Square-tiled surfaces & quasimodular forms

The results that will not be presented here have been obtained in collaboration with Samuel Lelièvre (Paris-Sud XI) inside the project *ANR Teichmüller*.

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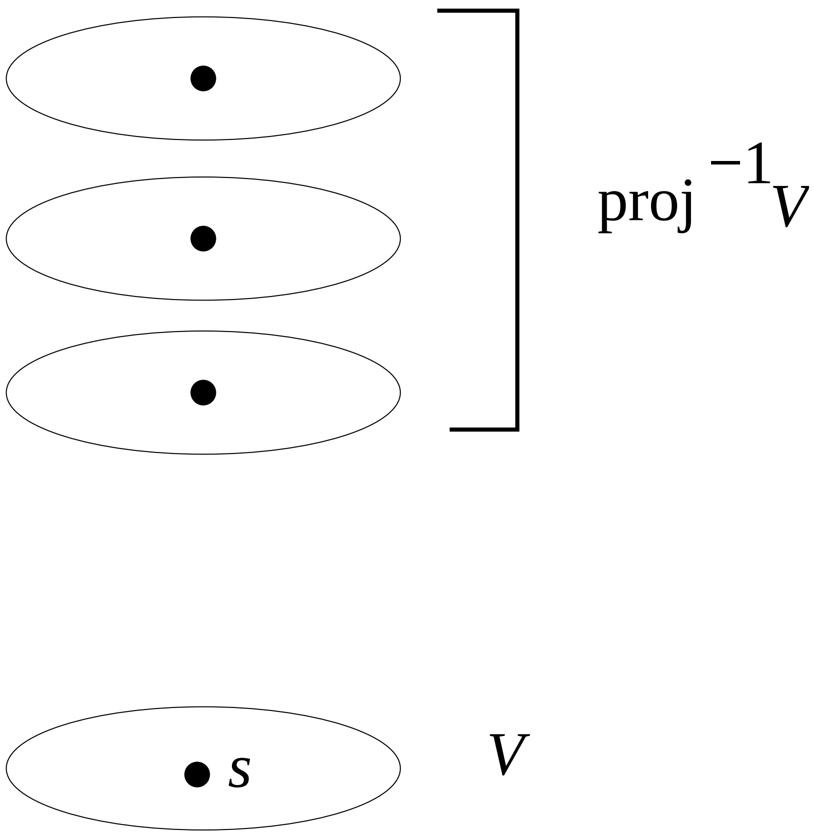
☺

Ramified coverings

Let S be a surface (2 dimensional variety, smooth, compact, connected, oriented, without boundary). A ramified covering of S by a surface C is a smooth application preserving the orientation $p\colon C \longrightarrow S$ such that

- each point of S has a finite number of preimages (by p) ;
- there exists a finite set $\mathcal{R} \subset S$ such that p is a covering of $S \setminus \mathcal{R}$.

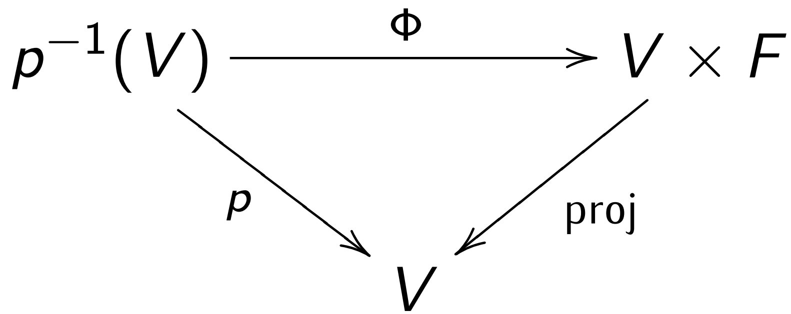
Coverings



Every $s \in S \setminus \mathcal{R}$ has

- a neighborhood V,
- a non empty finite a set F
- and a homeomorphism $\Phi \colon p^{-1}(V) \to V \times F$

such that the following diagram commutes :



Local behavior

Given $c \in C$, the coordinates may be chosen so that, locally, p is $z \mapsto z^k$ for some integer $k \geq 1$.

