

# Random matrix theory and $L$ -functions

Zeros of symmetric power  $L$  functions

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# What is a $L$ -function?

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$$\begin{aligned} L(\omega, s) &= \sum_{n=1}^{+\infty} \lambda_{\omega}(n) n^{-s} \\ &= \prod_{p \in \mathcal{P}} \prod_{j=1}^d \left[ 1 - \frac{\alpha_{\omega, j}(p)}{p^s} \right]^{-1} \quad (\alpha_{\omega, j}(p) \in \mathbb{C}) \end{aligned}$$

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We assume  $\alpha_{\omega, j}(p) \ll 1$  hence the series and product converge absolutely for  $\Re s > 1$ .

# What is a $L$ -function? (cont.)

- 2 A gamma factor:

$$\gamma(\omega, s) = \pi^{-ds/2} \prod_{j=1}^d \Gamma\left(\frac{s + \kappa_j}{2}\right)$$

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with

$$\kappa_j \in \mathbb{R}_{>-1} \text{ or } \left\{ \begin{array}{l} \kappa_j \in \mathbb{C} \\ \exists j' : \kappa_{j'} = \overline{\kappa_j} \\ \Re \kappa_j > -1. \end{array} \right.$$

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hence  $\gamma(\omega, s)$  has no zeros in  $\mathbb{C}$  and no pole in  $\Re s \geq 1$ .

# What is a $L$ -function? (cont.)

- 3 A positive integer  $q(\omega) \geq 1$ : the **conductor** of  $L(\omega, s)$  such that:

$$p \nmid q(\omega) \implies \alpha_{\omega, j}(p) \neq 0.$$

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- ④ A **complete**  $L$ -function:

$$\Lambda(\omega, s) = q(\omega)^{s/2} \gamma(\omega, s) L(\omega, s)$$

that is holomorphic in  $\Re s > 1$  and is required to admit analytic continuation to a meromorphic function (of order 1) on  $\mathbb{C}$  with at most poles at  $s = 0$  and  $s = 1$ .

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where  $\varepsilon(\omega)$  is a complex number of norm 1 and  $L(\bar{\omega}, s)$  is the “dual” of  $L(\omega, s)$ :

- $\lambda_{\bar{\omega}}(n) = \overline{\lambda_{\omega}(n)}$
- $\gamma(\bar{\omega}, s) = \gamma(\omega, s)$
- $q(\bar{\omega}, s) = q(\omega, s)$ .

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- hence we shall use it only in particular cases where it appears “natural”.
- In general, at least one parameter is to be taken in account, the conductor: a family  $\mathcal{F}_Q$  has a parameter  $Q$  such that:

