. This is true after tensoring with A' and the result follows from (1) since B is of finite presentation. □

Proposition 1.17. *Let $A \to B$ and $A \to C$ be ring morphisms. The A-algebra $B \otimes_A C$ is a-flat if and only if $A \to B$ and $A \to C$ are a-flat.*

Proof. If $A \to B$ and $A \to C$ are a-flat, so is $A \to B \otimes_A C$ by Proposition 1.14 (1), (2). Now, the a-flatness of $A \to B \otimes_A C$ implies its a-purity by Corollary 1.11 so that $A \to B$ is a-pure by Popescu [26]. Then Theorem 1.16 shows that $A \to C$ is a-flat and so is $A \to B$. □

Proposition 1.18. *Let $s_1, \ldots, s_n$ in a ring A be such that $(s_1, \ldots, s_n) = A$. Then $A \to \prod_{i=1}^n A_{s_i} = A'$ is of finite presentation, faithfully flat and locally retractable. It follows that if $A \to B$ is a ring morphism such that $A' \to B \otimes_A A'$ is a-flat, then $A \to B$ is locally a-flat.*

Proof. It is well known that $A \to A'$ is of finite presentation and faithfully flat. Now, let P be a prime ideal of A. There is some s_i such that $s_i \notin P$ so that $(A_{s_i})_P \simeq A_P$. It follows that A_P is a retract of A'_P. Now, if $A' \to B \otimes_A A' = B'$ is a-flat, so is $A'_P \to B'_P$. □

Proposition 1.19. *Let $A \to B$ and $A' \to B'$ be two a-flat ring morphisms. Then $R = A \times A' \to B \times B' = S$ is a-flat.*

Proof. There is an isomorphism of R-algebras for each integer n

$$f : R[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n] \times A'[X_1, \ldots, X_n]$$

where f is defined by $f(\sum (a_v, a'_v) X^v) = (\sum a_v X^v, \sum a'_v X^v)$. Then use the criterion $(\mathscr{A}\mathscr{F}'')$ of (1.7). □

Remark 1.20. If $A \to B$ is an a-flat morphism, then so is $A_S \to B_S$ for each multiplicative subset S of A (see Proposition 1.14 (2)). In particular, $A_P \to B_P$ is a-flat for each prime ideal P of A.

(1) We do not know whether a-flatness globalizes or not, although we suspect that the answer is negative. The following remarks show that a-flatness globalizes in some cases.
(2) Let $A \to B$ be a ring morphism of finite presentation. If $A \to B$ is locally polynomial (for every prime ideal P of A, there is some integer n such that $B_P \simeq A_P[X_1, \ldots, X_n]$), then a result of Bass, Connell and Wright says that $B \simeq \mathrm{S}_A(M)$ where M is a finitely generated projective module [3, 4.4]. It follows that such an algebra is a-flat. Actually, $A \to B$ is algebraically projective (see the next section).
(3) Let P be the set of all prime integers. Let S be a subset of P and set $B(S) = \mathbb{Z}[p^{-1}X; p \in S]$. Then $B(P)$ is the direct limit of the $\mathbb{Z}$-algebras $B(S)$ where S varies in the set of all finite subsets of P. It is straightforward to check that $\mathbb{Z} \to B(P)$ and $\mathbb{Z} \to B(S)$ are locally polynomial (for instance, see [10]). In view of (2), $B(S)$ is a-flat so that $B(P)$ is a-flat by Proposition 1.14 (3).

(4) Let A be a ring, I an ideal of A and $R_A(I) = A[IX]$ its Rees algebra. The ideal I is of linear type if the canonical surjective map $\mathrm{S}_A(I) \to R_A(I)$ is an isomorphism [13]. If I is of linear type and flat, then $R_A(I)$ is an a-flat algebra. Notice that $R_A(I)$ is a-flat only if I is flat. Indeed, I is a direct summand of $R_A(I)$ and an a-flat algebra is flat.
(5) In particular, assume that A is an integral domain and I is an invertible ideal whence projective, then I is of linear type [23, IV, 2, Theorem 2']. Moreover, $A \to R_A(I)$ is a-flat and locally polynomial since I is locally principal. Now, if I is a directed union of invertible ideals I_λ, then $R_A(I) = \varinjlim R_A(I_\lambda)$ shows that $R_A(I)$ is a-flat. If A is a Prüfer domain, each of its nonzero finitely generated ideals is invertible. Thus a Rees algebra over A is a-flat.
(6) Let A be a noetherian ring and I an ideal of A. Set $\mathrm{gr}_I(A) = \oplus_n I^n/I^{n+1}$. Then I is of linear type if and only if $\mathrm{S}_{A/I}(I/I^2) \simeq \mathrm{gr}_I(A)$ [13, 3.1]. Therefore, if I is of linear type and I/I^2 is (A/I)-flat, $\mathrm{gr}_I(A)$ is an a-flat (A/I)-algebra. Hence, if the ideal I is completely secant (see [8, 5.2, Theorem 1]), $\mathrm{gr}_I(A)$ is an a-flat (A/I)-algebra (actually, an a-projective algebra since I/I^2 is (A/I)-projective).

Proposition 1.21. *Let $A \to B$ be a flat ring morphism. If B is a direct limit of a system of A-algebras $\{B_i\}$ such that each $B_i \simeq A[X_1,\dots,X_n]/(f_1,\dots,f_p)$ and each f_j is a linear homogeneous polynomial, then $A \to B$ is a-flat.*

Proof. Denote by $\rho_i : B_i \to B$ the canonical morphisms. We can consider that each $B_i = \mathrm{S}_A(M_i)$ where M_i is an A-module of finite presentation. By flatness of $A \to B$ and Lazard's theorem, we get a factorization $M_i \to F_i \to B$ where F_i is free with finite rank. Taking symmetric algebras, we get a factorization $B_i \to L_i \to B$ where L_i is a-free. An appeal to Lemma 1.2 shows that $A \to B$ is a-flat. □

An a-flat A-algebra B is a direct limit of polynomial algebras L_λ with finite transcendence degree over A. We examine the situation when the set of integers $\mathrm{tr.deg}_A(L_\lambda)$ has an upper bound.

Definition 1.22. Let $A \to B$ be an a-flat morphism. We say that the flat rank f-rk(B) of B over A is $r \in \mathbb{N}$ if r is the least integer such that $B = \varinjlim L_\lambda$, $L_\lambda \in \mathscr{P}_A$ and $\mathrm{tr.deg}_A(L_\lambda) \leqslant r$ for each λ.