Different orders

$$\begin{array}{ccccc} C & \ni & c & \xmapsto[\text{around } c]{\text{coordinates}} & 0 \in \mathbb{C} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow & \circlearrowleft & \big\downarrow \qquad \big\downarrow{\scriptstyle z\mapsto z^k} \\ S & \ni & p(c) & \xmapsto[\text{around } p(c)]{\text{coordinates}} & 0 \in \mathbb{C} \end{array}$$

- for any $c \in C$, we define the preimage order:$\mathrm{ord\_inv}(c) = k$.
- if $k > 1$ then
  - c is said to be a critical point and its critical order is:

$$\mathrm{ord\_cri}(c) = k - 1 \geq 1.$$

  - $p(c)$ is said to be a ramification point and its ramification order is

$$\mathrm{ord\_ram}\,(p(c)) = \sum_{\substack{x \text{ critical} \\ p(x)=p(c).}} \mathrm{ord\_cri}(x)$$

Degree

The map

$$\begin{array}{rcl} C & \longrightarrow & \mathbb{N} \\ c & \longmapsto & \sum_{\substack{x \in C \\ p(x) = p(c)}} \mathrm{ord\_inv}(x) \end{array}$$

is constant.

This is called the degree of p: $\deg(p)$.

Riemann-Hürwitz formula (RH)

Let $p\colon C \to S$ be a ramified covering with ramification points $\mathcal{R} \subset S$, then

$$2\,\mathrm{genus}(C) - 2 = (2\,\mathrm{genus}(S) - 2)\,\deg(p) + \sum_{r\in\mathcal{R}} \mathrm{ord\_ram}(r).$$

Description of the preimages

For $y \in S$ and $i \in \{1, \ldots, \deg(p)\}$, let $q_i \geq 0$ be the number of preimages $x \in C$ of y having order

$$\text{ord\_inv}(x) = i.$$

Then

$$\sum_{i=1}^{\deg(p)} iq_i = \deg(p)$$

$$\sum_{i=1}^{\deg(p)} q_i = \text{number of distinct preimages of } y.$$

Partition associated to a ramification point

We can associate to (p, y) a partition of $\deg(p)$:

$$\Bigg(\underbrace{1,\ldots,1}_{q_1\text{times}},\ldots,\underbrace{\deg(p),\ldots,\deg(p)}_{q_p\text{times}}\Bigg) = \left(1^{q_1}\cdots \deg(p)^{\deg(p)}\right).$$

If

$$\Bigg(\underbrace{1,\ldots,1}_{q_1\text{times}},\ldots,\underbrace{\deg(p),\ldots,\deg(p)}_{q_p\text{times}}\Bigg) = (m_1,\ldots,m_d)$$

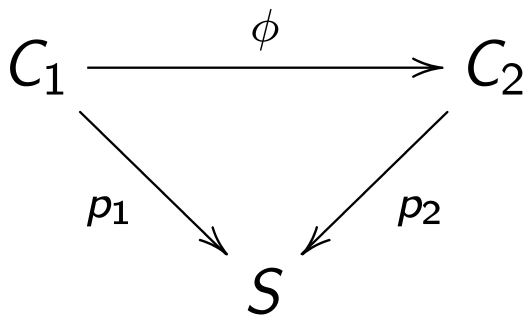
with $m_i \geq 1$ for any i, then d is the number of preimages of y.

Equivalent coverings

Two ramified coverings $p_1 \colon C_1 \to S$ and $p_2 \colon C_2 \to S$ of the same surface S are equivalent if there is a diffeomorphism $\phi \colon C_1 \to C_2$ such that:

$$p_2 \circ \phi = p_1 .$$

The following diagram commutes:

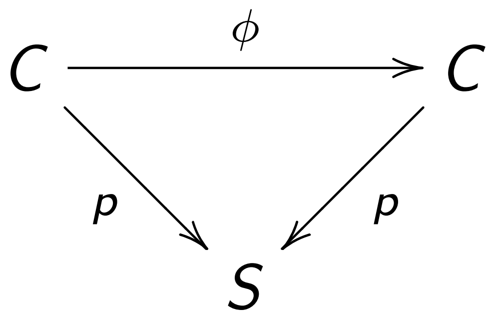


Automorphism of coverings

An automorphism of a ramified covering $p\colon C \to S$ is a diffeomorphism $\phi\colon C \to C$ which is invisible through p that is

$$p \circ \phi = p.$$

The following diagram commutes:



The group of automorphisms of p is denoted $\mathrm{Aut}(p)$.

Hürwitz Problem

Data

- a compact Riemann surface S with c marqued points $y_1, \ldots, y_c$;
- an integer n and, for every marqued point y_i a partition τ_i of n.

Question

How many non equivalent ramified coverings of S of degree n have

- the marqued points $y_1, \ldots, y_c$ for ramification points
- the partition τ_i associated to the ramification point y_i for any i ?

Each covering p is counted with weight $\frac{1}{|\mathrm{Aut}(p)|}$. We get the Hürwitz number associated to the data : h.