$$L(\omega, s) \in \mathcal{F}_Q \implies q(\omega) = Q.$$

# Random matrix theory and $L$ -functions: general view

$\mathcal{F}_Q$ : family

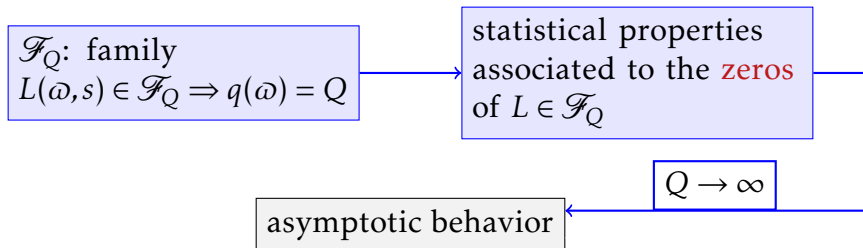
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statistical properties  
associated to the **zeros**  
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with measure  $G(N) \subset$   
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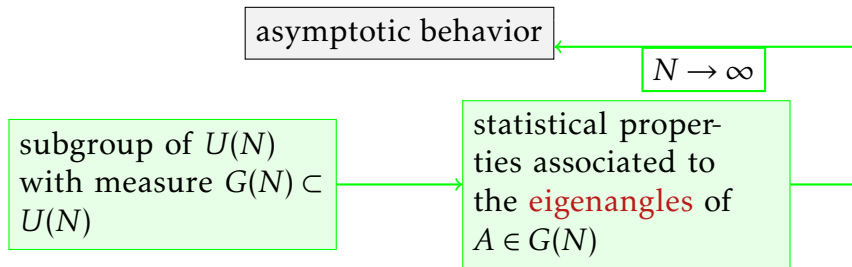
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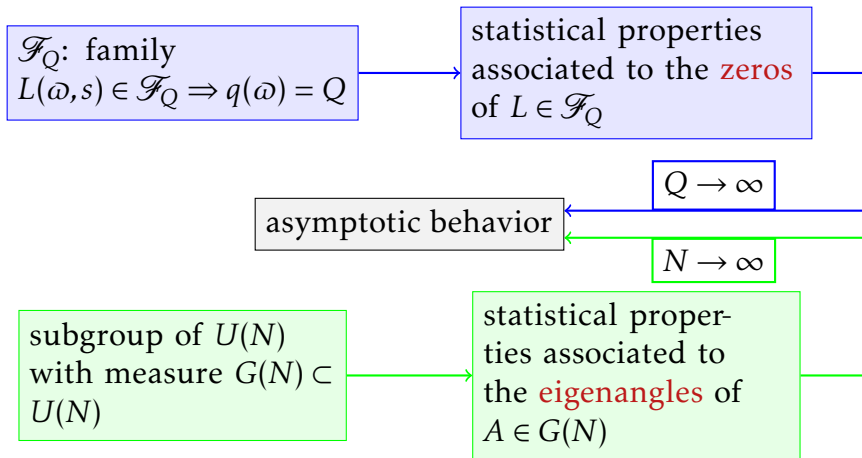


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# Parabolic forms

Let  $k \geq 2$  be an even integer and  $N \geq 1$  a **squarefree** integer.

A **parabolic (modular) form** of weight  $k$  and level  $N$  is a holomorphic function  $f$  on the Poincaré upper half-plane

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① For any  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : c \mid N \right\},$

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We get a finite dimensional vector space over  $\mathbb{C}$ .

# Dirichlet series of parabolic forms

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$$\Delta(f, s) = \left( \frac{N}{4\pi^2} \right)^{s/2} \Gamma(s) D(f, s).$$

# Dirichlet series of parabolic forms (cont.)

Let  $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$  and define

$$f|W_N(z) = (\det W_N)^{k/2} (Nz)^{-k} f\left(\frac{-1}{Nz}\right).$$

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$$\Delta(f, s) = i^k \int_1^{+\infty} f|W_N\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t} + \int_1^{+\infty} f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}$$

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hence

$$\Delta(f, s) = i^k \Delta(f|W_N, k-s).$$

# Wanted: Euler product

However these Dirichlet series lack an **Euler product** (multiplicativity properties for the Fourier coefficients) to be  $L$ -functions! To enter the world of  $L$ -functions, we need the theory of Atkin & Lehner. It will be possible only on a subspace of parabolic forms, called the subspace of **newforms**.

# Scalar product

On the space of modular forms, we have the **Petersson scalar product**:

$$(f, g) = \int_{\Gamma_0(N) \backslash \mathcal{H}} f(x + iy) \overline{g(x + iy)} y^k \frac{dx dy}{y^2}$$

where  $\Gamma_0(N) \backslash \mathcal{H}$  is any representative set of the homographic action of  $\Gamma_0(N)$  on  $\mathcal{H}$ :

$$M.z \sim z \Leftrightarrow M \in \Gamma_0(N).$$

# Hecke operators

For any integer  $n$ , the  $n$ th **Hecke operator**  $T_n$  defined by:

$$T_n f(z) = \frac{1}{n} \sum_{\substack{ad=n \\ (a,N)=1}} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{cz+d}\right)$$

acts on parabolic forms of weight  $k$  and level  $N$ . The Hecke operators commute, more precisely they enjoy the following multiplicativity relation:

$$T_m T_n = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d^{k-1} T_{mn/d^2}.$$

# Hecke operators (cont.)

Nearly all Hecke operators are selfadjoint: if  $f$  and  $g$  are parabolic forms of weight  $k$  and level  $N$  and if  $n$  is coprime to  $N$  then

$$(T_n f, g) = (f, T_n g).$$

Therefore, we can find an orthogonal basis of the space of parabolic forms of weight  $k$  and level  $N$ , of eigenvectors of  $\{T_n, (n, N) = 1\}$ .