Proposition 1.23. *Let $A \to B$ be an a-flat morphism. The following statements are equivalent*:
(1) f-rk$(B) \leqslant r$.
(2) *B is a direct limit of polynomial A-algebras with transcendence degree r.*
(3) *Each morphism of A-algebras $P \to B$ where P is of finite presentation can be factored $P \to L \to B$ where* $\mathrm{tr.deg}_A(L) = r$ *and* $L \in \mathscr{P}_A$.
(4) *Each finitely generated A-subalgebra of B is contained in an A-subalgebra of B generated by r elements.*

Proof. Assume that (1) holds and consider $C=A[b_1,\dots,b_n]\subset B$. Denote the canonical morphisms by $\rho_\lambda : L_\lambda \to B$. There is an index μ such that $C\subset \rho_\mu(L_\mu)$ and $\rho_\mu(L_\mu)$ can be generated by r elements. Hence, (1) implies (4). Assuming that (4) is verified, we show (3). Consider a morphism of A-algebras $P\to B$ where P is of finite presentation. It can be factored $P\to L\to B$ where $L\in\mathscr{P}_A$. Let C be the image of L in B, there is an A-subalgebra $C'=A[b_1,\dots,b_r]$ of B which contains C. Set $L'=A[X_1,\dots,X_r]$, there is a surjective morphism $L'\to C'$. Since L is a free object, $L\to C\to C'$ can be factored $L\to L'\to C'$. Therefore, we get a factorization $P\to L'\to B$ of $P\to B$. Now (3) implies (2), it is enough to mimic the proof of Theorem 1.3. Obviously, (2) implies (1). □

Remark 1.24. Assume that $A\to B$ is a-flat of finite type. The flat rank of B is the least number $\lambda(B)$ of elements required to generate B over A.

If A is a field, B is an integral domain since a direct limit of integral domains. Then $\operatorname{tr.deg}_A(B)$ is defined and is $\leqslant \lambda(B)$. Therefore, the flat rank and the transcendence degree of B are equal if and only if $B\in\mathscr{P}_A$.

We intend to give some homological properties of a-flat morphisms. The following definition may be found in McDonald's book [22, p. 328]. For the definition and properties of stably free modules, see for instance Lam's book [16, I.4].

Definition 1.25. A ring A is called projectively trivial if each idempotent matrix over A is diagonalizable under a similarity transform.

According to [22, IV.49], if A is a connected projectively trivial ring, each of its finitely generated projective modules is free. The converse can be easily shown.

Definition 1.26. A ring A is called PPF if for each integer n, every finitely generated projective $A[X_1,\dots,X_n]$-module is free.

Hence, if A is connected and PPF, $A[X_1,\dots,X_n]$ is projectively trivial. If A is a principal ideal domain (or a Bézout domain such that prime ideals have finite heights), the Quillen–Suslin's theorem states that A is a PPF ring ([16,19]).

Definition 1.27. Let $A\to B$ be a ring morphism and $\mathbb{P}$ a property of rings. We say that $A\to B$ is a universal $\mathbb{P}$-morphism if $B\otimes_A A'$ has $\mathbb{P}$ for any base change $A\to A'$ where A' has $\mathbb{P}$.

An a-flat morphism $A\to B$ is a universal $\mathbb{P}$-morphism for many properties $\mathbb{P}$ like reduced, (integral) domain since B is a direct limit of polynomial algebras over A. However, this definition is not identical to the following usual definition.

Definition 1.28. Let $\mathbb{P}$ be a property of rings. Then $A\to B$ is called a $\mathbb{P}$-morphism if $A\to B$ is flat and for each prime ideal P of A, the ring $B\otimes_A K$ has $\mathbb{P}$ for every finite field extension $k(P)\to K$.

Therefore, if $A \to B$ is a flat universal $\mathbb{P}$-morphism, $A \to B$ is a $\mathbb{P}$-morphism. Actually, the flatness condition is verified in many cases by universal $\mathbb{P}$-morphisms. Recall that a universal reduced morphism $A \to B$ is flat if A is reduced [18, II, Proposition 2].

Theorem 1.29. *Let $A \to B$ be an a-flat morphism.*
(1) *$A \to B$ is a universal connected PPF morphism.*
(2) *If A is a connected PPF ring, so is B. Hence, every stably free projective B-module is free so that B is a Hermite ring.*

Proof. We can assume that A is a connected PPF ring. First observe that B is connected by Proposition 1.12. According to Proposition 1.14 (1), $A \to B[X_1, \dots X_n]$ is a-flat. Thus it is enough to show that a finitely generated projective B-module is free. By virtue of [22, IV.G.1], B is projectively trivial because B is a direct limit of projectively trivial rings. Use Definition 1.25 to complete. The statement (2) follows since a non finitely generated stably free module is free. □

Lemma 1.30. *Let A be a PPF regular integral domain. Then A is a unique factorization domain.*

Proof. Consider a nonzero divisorial ideal I of A. Since I is finitely generated over A and A is regular, its projective dimension is finite. Thus, according to [5, X.8.1, Proposition 2], I has a finite free resolution of finite length by Definition 1.26. It follows from [4, 4.7, Corollary 3] that A is a unique factorization domain. □

For simplicity's sake, we give [32, 1.1] as a reference for the following Popescu–Spivakovsky's theorem or Spivakovsky's paper for a more recent treatment [31].

Theorem 1.31. *Let $A \to B$ be a ring morphism between Noetherian rings. The following statements are equivalent*:
(1) *$A \to B$ is a regular morphism.*
(2) *B is a direct limit of smooth A-algebras (of finite type).*
(3) *Every morphism of A-algebras $P \to B$ where P is an A-algebra of finite presentation can be factored $P \to S \to B$ where S is a smooth A-algebra.*

Corollary 1.32. *An a-flat morphism $A \to B$ between Noetherian rings is a regular morphism.*

Artamonov showed the following result for algebraically projective algebras of finite type over a field [1, Proposition 7]

Theorem 1.33. *Let $A \to B$ be an a-flat morphism between Noetherian rings. Then $A \to B$ is a (regular) factorial morphism. If in addition, $A \to B$ is essentially of finite type, then $K \otimes_A B$ is a regular UFD for every ring morphism $A \to K$ where K is a field.*

Proof. Let P be a prime ideal of A and $\mathrm{k}(P) \to K$ a finite extension of fields, then $K \otimes_A B = F$ is a regular integral domain since $K \to F$ is a universal integral domain morphism (see Definition 1.27). Then use Lemma 1.30. Now, if $A \to B$ is essentially of finite type, $A \to B$ is a universal Noetherian morphism and the proof is complete. □

We have just seen that differential properties of a-flat morphisms are involved. If $A \to B$ is a ring morphism, we denote by $\Omega_A(B)$ the B-module of Kähler differentials of B over A.