Weights in countings (1)

Forgett the symmetry

Given n coins, there is $\alpha(n) = \binom{n}{2}$ possible choices of 2 coins.
The generating function is

$$\mathrm{GF}(\alpha; x) = \sum_{n \in \mathbb{N}} \alpha(n) x^n = \frac{x^2}{(1-x)^3}.$$

The computation is not so easy : use inversion of summations on

$$\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} \binom{n}{k} x^n y^k$$

and specialize.

Weights in countings (2)

Take account of the symmetry

There is $|\mathfrak{S}_n| = n!$ numerotations of the coins. Put $1/|\mathfrak{S}_n|$ as a weight leads to trivial computations :

$$\mathrm{EGF}(\alpha; x) = \sum_{n \in \mathbb{N}} \frac{1}{n!} \alpha(n) x^n = \frac{x^2}{2} e^x.$$

Idea

Weighting by the symmetry group in countings leads to

- easier caluclations
- same information

if the symmetry group is easy to evaluate!

Hürwitz formula (1)

Data

- surfaces S and C are spheres
- there is $n-d+2$ ramification points with ord_ram $=1$ and only one ramification point can have ord_ram >1
- the partition of n associated to this point is $(k_1,\dots,k_d)=(1^{a_1},\dots,n^{a_n})$

Notation

We note $|\mathrm{Aut}(k_1,\dots,k_d)|$ the number of permutations σ of the elements $k_1,\dots,k_d$ such that $k_{\sigma(i)}=k_i$ for any i. We have

$$|\mathrm{Aut}(k_1,\dots,k_d)|=a_1!\cdots a_n!$$

Hürwitz formula (2)

The associated Hürwitz number is

$$n^{d-3}\frac{(n+d-2)!}{|\mathrm{Aut}(k_1,\ldots,k_d)|}\prod_{i=1}^{d}\frac{k_i^{k_i}}{k_i!}$$

Coverings of the torus

Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ and $\mathcal{R} \subset \mathbb{T}$ of finite cardinality r (necessary even).
Let $R(\mathbb{T}, \mathcal{R}, n)$ be the set of equivalent classes of coverings of degree n whose ramification points are in $\mathcal{R}$ and are all simple.

The dependance in $\mathcal{R}$ of the number

$$\sum_{p \in R(\mathbb{T},\mathcal{R},n)} \frac{1}{|\mathrm{Aut}(p)|}$$

is only in r. Denote it by $N_{\mathbb{T}}(r, n)$.
NB. If $p\colon C \to \mathbb{T}$ is in $R(\mathbb{T}, \mathcal{R}, n)$ then the genus of C is $(r+2)/2$.

Coverings of the torus

Dijkgraaf and Kaneko & Zagier

Fix $r \geq 2$. The generating function

$$\sum_{n \geq 1} N_{\mathbb{T}}(r, n) e^{2\pi i n z}$$

is a quasimodular form of weight $3r = 6g - 6$.

Generalisation

The result is still true when considering the other types of ramification :

- Eskin, Masur & Schmoll
- Bloch & Okounkov
- Eskin & Okounkov.

We will present a specific example.

What is a quasimodular forms ?

Quasimodularity condition

A holomorphic function f on the Poincaré half-plane is a quasimodular form of weight k and depth $s \geq 0$ if there exists holomorphic functions $f_0, \ldots, f_s$ ($f_s \neq 0$) such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz+d}\right)^i$$

for any

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

What is a quasimodular forms ?

Growth condition

Take the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ to see that f is 1-periodic. We assume that there are no Fourier coefficient of neative order

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n) e^{2i\pi nz}.$$

ATTENTION

Think on how we see that if f is a modular form then it has a Fourier development at any cusp. Remark that a quasimodular form does not satisfy this property. The growth condition is not a condition at the cusp. This is (a little) problematic for congruence subgroups.

Example

Eisenstein series of weight 2

Let

$$E_2(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) e^{2i\pi nz}.$$

It satisfies

$$(cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) + \frac{6}{i\pi}\frac{c}{cz+d}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Hence this is a quasimodular form of weight 2 and depth 1.

Example

Modular forms and differentation

- a modular form is a quasimodular form of depth 0.
- the derivative of a modular form of weight k is a quasimodular form of weight $k+2$ and depth 1.
- the derivative of a quasimodular form of weight k and depth s is a quasimodular form of weight $k+2$ and depth $s+1$.