# Hecke operators (cont.)

The  $m$ th Fourier coefficient of  $T_n f$  is given in terms of the Fourier coefficients of  $f$  by:

$$\widehat{T_n f}(m) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d^{k-1} \widehat{f}\left(\frac{mn}{d^2}\right)$$

hence, if  $f$  is an eigenvector of any  $T_n$  with  $(n, N) = 1$  the eigenvalue  $t_f(n)$  satisfies

$$\widehat{f}(n) = \widehat{f}(1)t_f(n).$$

Since we can build parabolic forms  $f \neq 0$  with  $\widehat{f}(1) = 0$  that are eigenvectors of  $\{T_n, (n, N) = 1\}$  this is **not** an **interesting** information. L'Université est une chance. *Université de Savoie*

# Atkin-Lehner theory

Define the space of **old forms** of level  $N$ :

$$\text{Old}_k(N) = \text{Vect}\{z \mapsto f(Lz) : LM \mid N, M \neq N, f \text{ of level } M\}$$

and the space of **newforms**:

$$\text{New}_k(N) = \text{Old}_k(N)^\perp.$$

Atkin & Lehner theory provides an orthogonal basis of  $\text{New}_k(N)$  made of eigenvectors of **all** the Hecke operators. We can normalize these eigenvectors to have first Fourier coefficient equal to 1. This basis is denoted by  $H_k^*(N)$  and its member are called **primitive forms**

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We normalize the Fourier coefficients  $\lambda_f(n) = \frac{\widehat{f}(n)}{n^{(k-1)/2}}$ .

# $L$ -functions of primitive forms

Let  $f \in H_k^*(N)$  and define its  $L$ -function by

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- Euler product:

$$L(f, s) = \prod_{p \in \mathcal{P}} \left( 1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_f(p)}{p^s} \right)^{-1}$$

where  $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ ,  $\alpha_f(p) = \beta_f(p)^{-1} = \overline{\beta_f(p)}$  if  $(p, N) = 1$  and  $\beta_f(p) = 0$  if  $p \mid N$ .

# $L$ -functions of primitive forms (cont)

- gamma factor

$$\gamma(f, s) = \pi^{-s} \Gamma\left(\frac{s + (k-1)/2}{2}\right) \Gamma\left(\frac{s + (k+1)/2}{2}\right)$$

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- the completed  $L$  function admits an entire continuation
- functional equation:  $\Lambda(f, s) = i^k \varepsilon_f(N) \Lambda(f, 1 - s)$ .

# $L$ -functions of primitive forms: zeros

The zeros of  $\Lambda(f, s)$  – that is the zeros of  $L(f, s)$  that do not compensate the poles of the gamma factor – are inside the **critical strip**:

$$0 \leq \Re s \leq 1$$

and if  $\rho$  is a zero, then  $\bar{\rho}$ ,  $1 - \rho$  and  $1 - \bar{\rho}$  also are.

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If  $\Lambda(f, 1/2 + i\gamma_f) = 0$ , we associate a normalised zero:

$$\tilde{\gamma}_f = \frac{\log(k^2 N)}{2\pi} \gamma_f.$$

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# Mean spacing

Let  $\mathcal{Z}(f)$  the set imaginary parts of normalised zeros of  $\Lambda(f, s)$  (repeated with multiplicities). If  $\gamma \in \mathcal{Z}(f)$  is nonnegative, we define the spacing:

$$E(\gamma) = \min\{\gamma - \gamma' : \gamma' \in \mathcal{Z}(f) \setminus \{\gamma\}, 0 \leq \gamma' \leq \gamma\} \quad (\min(\emptyset) = 0).$$

The normalisation of the zeros is chosen as to obtain a unit mean spacing:

$$\lim_{T \rightarrow +\infty} \frac{1}{\#\{\gamma \in \mathcal{Z}(f) : 0 \leq \gamma \leq T\}} \sum_{\substack{\gamma \in \mathcal{Z}(f) \\ 0 \leq \gamma \leq T}} E(\gamma) = 1.$$