Proposition 1.34. *Let $A \to B$ be an a-flat morphism. Then $\Omega_A(B)$ is a flat B-module.*

Proof. Since B is the direct limit of polynomial A-algebras L_λ in finitely many indeterminates, the conclusion follows from $\Omega_A(B) = \varinjlim(\Omega_A(L_\lambda) \otimes_{L_\lambda} B)$ and $\Omega_A(L_\lambda)$ is a free L_λ-module with finite rank. □

2. Algebraically projective morphisms

Definition 2.1. An A-algebra B is called algebraically projective (a-projective), if the natural map $\mathrm{Hom}_{A\text{-alg}}(B, C) \to \mathrm{Hom}_{A\text{-alg}}(B, D)$ is surjective for every surjective morphism of A-algebras $C \to D$.

The symmetric A-algebra $\mathrm{S}_A(P)$ of a projective A-module P is a-projective [9].

In the literature, a-projective algebras are called projective algebras or weakly projective algebras. The word projective has many meanings. So we have preferred to introduce another name. In this paper, a projective algebra is an A-algebra B such that the A-module B is projective. Retracts of algebras are defined in Definition 1.9.

Proposition 2.2. *Let B be an A-algebra. The following statements are equivalent*:
(1) *B is a-projective.*
(2) *B is a retract of a polynomial algebra.*
(3) *For every A-algebra R and every surjective morphism $C \to B$, the natural map $\mathrm{Hom}_{A\text{-alg}}(R, C) \to \mathrm{Hom}_{A\text{-alg}}(R, B)$ is surjective.*

Therefore, an a-projective morphism is universally a-projective, projective and faithfully flat.

Proof. (1) $\Leftrightarrow$ (2) is well known (for instance, see [9]). Assume that (1) holds and consider morphisms $R \to B$ and $C \to B$ where $C \to B$ is surjective. Then Id_B can be factored $B \to C \to B$ and (3) is proved. Clearly, (3) implies that B is a retract of a polynomial ring and (2) is shown. Assume that $f : B \to L$ defines B as a retract of a polynomial ring L. Then by (1.9), there is a direct sum $L = f(B) \oplus J$ of B-modules whence a direct sum of A-modules. Since L is free over A, B is projective over A. □

Lemma 2.3. *Let C be an A-algebra of finite presentation and B a retract of C. Then B is of finite presentation. Hence, an a-projective algebra of finite type is of finite presentation.*

Proof. Let $f : B \to C$ and $s : C \to B$ be the morphisms defining B as a retract. We need only to show that $J = \mathrm{Ker}(s)$ is of finite type since finite presentation is stable under composition. Let $\{c_1, \dots, c_n\}$ be a system of generators of the algebra C. From $C = f(B) \oplus J$ we deduce $c_i = b_i + x_i$ where $b_i \in B$ and $x_i \in J$ so that $\sum Cx_i \subset J$. Now let $x = p(x_1, \dots, x_n) \in J$ where $p(X_1, \dots, X_n) \in B[X_1, \dots X_n]$. Observe that $p(0, \dots, 0) = 0$ because $J \cap B = \{0\}$. It follows that $J \subset \sum Cx_i$. Now, an a-projective A-algebra B of finite type is a retract of a polynomial ring $L \in \mathscr{P}_A$ over A. □

The following result gives a partial answer to the question: is a projective algebra of finite type a polynomial algebra?

Theorem 2.4. *Let $A \to B$ be a ring morphism.*

(1) *If B is a-projective, then B is a-flat.*

(2) *If $A \to B$ is of finite presentation and a-flat, B is a-projective. In particular, if A is Noetherian or an integral domain and $A \to B$ is of finite type and a-flat, then B is a-projective.*

Hence, if $A \to B$ is of finite presentation, B is a-projective if and only if B is a-flat. In this case, B is a direct limit of polynomial algebras over A with transcendence degree $\text{f-rk}(B) = \lambda(B)$.

Proof. To show (1), use Proposition 2.2 (3) and the a-flatness definition. If $A \to B$ is of finite presentation and a-flat, Id_B can be factored $B \to L \to B$ where L is a polynomial algebra. Hence B is a-projective. Now, if $A \to B$ is of finite type and flat and A is Noetherian or an integral domain, $A \to B$ is of finite presentation [12, I.3.4.7]. For the last statement, see Proposition 1.23 and Remark 1.24. □

Ohm and Rush defined content modules [24]. A projective module is a content module. Moreover, Rush introduced weak content algebras [28]. We will use the following characterization. If B is an A-algebra such that B is a content module, B is weak content if and only if $PB = B$ or PB is a prime ideal for each prime ideal P of A [28, 1.2]. For instance, a polynomial algebra is weak content.

Proposition 2.5. *Let $A \to B$ be a ring morphism.*

(1) *If $A \to B$ is a weak content injective morphism, every finitely generated flat module over A is projective if and only if every finitely generated flat module over B is projective.*

(2) *If $A \to B$ is a-projective, then $A \to B$ is weak content and injective.*

Proof. The first result is quoted in [28, Note, p. 333] while the lacking proof is a consequence of [27]. Assume that B is a-projective. Then B is a content module over A because B is projective over A (see Proposition 2.2). Since B is a-flat, $A \to B$ is a universal domain morphism. It follows that PB is a prime ideal for each prime ideal P of A. Therefore, $A \to B$ is weak content. □

A ring A is called FGFP if each of its finitely generated flat modules is projective. Jöndrup showed that the FGFP property is stable under flat and finite morphisms [14].

An integral domain or a semilocal ring is FGFP. Moreover, A is FGFP if and only if $A[X]$ is FGFP [14].

Theorem 2.6. *Let $A \to B$ be an a-projective morphism. If A is a PPF connected FGFP ring (for instance a PID), then every finitely generated flat module over $B[X_1,\dots,X_n]$ is free.*

Proof. Use Proposition 2.5 and Theorem 1.29 since a-projective implies a-flat. □

We look at the differential properties of a-projective morphisms. In order to avoid many references, we use the definitions and results of [8] although they may be found elsewhere.