Structure theorem

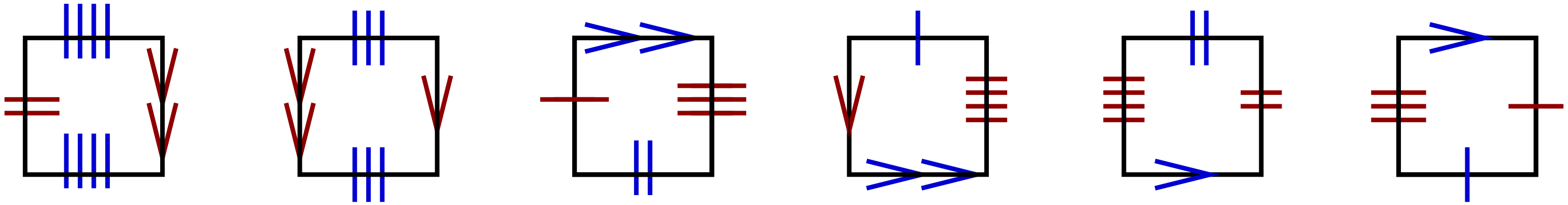
Denote by M_k the vector space of modular forms of weight k, and by $M_k^{\leq s}$ the vector space of quasimodular forms of weight k and depth $\leq s$.

$$M_k^{\infty} = \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i} \bigoplus \mathbb{C} D^{k/2-1} E_2.$$

$$M_k^{\infty} = \bigoplus_{i=0}^{k/2} M_{k-2i} E_2^i.$$

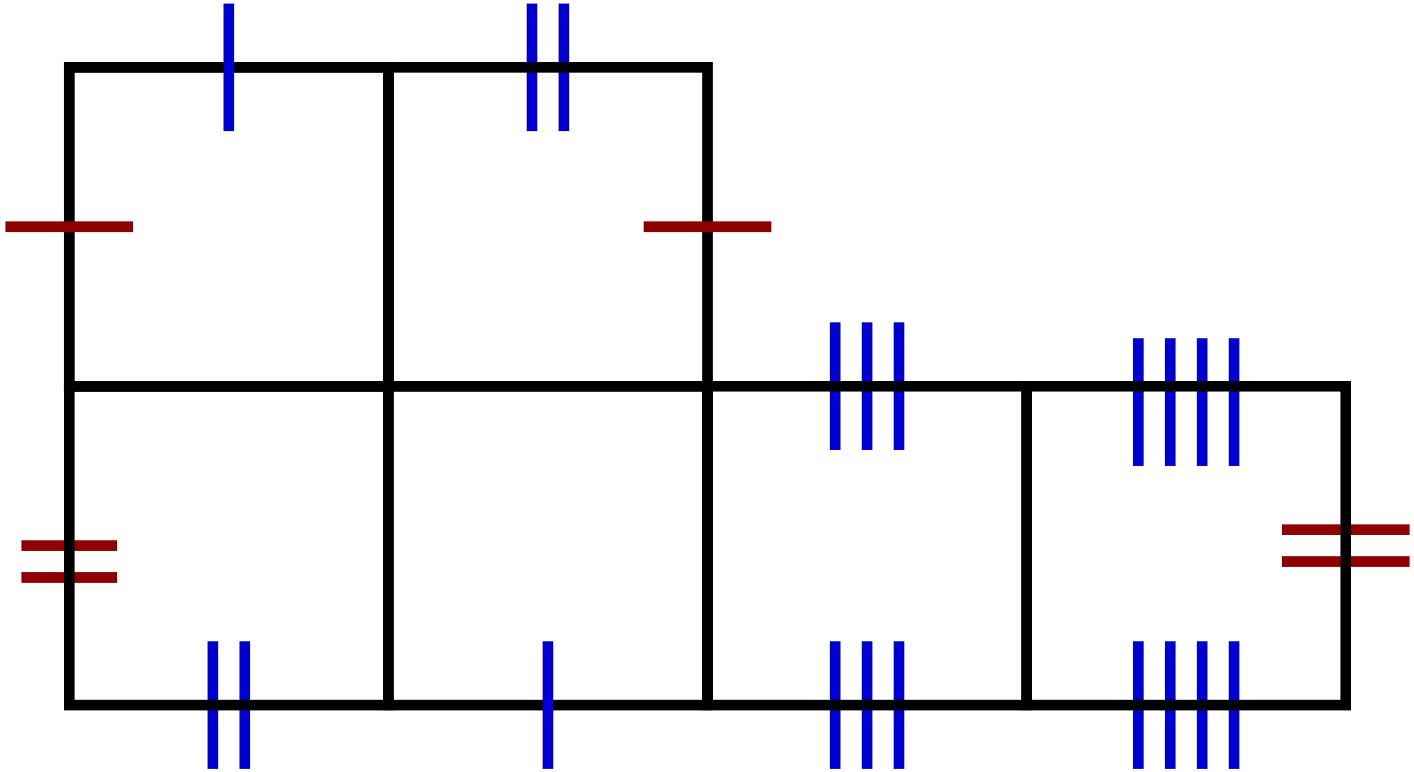
Definition

A square-tiled surface is a collection of unit squares with identifications of the opposite sides.