# One-level density

Let  $\Phi$  be a Schwartz function whose Fourier transform

$$\widehat{\Phi}(\xi) = \int_{\mathbb{R}} \Phi(x) \exp(-2\pi i x \xi) dx$$

has Fourier compact support. We define the **one-level density** of  $L(f, s)$  by:

$$D_1[\Phi](f) = \sum_{\gamma \in \mathcal{Z}(f)} \Phi(\gamma).$$

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Since  $\Phi$  is of fast decreasing, this evaluates the number of zeros of  $\Lambda(f, s)$  in bounded intervals. Due to the normalisation (mean spacing 1) this is expected to be bounded, regardless of the value of  $N$ .

# One-level density (cont.)

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Theorem (Iwaniec, Luo & Sarnak, 2000)

Assume that  $\widehat{\Phi}$  has support in  $(-2, 2)$ . Then

$$\lim_{N \rightarrow \infty} \frac{1}{\#H_k^*(N)} \sum_{f \in H_k^*(N)} D_1[\Phi](f) = \int_{\mathbb{R}} \Phi(x) W[O](x) dx$$

with

$$W[O](x) = 1 + \frac{1}{2} \text{Dirac}_0(x).$$

# One-level density on RMT side

Let  $A \in U(N)$ . Its spectrum is

$$\{e^{i\varphi_j(A)} : 1 \leq j \leq N\}$$

for some choice of angles  $\{\varphi_j(A) : 1 \leq j \leq N\}$ .

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for some choice of angles  $\{\varphi_j(A) : 1 \leq j \leq N\}$ . We define its “full anglespectrum”:

$$\text{Fas}(A) = \bigcup_{\ell \in \mathbb{Z}} \{\varphi_j(A) + 2\ell\pi : 1 \leq j \leq N\}$$

which becomes independant on any choice.

# One-level density on RMT side (cont.)

The one level density of  $A \in U(N)$  is

$$D_1[\Phi](A) = \sum_{\varphi \in \text{Fas}(A)} \Phi\left(\frac{N}{2\pi}\varphi\right).$$

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Theorem (Katz & Sarnak, 1999)

*We have*

$$\lim_{N \rightarrow \infty} \int_{O(N)} D_1[\Phi](A) d\text{Haar}_{O(N)}(A) = \int_{\mathbb{R}} \Phi(x) W[O](x) dx$$

*with*

$$W[O](x) = 1 + \frac{1}{2} \text{Dirac}_0(x).$$

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These two sets have (asymptotically in  $N$ ) the same size: half the one of  $H_k^*(N)$ . Note that, thanks to functional equation:

$$f \in H_k^-(N) \Rightarrow L\left(f, \frac{1}{2}\right) = 0.$$

# One-level density: smaller families (cont.)

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# One-level density: smaller families (cont.)

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with

$$W[SO^-](x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \text{Dirac}_0(x).$$

# One-level density on RMT side: smaller subgroups

Theorem (Katz & Sarnak, 1999)

*We have*

$$\lim_{N \rightarrow \infty} \int_{SO(2N)} D_1[\Phi](A) d\text{Haar}_{SO(2N)}(A) = \int_{\mathbb{R}} \Phi(x) W[SO^+](x) dx$$

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Theorem (Katz & Sarnak, 1999)

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with

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# Importance of the support of $\widehat{\Phi}$

The model for a family of  $L$ -function is provided by one of the three integrals

$$\int_{\mathbb{R}} \Phi W[O], \int_{\mathbb{R}} \Phi W[SO^+], \int_{\mathbb{R}} \Phi W[SO^-]$$

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and

$$\widehat{W}[O]|_{(-1,1)} = \widehat{W}[SO^-]|_{(-1,1)} = \widehat{W}[SO^+]|_{(-1,1)}.$$

# Even smaller families

If  $N$  is squarefree, write  $N = p_1 \cdots p_\ell$  with  $p_1 < \cdots < p_\ell$  its prime decomposition. We have defined an operator  $W_N: f \mapsto f|W_N$ . Actually, we have  $W_N = W_{p_1} \cdots W_{p_\ell}$  where:

$$f|W_p(z) = p^{k/2} (Nz + pd)^{-k} f\left(\frac{paz + b}{Nz + pd}\right)$$

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$$(a, b, d) \in \mathbb{Z}^3, d \equiv 1 \pmod{N/p}, p^2 ad - bN = p.$$

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If  $f \in H_k^*(N)$  then  $f|W_{p_j} = \varepsilon_f(p_j)f$  where  $\varepsilon_f(p_j) = \pm 1$ .