Proposition 2.7. *Let $A \to B$ be an a-projective morphism which is a retract of a polynomial algebra L. Denote by J the kernel of $L \to B$.*

(1) *$A \to B$ is formally smooth.*
(2) *There is a left-invertible morphism of B-modules $\Omega_A(B) \to \Omega_A(L) \otimes_L B$. Hence, $\Omega_A(B)$ is a projective B-module.*
(3) *There is an isomorphism of B-modules $J/J^2 \simeq \Omega_B(L) \otimes_L B$.*

Proof. Let C be an A-algebra and I an ideal of C such that $I^2 = 0$. The natural map $\mathrm{Hom}_{A\text{-alg}}(B,C) \to \mathrm{Hom}_{A\text{-alg}}(B,C/I)$ is surjective. Hence, $A \to B$ is formally smooth [8, X.7.2, Definition 1]. Then (2) can be shown in the same way as in [2, 6.5]. Consider the factorization $B \to L \to B$ of Id_B. Since Id_B is formally smooth, there is an exact sequence of B-modules $0 \to J/J^2 \to \Omega_B(L) \otimes_L B \to \Omega_B(B) \to 0$ [8, X.7.2, Remarques]. Since $\Omega_B(B)$ is zero, (3) follows. □

In the following, we consider only a-projective morphisms of finite type, hence of finite presentation by Lemma 2.3.

Theorem 2.8. *Let $A \to B$ be an a-projective morphism of finite type.*

(1) *$A \to B$ is a projective smooth morphism.*
(2) *$A \to B$ is a universal regular morphism.*
(3) *$K \otimes_A B$ is a regular unique factorization domain for every ring morphism $A \to K$ where K is a field.*
(4) *$R \otimes_A B$ is a connected PPF ring for every ring morphism $A \to R$ where R is a connected PPF ring. In this case, $\Omega_R(R \otimes_A B)$ is a free $R \otimes_A B$-module with finite rank.*

Proof. In view of Proposition 2.7, $A \to B$ is formally smooth and is of finite presentation. Thus, $A \to B$ is smooth. Now, if A is Noetherian, $A \to B$ is a universal regular morphism by [8, X.7.10, Theorem 4]. We can reduce to the Noetherian case by virtue of Proposition 2.9 (4). Hence (1) and (2) are proved. Now (3) follows from Theorem 1.33 since an a-projective morphism is a-flat. The first part of (4) is a consequence of Theorem 1.29. Set $S = R \otimes_A B$, then $R \to S$ is of finite type so that $\Omega_R(S)$ is a finitely generated S-module. By Proposition 2.7, $\Omega_R(S)$ is a projective S-module and hence is free with finite rank according to the first part of (4). □

The following result is well known (except (4)) and defines representations of a-projective algebras of finite type [9].

Proposition 2.9. *Let $A \to B$ be an a-projective morphism of finite type and $L = A[X_1,\dots,X_n] \to B$ defining B as a retract of L. Let J be the kernel of $L \to B$ and $u: A[X_1,\dots,X_n] \to A[X_1,\dots,X_n]$ the associated idempotent endomorphism of A-algebras. Then $\{f_i = u(X_i) | i = 1,\dots,n\}$ verifies*:

(1) $f_i(f_1,\dots,f_n) = f_i$ *for* $i = 1,\dots,n$.
(2) $J = \mathrm{Ker}(u) = (X_1 - f_1,\dots,X_n - f_n)$.
(3) $B \simeq \mathrm{Im}(u) = A[f_1,\dots,f_n] \simeq A[X_1,\dots,X_n]/J$.
Conversely, a sequence of polynomials $f_1,\dots,f_n \in A[X_1,\dots,X_n]$ verifying (1) *defines an a-projective algebra $A \to A[X_1,\dots,X_n](X_1 - f_1,\dots,X_n - f_n)$.*
(4) *There exist a Noetherian ring R, an a-projective ring morphism of finite type $R \to S$ and a ring morphism $R \to A$ such that $B = A \otimes_R S$.*

Proof. To show (4), consider the set G of all the coefficients of f_i. It is enough to take $\mathbb{Z}[G] = R \subset A$ and $S = R[X_1,\dots,X_n]/(X_1 - f_1,\dots,X_n - f_n)$. □

A sequence $\{f_1,\dots,f_n\}$ is called a representation of B and the ideal of the representation is $J = (X_1 - f_1,\dots,X_n - f_n)$. A representation $\{f_1,\dots,f_n\}$ is called standard if $f_i(0,\dots,0) = 0$ for each i.

Thanks to the following results, we can get more interesting representations.

Lemma 2.10. *Let $A \to B$ be an a-projective morphism of finite type, u an associated idempotent endomorphism defining a representation $\{f_1,\dots,f_n\}$ of B. Let φ be an A-automorphism of the algebra $A[X_1,\dots,X_n]$ and set $v = \varphi \circ u \circ \varphi^{-1}$.*

(1) *v is an idempotent endomorphism of the A-algebra $A[X_1,\dots,X_n]$ defining a representation $\{g_1,\dots g_n\}$ of B.*
(2) $(\varphi(X_1 - f_1),\dots,\varphi(X_n - f_n)) = (X_1 - g_1,\dots,X_n - g_n)$.

Proof. Obviously, φ induces an isomorphism of A-algebras $\mathrm{Im}(u) \to \mathrm{Im}(v)$ and we have $\varphi(\mathrm{Ker}(u)) = \mathrm{Ker}(v)$. □

Proposition 2.11. *Let $A \to B$ be an a-projective morphism of finite type which is not a polynomial algebra.*

(1) *B has a standard representation $\{g_1,\dots,g_n\} \subset A[X_1,\dots,X_n]$ such that its ideal contains a polynomial of the form $aX_n^s + p_{s-1}X_n^{s-1} + \cdots + p_1X_n$ where $a \in A$ is nonzero, $s \neq 0$ is an integer and $p_i \in A[X_1,\dots,X_{n-1}]$.*
(2) *Moreover, if A is Noetherian and B has a representation ideal J such that* $\mathrm{ht}(J) > \dim(A)$, *we can assume that $a = 1$.*

Proof. Let $\{f_1,\dots,f_n\}$ be a representation of B. We can assume that $f_n \notin A$. Let a_i be the constant term of f_i and define φ by $\varphi(X_i)=X_i+a_i$. We get $g_i=v(X_i)=\varphi(u(X_i-a_i))=f_i(X_1+a_1,\dots,X_n+a_n)-a_i$. Arguing as in [9, 3.2], we find that the constant term of g_i is zero. Thus we can assume that the representation is standard. Now, define ψ by $\psi(X_n)=X_n$ and $\psi(X_i)=X_i+X_n^{n_i}$. The constant term of each polynomial $v(X_i)$ is still zero. Following Nagata's proof of the Noether normalization Lemma, we can choose integers n_i such that $\psi(X_n-f_n)$ has the required form. Thus, (1) is shown. Now, (2) is an immediate consequence of a Suslin's result involving the same automorphism [20, 6.1.5]). □

Proposition 2.12. *Let A be a UFD and Q a prime ideal of $A[T]$ such that $Q\cap A=P$ and $P[T]\neq Q$. There is some irreducible polynomial $f(T)\in A[T]\backslash A$ such that $Q=P[T]+A[T]f(T)$.*