- each top side is identified with a bottom side
- each left side is identified with a right side.

We moreover ask to obtain a connex surface.



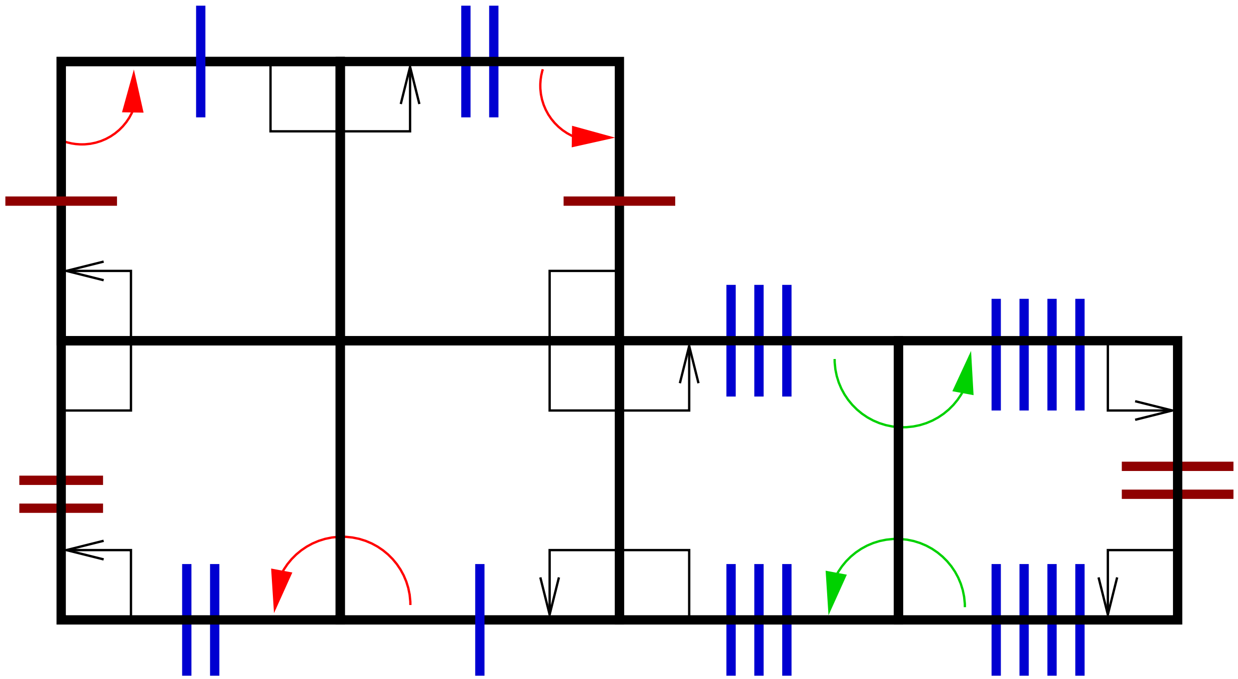
We obtain a ramified covering of the torus $\mathbb{R}/\mathbb{Z}$ with a single ramification point (the origin of the torus).

Ramification type

- The image of the unit circle centered at 0 covered once is this circle covered k times.
- Hence the order ord_inv(x) of a preimage x of the ramification point is determined by its angle $2\pi\,\mathrm{ord\_inv}(x)$.
- Denote by $2\pi(k_i+1)$ the angles of the preimages, Riemann-Hürwitz formula leads to

$$\sum_i k_i = 2g - 2$$

  where g is the genus of the square-tiled surface.
- We denote by $\mathcal{H}(k_1,\dots,k_d)$ the set of non-equivalent surfaces with angles $2\pi(k_i+1)$.