# Even smaller families (cont.)

Let  $\sigma \in \{-1, 1\}^\ell$ . We define

$$H_k^\sigma(N) = \left\{ f \in H_k^*(N) : (\varepsilon_f(p_1), \dots, \varepsilon_f(p_\ell)) = \sigma \right\}.$$

It can be shown that

$$\#H_k^\sigma(N) \sim \frac{1}{2^\ell} \#H_k^*(N) \quad (N \rightarrow +\infty, \omega(N) = \ell)$$

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The one-level density of the zeros of  $L$ -functions of forms in  $H_k^\sigma(N)$  only depends of the **sign of the functional equation**.

# Even smaller families (cont.)

## Theorem (Royer, 2001)

Let  $\ell$  a fixed positive integer and  $\kappa \in (0, 1/\ell)$ . Let  $\sigma \in \{-1, 1\}^\ell$  and  $\Phi$  whose Fourier transform has compact support in  $(-2, 2)$ . Consider an infinite sequence  $\mathcal{N}$  of squarefree integers having  $\ell$  prime divisors and such that  $N^\kappa < \min(p \mid N)$ . Then

$$\lim_{\substack{N \rightarrow \infty \\ N \in \mathcal{N}}} \frac{1}{\#H_k^\sigma(N)} \sum_{f \in H_k^\sigma(N)} D_1[\Phi](f) = \int_{\mathbb{R}} W[SO^\varepsilon] \Phi(x) dx$$

where  $\varepsilon = i^k \sigma_1 \cdots \sigma_\ell$ .

# Higher moments

To simplify notations, we define three expectation operators for  $X: H_k^*(N) \rightarrow \mathbb{C}$ :

$$\mathbb{E}[X] = \frac{1}{H_k^*(N)} \sum_{f \in H_k^*(N)} X(f)$$

$$\mathbb{E}^+[X] = \frac{1}{H_k^+(N)} \sum_{f \in H_k^+(N)} X(f)$$

$$\mathbb{E}^-[X] = \frac{1}{H_k^-(N)} \sum_{f \in H_k^-(N)} X(f)$$

# Higher moments (cont.)

## Theorem (Hughes & Miller, 2007)

Let  $n \geq 2$  and assume that  $\widehat{\Phi}$  has compact support in  $(-\frac{1}{n-1}, \frac{1}{n-1})$ . Then

$$\lim_{\substack{N \rightarrow \infty \\ N \in \mathcal{P}}} \mathbb{E}^{\pm} \left[ \left( D_1[\Phi] - \mathbb{E}^{\pm} [D_1[\Phi]] \right)^n \right] = \alpha(n) \pm R_n(\Phi)$$

where  $\alpha(2m) = (2m-1)!! \sigma_{\Phi}^{2m}$  and  $\alpha(2m+1) = 0$ ,

$$\sigma_{\Phi}^2 = 2 \int_{-1}^1 |y| \widehat{\Phi}(y)^2 dy,$$

$$R_n(\Phi) = (-2)^{n-1} \int_{\mathbb{R}} \Phi(x)^n \left[ \frac{\sin(2\pi x)}{2\pi x} - \frac{1}{2} \text{Dirac}_0(x) \right] dx.$$

# Higher moments: RMT settings

Exactly the same result holds for subgroups of  $U(N)$  when defining for  $X: U(N) \rightarrow \mathbb{C}$ :

$$\mathbb{E}^+[X] = \int_{SO(2N)} X(A) d\text{Haar}_{SO(2N)}(A)$$

$$\mathbb{E}^-[X] = \int_{SO(2N+1)} X(A) d\text{Haar}_{SO(2N+1)}(A).$$

# Mock Gaussian behavior

We have:

$$\frac{1}{(-2)^{n-1}} R_n(\Phi) = \int_{\mathbb{R}} \Phi^n S - \frac{1}{2} \Phi(0)^n = \int_{\mathbb{R}} \widehat{\Phi^n S} - \frac{1}{2} \Phi(0)^n$$

with  $S(x) = \frac{\sin(2\pi x)}{2\pi x}$ . Assume that the support of  $\widehat{\Phi}$  is in  $(-\frac{1}{n}, \frac{1}{n})$  so that the one of  $\widehat{\Phi^n}$  is in  $(-1, 1)$ . Then,