Proof. Set $B=A/P$ and consider the prime ideal Q' of $B[T]$ lying over Q so that $Q'\neq 0$ and $Q'\cap B=0$. Let $\bar{g}(T)$ be a polynomial of least positive degree in Q' (hence, $g(T)\in Q\backslash P[T]$). Pick an irreducible polynomial $f(T)$ in $Q\backslash P[T]$ dividing $g(T)$. The content ideal of $f(T)$ is A and thus the content ideal of $\bar{f}(T)$ is B. Then $\bar{f}(T)$ cannot lie in B and we can write $\bar{g}(T)=\bar{f}(T)\bar{h}(T)$. In this case, the degree of $\bar{h}(T)$ is zero (if not, we get $0<d^\circ(\bar{f}(T))<d^\circ(\bar{g}(T))$, contradicting the definition of $\bar{g}(T)$ since $\bar{f}(T)\in Q'$). It follows that $\bar{h}(T)=\bar{a}\in B$. Therefore, $\bar{f}(T)$ is a polynomial of least positive degree in Q' with content ideal B. A result of Sharma shows that $Q'=(\bar{f}(T))$ [30, Corollary 3] and the proof is complete. □

When M is a finitely generated A-module, we denote by $\mu(M)$ the minimal number of generators of M.

Proposition 2.13. *Let A be a Noetherian UFD and Q a prime ideal of $R=A[T]$ lying over P in A. Assume that Q contains a monic polynomial, Q/Q^2 is R/Q-free and that stably free R/Q-modules are free, then $\mu(Q)=\mu(Q/Q^2)$.*

Proof. By Proposition 2.12, we have $Q=(P,f(T))$ since Q contains a monic polynomial. A result by Mandal and Roy gives the conclusion [21, 3.6]. □

Theorem 2.14. *Let $A\to B$ be an a-projective morphism of finite type with representation ideal J in $L=A[X_1,\dots,X_n]$.*
(1) *If A is a connected PPF ring, then J/J^2 and $\Omega_A(B)$ are free B-modules such that $n=\mathrm{rk}_B(\Omega_A(B))+\mathrm{rk}_B(J/J^2)$.*
(2) *If A is a field, J is a completely secant prime ideal so that $\mathrm{S}_A(J/J^2)\simeq \mathrm{gr}_J(L)$. Moreover, J/J^2 and $\Omega_A(B)$ are free B-modules such that $\mathrm{rk}_B(\Omega_A(B))=\dim(B)$ and $\mathrm{ht}(J)=\mathrm{rk}_B(J/J^2)=n-\dim(B)$.*
(3) *If A is a connected Noetherian PPF ring, $\dim(B)\leqslant\dim(A)+\mathrm{rk}_B(\Omega_A(B))$.*
(4) *If A is an affine PPF integral domain over a field K, so is the ring B and $\mathrm{rk}_B(J/J^2)\leqslant\mathrm{ht}(J)\leqslant\mu(J)$ holds for the prime ideal J.*

An example of affine PPF integral domain A over a field is given by an a-projective algebra of finite type over a field.

Proof. Let A be a connected PPF ring. Since $A \to B$ is smooth by Theorem 2.8, there is an isomorphism of B-modules $\Omega_A(L) \otimes_L B \simeq J/J^2 \oplus \Omega_A(B)$ induced by the split exact sequence $0 \to J/J^2 \to \Omega_A(L) \otimes_L B \to \Omega_A(B) \to 0$ [8, X.7.2, Remark 1]. Observe that $\Omega_A(L) \otimes_L B$ is a free B-module with rank n. Therefore, J/J^2 and $\Omega_A(B)$ are finitely generated projective B-modules. These B-modules are free with finite rank by Theorem 2.8 (4) and we get $n = \mathrm{rk}_B(\Omega_A(B)) + \mathrm{rk}_B(J/J^2)$. If A is a field, B is an integral domain so that J is a prime ideal. From Theorem 2.8 (2), we deduce that B is a regular ring. Now L is a regular ring as well as B. It follows that J is completely secant by [8, X.5.3, Proposition 2] and $\mathrm{S}_A(J/J^2) \simeq \mathrm{gr}_J(L)$ is a consequence of [8, X.5.2,Theorem 1]. To complete the proof of (2), it is enough to show that $\mathrm{rk}_B(\Omega_A(B)) = \dim(B)$. If M is a maximal ideal of the affine integral domain B with quotient field K, then $\dim(B_M) = \dim(B) = \mathrm{tr.deg}_A(K)$ (the quotient field of B_M is K) [7, VIII.2.4, Theorem 3]. From $\Omega_A(B_M) \simeq \Omega_A(B)_M$ and [8, X.6.5, Theorem 1], we deduce that $\mathrm{rk}_B(\Omega_A(B)) = \mathrm{rk}_{B_M}(\Omega_A(B)_M) = \mathrm{tr.deg}_A(K)$ because $A \to B_M$ is a regular morphism [8, X.6.4, Proposition 6]. Now assume that A is a connected PPF Noetherian ring. In view of (1), the B-module $\Omega_A(B)$ is free with finite rank. Let P be a prime ideal of A and set $F(P) = B \otimes_A \mathrm{k}(P)$. Then $\mathrm{rk}_{F(P)}(\Omega_{\mathrm{k}(P)}(F(P))) = \mathrm{rk}_B(\Omega_A(B))$ follows from $\Omega_A(B) \otimes_B F(P) \simeq \Omega_{\mathrm{k}(P)}(F(P))$. According to (2), we get $\mathrm{rk}_{F(P)}(\Omega_{\mathrm{k}(P)}(F(P))) = \dim(F(P))$ since $\mathrm{k}(P) \to F(P)$ is a-projective so that $\dim(F(P)) = \mathrm{rk}_B(\Omega_A(B))$. It follows that $\dim(B) \leqslant \dim(A) + \mathrm{rk}_B(\Omega_A(B))$ by [7, VIII.3.4, Corollary 2]. Thus (3) is shown. If A is an affine PPF integral domain, so are L and B because $K \to L$ is of finite type as well as $K \to B$. Since L is an affine integral domain, we get from (3) that $\dim(B) - \dim(A) = n - \mathrm{ht}(J) \leqslant \mathrm{rk}_B(\Omega_A(B)) = n - \mathrm{rk}_B(J/J^2)$. Therefore, (4) is proved since $\mathrm{ht}(J) \leqslant \mu(J)$ holds for an arbitrary Noetherian ring. □

Theorem 2.15. *Let K be a PPF affine regular integral domain* (*for instance*, *an a-projective algebra of finite type over a field*) *and $K \to B$ an a-projective morphism of finite type which is not a polynomial algebra. Then each representation ideal J of B such that* $\mathrm{ht}(J) > \dim(K)$ *is a complete intersection and* $\dim(B) = \mathrm{rk}_B(\Omega_K(B))$.