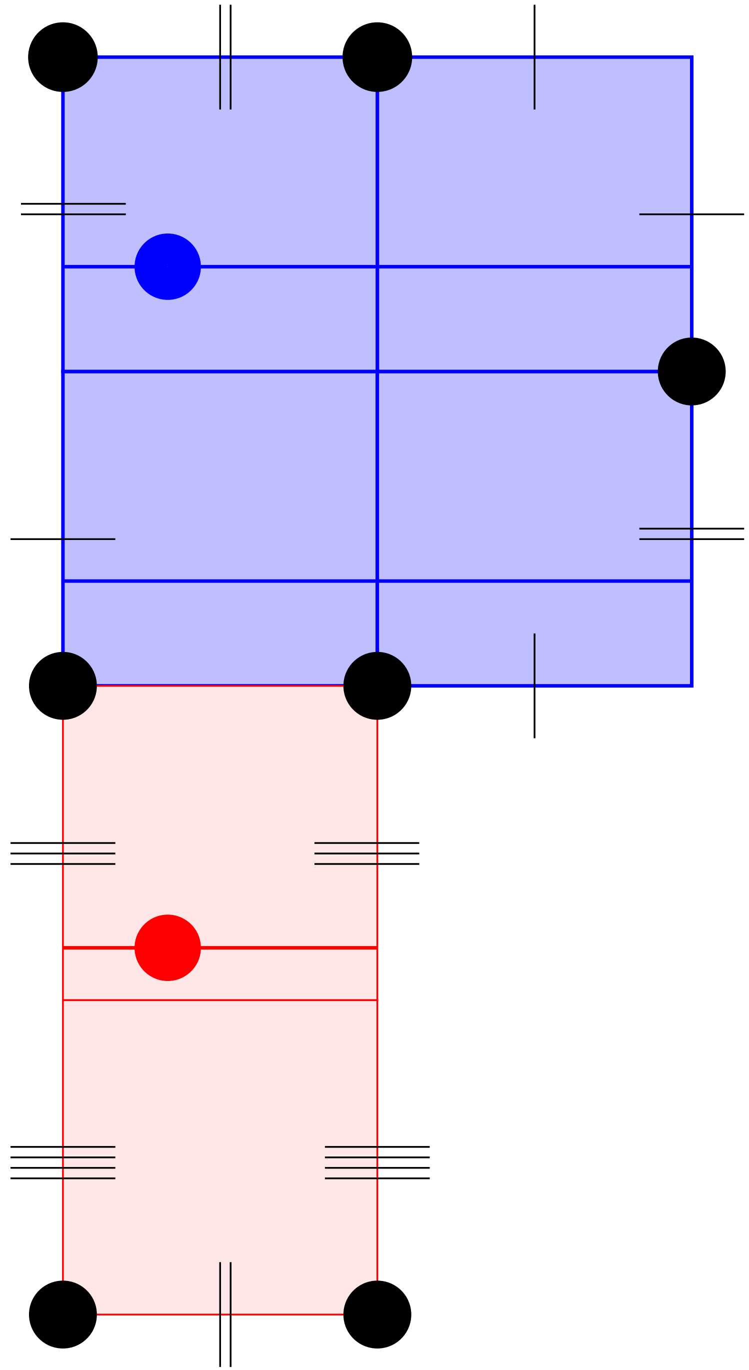


Only one preimage is critical, its angle is 6π, its critical order is 2. The genus of the square-tiled surface is 2. This surface is in $\mathcal{H}(2)$.

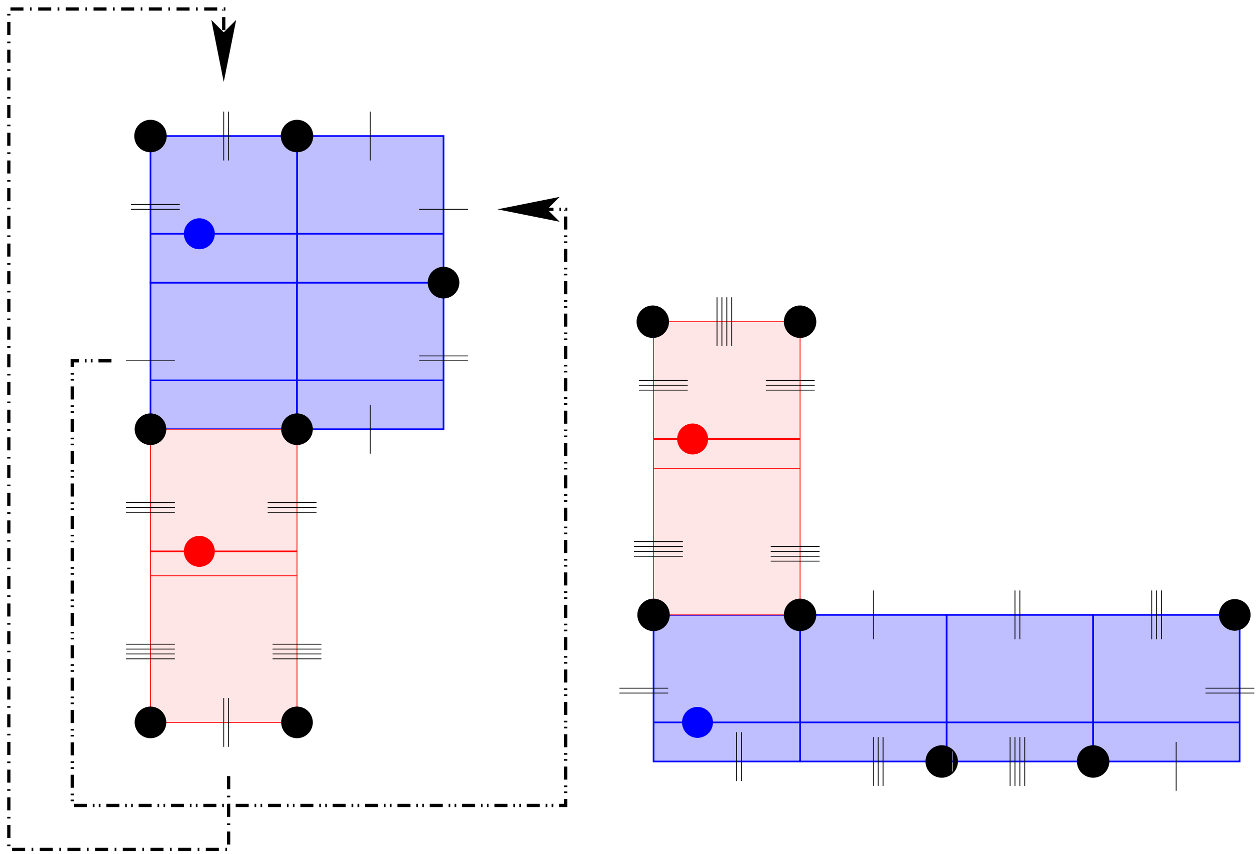
Zorich coordinates in $\mathcal{H}(2)$: cylinder decomposition