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so that

$$R_n(\Phi) = 0.$$

# Mock Gaussian behavior (cont.)

We deduce that the first moments (comparing to the support of  $\widehat{\Phi}$ ) are Gaussian, but that this does not remain for highest moments.

# Mock Gaussian behavior (cont.)

We deduce that the first moments (comparing to the support of  $\widehat{\Phi}$ ) are Gaussian, but that this does not remain for highest moments. This phenomenon, first discovered by Hughes & Rudnick for  $L$ -functions of Dirichlet characters is called **Mock Gaussian behavior**.

# Symmetric power $L$ -functions

For any natural integer  $r \geq 1$ , the symmetric power  $r$ th function of associated to  $f \in H_k^*(N)$  is the following Euler product:

$$L(\text{Sym}^r f, s) = \prod_{p \in \mathcal{P}} \prod_{j=0}^r \left( 1 - \frac{\alpha_f(p)^j \beta_f(p)^{r-j}}{p^s} \right)^{-1}.$$

# Symmetric power $L$ -functions: gamma factor

If  $r$  is odd, the gamma factor is

$$\gamma(\mathrm{Sym}^r f, s) = \pi^{-(r+1)s/2} \times \prod_{\ell=0}^{(r-1)/2} \Gamma\left(\frac{s + (2\ell + 1)(k-1)/2}{2}\right) \Gamma\left(\frac{s + 1 + (2\ell + 1)(k-1)/2}{2}\right)$$

# Symmetric power $L$ -functions: gamma factor

If  $r$  is even, the gamma factor is

$$\gamma(\mathrm{Sym}^r f, s) = \pi^{-(r+1)s/2} \times \\ \Gamma\left(\frac{s + \mu_{k,r}}{2}\right) \prod_{\ell=1}^{r/2} \Gamma\left(\frac{s + \ell(k-1)}{2}\right) \Gamma\left(\frac{s + 1 + \ell(k-1)}{2}\right)$$

with

$$\mu_{k,r} = \begin{cases} 1 & \text{if } r(k-1)/2 \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

# Symmetric power $L$ -functions: functional equation

The conductor is  $N^r$  and the functional equation is

$$\Lambda(\mathrm{Sym}^r f, s) = \varepsilon_{\mathrm{Sym}^r f}(N) \Lambda(\mathrm{Sym}^r f, s)$$

where  $\varepsilon_{\mathrm{Sym}^r f}(N)$  is  $\varepsilon_f(N)$  up to a sign depending only on the fixed variables  $k$  and  $r$ . This symmetric power  $L$ -function admits an entire continuation.

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The functional equation and continuation are known if  $1 \leq r \leq 4$  (Hecke, Gelbart & Jacquet, Kim & Shahidi) and conjectural for  $r > 4$ . This is a consequence of the Langlands modularity conjecture.

# Symmetric power $L$ -functions: symmetry type

We want to determine which subgroup of  $U(N)$  can be used to modelise the zeros of symmetric power  $L$ -functions. We assume Riemann hypothesis for these  $L$ -functions and define the one-level density by

$$D_1[\Phi; r](f) = \sum_{\gamma, \Lambda(\text{Sym}^r f, 1/2+i\gamma)=0} \Phi\left(\frac{\log(N^r)}{2\pi} \gamma\right)$$

# Symmetric power $L$ -functions: symmetry type

## Theorem (Ricotta & Royer, 2007)

Assume  $\widehat{\Phi}$  has support in  $(-v, v)$ . Let  $\theta = 7/64$  and

$$v_{1,\max}(r, \kappa, \theta) := \left(1 - \frac{1}{2(\kappa - 2\theta)}\right) \frac{2}{r^2}.$$

If  $v < v_{1,\max}(r, \kappa, \theta)$  then the asymptotic expectation of the one-level density is

$$\widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0).$$

# Symmetric power $L$ -functions: symmetry type

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The support is too small to determine the symmetry type in the case of odd  $r \geq 3$ . We shall use the two level density.