In particular, *if K is a field then each representation ideal of B is a complete intersection ideal.*

Proof. Let $\{f_1,\dots,f_n\} \subset K[X_1,\dots,X_n]$ be a representation of B and denote by J the associated representation ideal. First assume that $n = 1$. In this case $f_1(X_1) = a \in K$ or $f_1(X_1) = X$ [9, 3.4] which yields $J = (X - a)$ or $J = 0$. Now assume that $n > 1$. We set $A = K[X_1,\dots,X_{n-1}]$, $X_n = T$ so that $B = A[T]/J$ where A is a Noetherian UFD since K is a UFD by (1.30). According to (2.11)(2), we can assume that J contains a monic polynomial of $A[T]$. Hence, $\mu(J) = \mu(J/J^2)$ follows from (2.13). Now, $\mathrm{rk}_B(J/J^2) \leqslant \mathrm{ht}(J) \leqslant \mu(J)$ is a consequence of (2.14)(4) and then $\mu(J/J^2) = \mathrm{rk}_B(J/J^2)$ implies $\mathrm{ht}(J) = \mu(J)$. It follows that J is a complete intersection ideal. Moreover, B is an affine integral domain and we have $n = \mathrm{ht}(J) + \mathrm{rk}_B(\Omega_K(B))$ so that $\dim(B) = \mathrm{rk}_B(\Omega_K(B))$. □

Definition 2.16. We call a ring B a global complete intersection ring if $B \simeq A[X_1,\dots,X_n]/J$ where A is a regular ring and J is a complete intersection ideal (generated by a regular sequence).

It follows that $A[X_1,\dots X_n]$ is a global complete intersection ring when A is a regular ring. We do not know whether the previous definition is independent of the presentation of the ring B although this is known for local rings.

The adjective global is added because of possible confusions with complete intersection rings (rings which are locally complete intersection).

Corollary 2.17. *Let $A \to B$ be an a-projective morphism of finite type. Then $A \to B$ is a global complete intersection morphism.*

Let $A \to B$ be an a-projective morphism of finite type. In view of Proposition 2.11 (1), $A \to B$ has a retract $B \to A$ with kernel $I = (f_1,\dots,f_n)$. Tronin used this fact to exhibit some morphims [33]. Consider the ideal $M = (X_1,\dots,X_n)$ of $L = A[X_1,\dots,X_n]$. There is a factorization $B = A \oplus I \xrightarrow{\varphi} L = A \oplus M \xrightarrow{\sigma} B = A \oplus I$ of Id_B where $\varphi : B \to L$ is the canonical injection and σ is defined by $\sigma(X_i) = f_i$. This factorization induces injective morphisms of A-algebras

$$\Phi : B \xrightarrow{\varphi} L \xrightarrow{\simeq} \mathrm{S}_A(M/M^2) \xrightarrow{\mathrm{S}(\bar\sigma)} \mathrm{S}_A(I/I^2) = B',$$

$$\Psi : B' = \mathrm{S}_A(I/I^2) \xrightarrow{\mathrm{S}(\bar\varphi)} \mathrm{S}_A(M/M^2) \xrightarrow{\simeq} L \xrightarrow{\sigma} B.$$

Now, observe that $B' \simeq \mathrm{S}_A(\Omega_A(B) \otimes_B A)$ since the exact sequence

$$0 \to I/I^2 \to \Omega_A(B) \otimes_B A \to \Omega_A(A) \to 0,$$

ensures us that $I/I^2 \simeq \Omega_A(B) \otimes_B A$ and $\mathrm{rk}_B(\Omega_A(B)) = \mathrm{rk}_A(I/I^2)$.

Using our previous results, we can improve a result by Tronin [33].

Proposition 2.18. *Let $A \to B$ be an a-projective morphism of finite type.*

(1) *The following sequences are exact*

$$0 \to \Omega_A(B) \otimes_B B' \to \Omega_A(B') \to \Omega_B(B') \to 0,$$

$$0 \to \Omega_A(B') \otimes_{B'} B \to \Omega_A(B) \to \Omega_{B'}(B) \to 0.$$

(2) *If A is a connected PPF ring and $\mathrm{rk}_B(\Omega_A(B)) = r$, then $\mathrm{rk}_A(I/I^2) = r$ and there are two injective morphisms of A-algebras*

$$B \xrightarrow{\alpha} A[X_1,\dots,X_r] = B' \text{ and } B' = A[X_1,\dots,X_r] \xrightarrow{\beta} B,$$

where $r = \dim(B)$ when A is a field.

(3) *If A is a PPF integral domain, α and β induce separable algebraic extensions of the quotient fields.*

Proof. See [33] for a proof of (1). To show (2), observe that $\Omega_A(B)$ is a free B-module of rank r by (2.14) while $I/I^2 \simeq \Omega_A(B) \otimes_B A$ and $B' = \mathrm{S}_A(I/I^2) \simeq A[X_1,\dots,X_r]$. Next notice that $\dim(B) = \mathrm{rk}_B(\Omega_A(B))$ when A is a field by Theorem 2.14. We prove (3). Let K and K' be the respective quotient fields of B and B'. Tensoring the first exact sequence with $\otimes_{B'} K'$ gives an exact sequence of K'-vector spaces since $B' \to K'$ is flat. The first two K'-vector spaces have the same rank r so that $\Omega_K(K') \simeq \Omega_B(B') \otimes_{B'} K' = 0$. The conclusion follows from [6, V.16.6, Corollary 2]. □

Remark 2.19. Costa proved that when A is a field and $A \to B$ is a-projective of finite type with representation $\{f_1, f_2\} \subset A[X_1, X_2]$ or such that $\dim(B)=2$, then $B=A[X_1, X_2]$ (see [9, 3.5]). We can recover this result thanks to Proposition 2.18. Let A be a perfect field and $A \to B$ an a-projective algebra of finite type with $\dim(B) = 2$. The A-algebra B is isomorphic to $A[X_1, X_2]$. Indeed, the hypotheses of Castelnuovo's affine theorem are fulfilled [29, Theorem 3] since in this case $B' = A[X_1, X_2]$, B is regular, $K \otimes_A B$ is a UFD for every morphism $A \to K$ where K is a field and the quotient fields extension is separable by Proposition 2.18. If A is not perfect, let $A \to C$ where C is an algebraic closure of A. Then $A \to C$ is faithfully flat and we can use the descent result of Proposition 2.23.