By drawing geodesics (for the flat metric) passing through interior points, it is possible to decompose a surface in $\mathcal{H}(2)$ into cylinders.

- Take a point inside the surface.
- Draw the horizontal line joining this point to itself.
- Continuously move this line until crossing a link between saddle points.
- repeat for a point not already in a founded cylinder



One can show that one always get a surface with 1 or 2 cylinders.

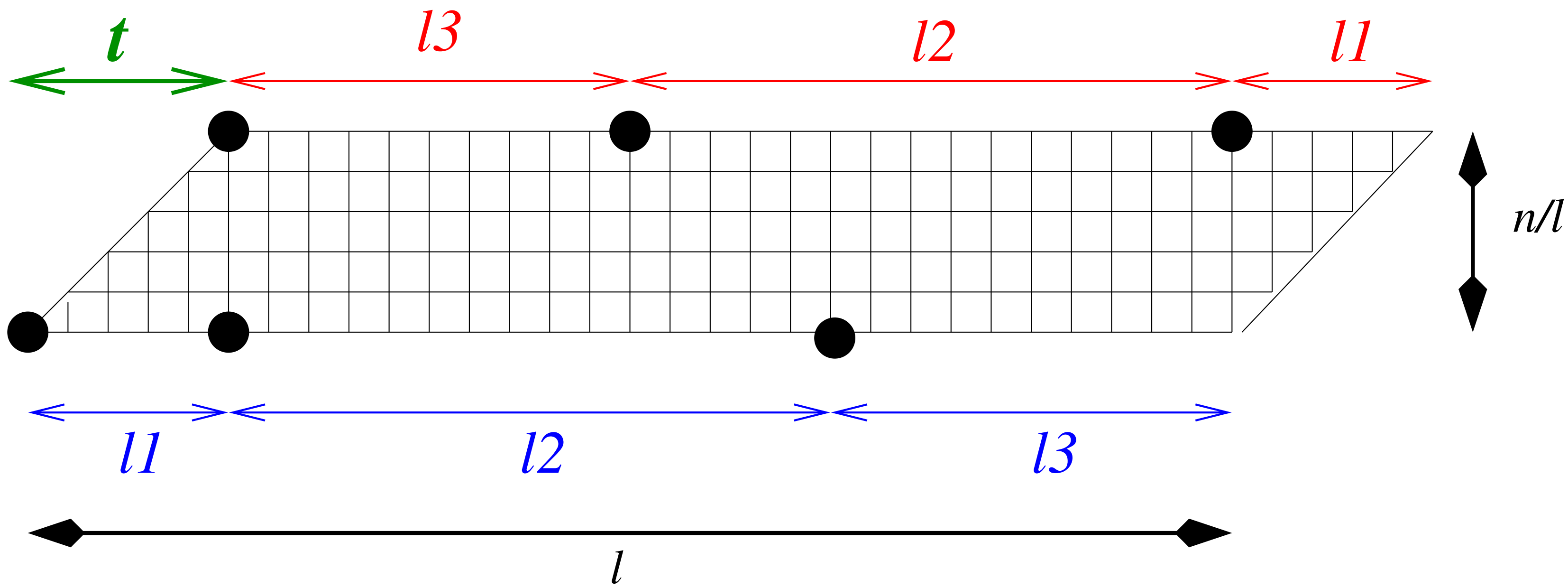