# Numbering of the zeros

The set of the zeros of  $\Lambda(\text{Sym}^r f, s)$  counted with multiplicities is given by

$$\left\{ \frac{1}{2} + i\gamma_{f,r}^{(j)} : j \in \mathcal{E}(f, r) \right\}$$

where

$$\mathcal{E}(f, r) := \begin{cases} \mathbb{Z} & \text{if } \varepsilon(\text{Sym}^r f) = -1 \\ \mathbb{Z} \setminus \{0\} & \text{if } \varepsilon(\text{Sym}^r f) = 1 \end{cases}$$

# Numbering of the zeros (cont.)

We enumerate the zeros such that

- 1 the sequence  $j \mapsto \gamma_{f,r}^{(j)}$  is increasing
- 2 we have  $j \geq 0$  if and only if  $\gamma_{f,r}^{(j)} \geq 0$
- 3 we have  $\gamma_{f,r}^{(-j)} = -\gamma_{f,r}^{(j)}$ .

# Numbering of the zeros (cont.)

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Remember that if  $\varepsilon(\text{Sym}^r f) = -1$  then the functional equation of  $\Lambda(\text{Sym}^r f, s)$  evaluated at the critical point  $s = 1/2$  provides a zero denoted by  $\frac{1}{2} + i\gamma_{f,r}^{(0)}$ .

# Two level density

The **two-level density** of  $\text{Sym}^r f$  (relatively to  $\Phi_1$  and  $\Phi_2$ ) is defined by

$$D_2[\Phi_1, \Phi_2; r](f) = \sum_{\substack{(j_1, j_2) \in \mathcal{E}(f, r)^2 \\ j_1 \neq \pm j_2}} \Phi_1 \left( \frac{\log N^r}{2\pi} \gamma_{f, r}^{(j_1)} \right) \Phi_2 \left( \frac{\log N^r}{2\pi} \gamma_{f, r}^{(j_2)} \right).$$

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It has been shown by Miller that, this statistics for subgroups of  $U(N)$  allow to distinguish the subgroups, regardless to the support of  $\Phi_1$  and  $\Phi_2$ . Hence, it should allow us to determine the symmetry type of our symmetric  $L$ -functions.

# Two level density

## Theorem (Ricotta & Royer, 2007)

Assume  $\widehat{\Phi}_1$  and  $\widehat{\Phi}_2$  have support in  $(-v, v)$ . If  $v < 1/r^2$  then the asymptotic expectation of the two-level density is

$$\begin{aligned} & \left[ \widehat{\Phi}_1(0) + \frac{(-1)^{r+1}}{2} \Phi_1(0) \right] \left[ \widehat{\Phi}_2(0) + \frac{(-1)^{r+1}}{2} \Phi_2(0) \right] \\ & + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du - 2 \widehat{\Phi}_1 \widehat{\Phi}_2(0) \\ & + \left( (-1)^r + \frac{\chi_{2\mathbb{N}+1}(r)}{2} \right) \Phi_1(0) \Phi_2(0). \end{aligned}$$

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It follows that:

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- if  $r$  is odd, the zeros are modelised by matrices in  $O$ .

# Symmetric power $L$ -functions: moments

Finally, we are able to compute the moments of the one-level density. However, not in a large enough range to exhibit the Mock Gaussian behavior.

# Symmetric power $L$ -functions: moments

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## Theorem (Ricotta & Royer, 2007)

*Assume  $\widehat{\Phi}$  has support in  $(-v, v)$ . If  $mv < 4/(r(r+2))$  then the asymptotic  $m$ -th moment of the one-level density is*

$$\begin{cases} 0 & \text{if } m \text{ is odd,} \\ \left(2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) du\right)^{m/2} \times \frac{m!}{2^{m/2} \left(\frac{m}{2}\right)!} & \text{otherwise.} \end{cases}$$