The previous proposition cannot be used to prove that B is isomorphic to a polynomial algebra when $\dim(B) > 2$ since Castelnuovo's Theorem is no longer true when $d > 2$ [15, p. 297].

We give here some descent results.

Proposition 2.20. *Algebraically pure morphisms descend a-projective algebras of finite presentation.*

Proof. Observe that a pure morphism descends algebras of finite presentation [25, 5.3]. To conclude use Theorems 2.4 and 1.16. □

A ring morphism $A \to A'$ is called strongly Nakayama if for every A-module M, the equation $M \otimes_A A' = 0$ implies $M = 0$. A strongly Nakayama morphism $A \to A'$ descends the surjectivity of A-module morphisms [25].

Lemma 2.21. *Let $A \to B$ be a ring morphism and $A \to A'$ a strongly Nakayama morphism. If $\{b_\lambda\}$ is a family of elements in B such that $\{b_\lambda \otimes 1\}$ generates the A'-algebra $B \otimes_A A'$, then so does $\{b_\lambda\}$ in B.*

Proof. Consider the morphism of A-algebras $A[X_\lambda] \to B$ defined by $X_\lambda \mapsto b_\lambda$. Then $A[X_\lambda] \otimes_A A' \to B \otimes_A A'$ is surjective and so is $A[X_\lambda] \to B$. □

Proposition 2.22. *Let $A \to B$ and $A \to A'$ be ring morphisms. If $\{b_\lambda\} \subset B$ is a family such that $B \otimes_A A' = A'[b_\lambda \otimes 1]$ is a polynomial A'-algebra with respect to the elements $b_\lambda \otimes 1$, then $B = A[b_\lambda]$ is a polynomial A-algebra with respect to the elements b_λ in the following cases*:

(1) *$A \to A'$ is faithfully flat.*
(2) *The kernel of the morphism $A[X_\lambda] \to B$ defined by $X_\lambda \mapsto b_\lambda$ is a pure A-submodule of $A[X_\lambda]$ and $A \to A'$ is strongly Nakayama.*

Proof. In both cases, $A[X_\lambda] \otimes A' \to B \otimes A'$ is bijective so that $A[X_\lambda] \to B$ is surjective with kernel I. Then tensor the exact sequence $0 \to I \to A[X_\lambda] \to B \to 0$ by $\otimes_A A'$. The new sequence is exact and then $I \otimes_A A' = 0$ implies $I = 0$. □

Proposition 2.23. *Let $A \to B$ and $A \to A'$ be ring morphisms such that the A'-algebra $B \otimes_A A'$ is isomorphic to $A'[X_1, \ldots, X_n]$. The A-algebra B is isomorphic to $A[X_1, \ldots, X_n]$ in the following two cases*:
(1) *$A \to A'$ is faithfully flat.*
(2) *$A \to B$ is projective of finite type and $A \to A'$ is strongly Nakayama.*

Proof. Denote by f the isomorphism $B \otimes A' \to A'[X_1, \ldots, X_n]$ and set $f^{-1}(X_i) = \sum b_j \otimes a'_j$. Then $\{b_j \otimes 1\}$ generates the A'-algebra $B \otimes A'$ and $\{b_j\}$ generates the A-algebra B by Lemma 2.21. Let $u : A[X_1, \ldots, X_n] \to B$ be the surjective morphism defined by $X_j \mapsto b_j$ with kernel I. The composite morphism $f \circ (u \otimes \mathrm{Id}_{A'})$ is a surjective endomorphism of the A'-algebra of finite type $A'[X_1, \ldots, X_n]$, whence an isomorphism. Thus, $u \otimes \mathrm{Id}_{A'}$ is an isomorphism and so is u thanks to (2.22) if $A \to A'$ is faithfully flat. If $A \to B$ is projective, $A[X_1, \ldots, X_n] = I \oplus B$ implies that I is a pure A-submodule of $A[X_1 \ldots, X_n]$ and the proof can be completed as above. □

Next we give some informations on differential properties of a-projective algebras. For each positive integer m, we denote by $\mathrm{M}_m(R)$ the ring of all size m squared matrices with entries in the ring R and by $\mathrm{LG}_m(R)$ the set of all units in $\mathrm{M}_m(R)$. A ring morphism $\varphi : R \to S$ induces a ring morphism $\varphi_m : \mathrm{M}_m(R) \to \mathrm{M}_m(S)$ with kernel $\mathrm{M}_m(\mathrm{Ker}(\varphi))$. Let A be a ring and $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ defining an A-endomorphism $u : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n]$ by $u(X_i) = f_i$. We consider the jacobian matrix $\mathrm{J}_u = (\partial f_j / \partial X_i) \in \mathrm{M}_n(A[X_1, \ldots, X_n])$ where i is the index of the row and j the index of the column. Now let u, v be two A-endomorphisms of $A[X_1, \ldots, X_n]$. The rule of chained derivations gives here $\mathrm{J}_{v \circ u} = \mathrm{J}_v v(\mathrm{J}_u)$.

Let $A \to B$ be an a-projective morphism of finite type with representation $\{f_1, \ldots, f_n\} \subset A[X_1, \ldots, X_n]$ and $u : A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n]$ the associated idempotent endomorphism defined by $u(X_i) = f_i$. We get $\mathrm{J}_u = \mathrm{J}_{u^2} = \mathrm{J}_u u(\mathrm{J}_u)$ so that $u(\mathrm{J}_u)$ is an idempotent matrix of $\mathrm{M}_n(A[X_1, \ldots, X_n])$ and its determinant lies in $\mathrm{Bool}(A)$. The ideal of $A[X_1, \ldots, X_n]$ generated by the entries of $u(\mathrm{J}_u)$ is idempotent whence generated by an element of $\mathrm{Bool}(A)$.

Now assume that A is a connected PPF ring. Then $u(\mathrm{J}_u)$ is diagonalizable under a similarity transform. Thus there is some $M \in \mathrm{LG}_n(A[X_1, \ldots, X_n])$ such that $Mu(\mathrm{J}_u)M^{-1} = \mathrm{Diag}(1, \ldots, 1, 0, \ldots, 0)$ where the last matrix is diagonal with r nonzero entries. The kernel of the canonical surjective morphism $p : A[X_1, \ldots, X_n] \to B$ is $(X_1 - f_1, \ldots, X_n - f_n)$ and $p(M)p(u(\mathrm{J}_u))p(M)^{-1} = \mathrm{Diag}(1, \ldots, 1, 0, \ldots, 0)$. As usual, set $p(\partial f_j / \partial X_i) = \partial f_j / \partial x_i$ where x_i denotes the class of X_i in B. Therefore, the relation $\mathrm{Diag}(1, \ldots, 1, 0, \ldots, 0) = p(M)(\partial f_j / \partial x_i)p(M)^{-1}$ where $p(M) \in \mathrm{LG}_n(B)$ follows from $p(X_i) = p(f_i)$.

Proposition 2.24. *Let B be an a-projective algebra of finite type over a connected PPF ring A and u an associated idempotent endomorphism defining a representation $\{f_1, \ldots, f_n\}$. Then $u(\mathrm{J}_u)$ is similar to the matrix* $\mathrm{Diag}(1, \ldots, 1, 0, \ldots, 0)$ *with*

$\mathrm{rk}_B(\Omega_A(B))$ *nonzero entries. If the representation is standard*, $\mathrm{rk}_B(\Omega_A(B)) = \mathrm{rk}_A((f_1,\dots,f_n)/(f_1,\dots,f_n)^2)$.

Proof. Let α be the B-module endomorphism of B^n with matrix $(\partial(x_j - f_j)/\partial x_i) = \mathrm{I}_n - (\partial f_j/\partial x_i)$ in the canonical basis of B^n . Since α is idempotent, we get $B^n = \mathrm{Im}(\alpha) \oplus \mathrm{Ker}(\alpha)$. Then observe that $\Omega_A(B) \simeq B^n/\mathrm{Im}(\alpha) \simeq \mathrm{Ker}(\alpha)$. The result follows immediately, the last statement being a consequence of (2.18)(2). □

We come back to Lemma 2.10, where an A-automorphism φ of $A[X_1,\dots,X_n]$ is considered as well as $v=\varphi\circ u\circ\psi$ where $\psi=\varphi^{-1}$. Then $\varphi\circ u=v\circ\varphi$ gives $\mathrm{J}_\varphi\varphi(\mathrm{J}_u)=\mathrm{J}_v v(\mathrm{J}_\varphi)$ while $\varphi\circ\psi=\mathrm{Id}=\psi\circ\varphi$ gives $\mathrm{J}_\varphi\varphi(\mathrm{J}_\psi)=I_n$ and $\mathrm{J}_\psi\psi(\mathrm{J}_\varphi)=I_n$ so that $\varphi(\mathrm{J}_\psi)\mathrm{J}_\varphi=I_n$. It follows that $\mathrm{J}_v=\mathrm{J}_\varphi\varphi(\mathrm{J}_u)v(\mathrm{J}_\varphi)^{-1}=\mathrm{J}_\varphi\varphi(\mathrm{J}_u)v(\varphi(\mathrm{J}_\psi))=\mathrm{J}_\varphi\varphi(\mathrm{J}_u)\varphi(u(\mathrm{J}_\psi))$.

Now consider a matrix $M=(\alpha_{ij})\in\mathrm{M}_n(A)$ and the associated A-endomorphism φ defined by $\varphi(X_j)=\sum_i\alpha_{ij}X_i$ for $j=1,\dots,n$ that is to say φ is defined by the matrix equation $(\varphi(X_1)\dots\varphi(X_n))=(X_1\dots X_n)M$. Obviously, we have $M=\mathrm{J}_\varphi$. Now assume that $M\in\mathrm{LG}_n(A)$. With the previous notation, we get that $v(\mathrm{J}_\varphi)=\mathrm{J}_\varphi$ so that $\mathrm{J}_v=\mathrm{J}_\varphi\varphi(\mathrm{J}_u)\mathrm{J}_\varphi^{-1}$ and $v(\mathrm{J}_v)=\mathrm{J}_\varphi\varphi(u(\mathrm{J}_u))\mathrm{J}_\varphi^{-1}$.

Proposition 2.25. *Let $A\to B$ be an a-projective morphism of finite type with a standard representation $\{f_1,\dots,f_n\}$ associated to the idempotent endomorphism u. Let h_i be the degree one homogeneous component of f_i so that there is a matrix equation* $(h_1\dots h_n)=(X_1\dots X_n)\mathrm{J}_u(0,\dots,0)$.

(1) *$\{h_1,\dots,h_n\}$ defines a representation of an a-projective algebra B_1. Its associated idempotent endomorphism h is defined by* $\mathrm{J}_h=\mathrm{J}_u(0,\dots,0)$.
(2) *If in addition A is a connected PPF ring, the A-algebra B_1 is isomorphic to $A[X_1,\dots,X_r]$ where* $r=\mathrm{rk}_B(\Omega_A(B))$

Proof. (1) is obvious since $h_i(h_1,\dots,h_n)=h_i$. Assume that A is a connected PPF ring. Denote by $s: A[X_1,\dots,X_n]\to A$ the substitution morphism defined by $s(X_i)=0$ and observe that $\mathrm{J}_h=s(\mathrm{J}_u)=s(u(\mathrm{J}_u))$. There is an equation $Mu(\mathrm{J}_u)M^{-1}=\mathrm{Diag}(1,\dots,1,0,\dots,0)$ where $M\in\mathrm{LG}_n(A[X_1,\dots,X_n])$. Thus we get $s(M)\mathrm{J}_h s(M)^{-1}=\mathrm{Diag}(1,\dots,1,0,\dots,0)$ where the number of nonzero entries is $r=\mathrm{rk}_B(\Omega_A(B))$ and $s(M)\in\mathrm{LG}_n(A)$. Now $s(M)$ defines an A-automorphism φ of $A[X_1,\dots,X_n]$. Then $k=\varphi\circ h\circ\varphi^{-1}$ is an A-endomorphism associated to the matrix $\mathrm{Diag}(1,\dots,1,0,\dots,0)$ so that $k(X_1)=X_1,\dots,k(X_r)=X_r$ and $k(X_i)=0$ for $i>r$. Hence B_1 is isomorphic to $A[X_1,\dots,X_r]$. □

Remark 2.26. If A is a PPF affine regular integral domain, $\dim(B)=\dim(B_1)$.

Remark 2.27. Assume that A is a connected PPF ring. Consider the A-automorphism φ defined in Proposition 2.25, $v=\varphi\circ u\circ\varphi^{-1}$ and set $v(X_i)=g_i$. From $f_i=h_i+t_i$ where $t_i\in(X_1,\dots,X_n)^2$, we get that $X_1-g_1,\dots,X_r-g_r\in(X_1,\dots,X_n)^2$ and $g_{r+1},\dots,g_n\in(X_1,\dots,X_n)^2$. It follows that $g_{r+1},\dots,g_n\in(g_1,\dots,g_n)^2$. Hence the classes of $g_1,\dots,g_r$ in $(g_1,\dots,g_n)/(g_1,\dots,g_n)^2$ give a basis of this A-module (see Proposition 2.18 (2)).

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