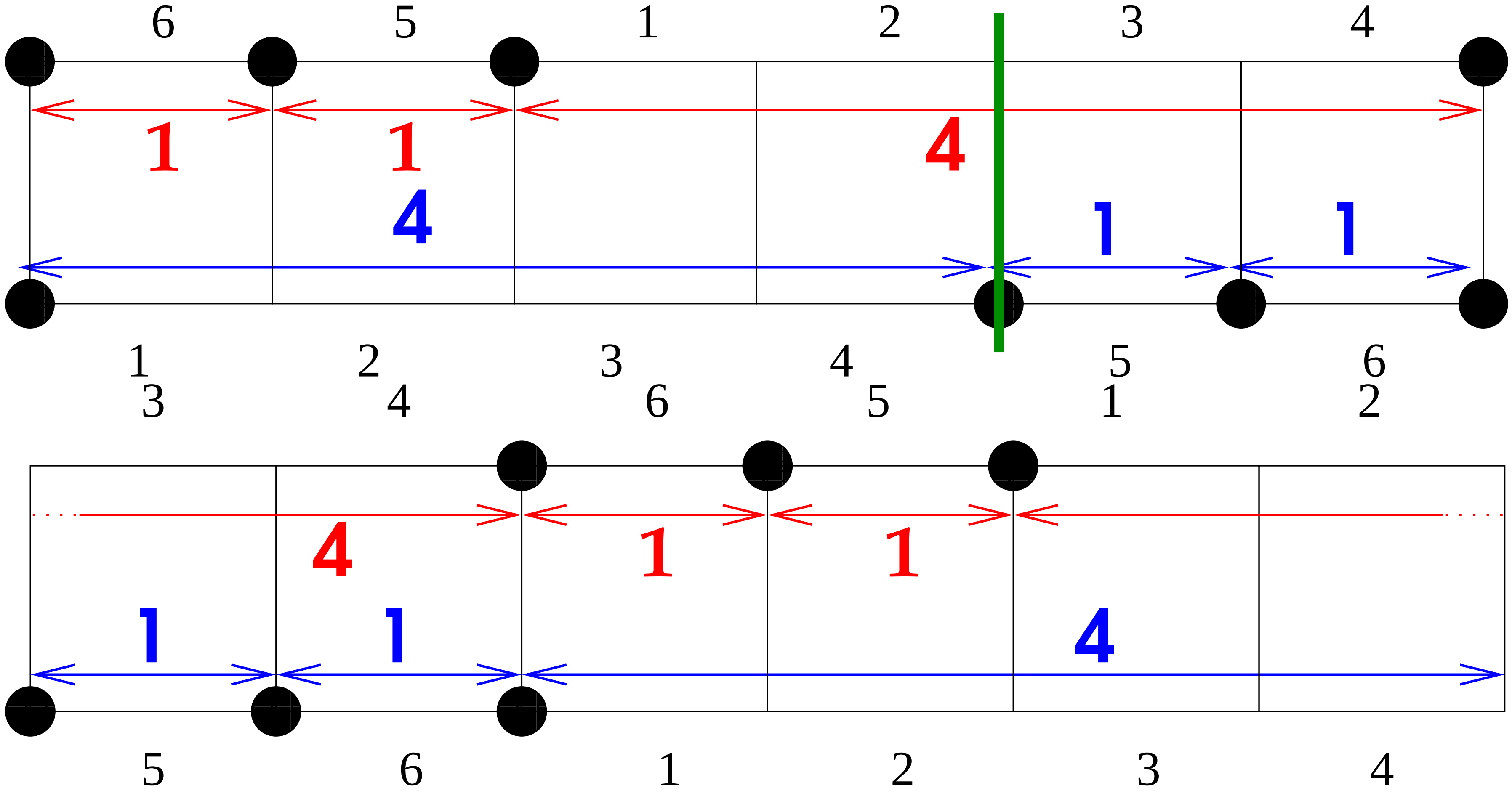
Zorich coordinates in $\mathcal{H}(2)$: torsion for one cylinder surfaces

By cutting & gluing, unit squares may be transformed in parralelogram such that a one cylinder surface has the following shape with

$$(\ell_1, \ell_2, \ell_3) = \min\{(\ell_1, \ell_2, \ell_3), (\ell_2, \ell_3, \ell_1), (\ell_3, \ell_1, \ell_2)\}$$

for the lexicographical order. This allow to define a torsion $t \in [0, \ell[$.





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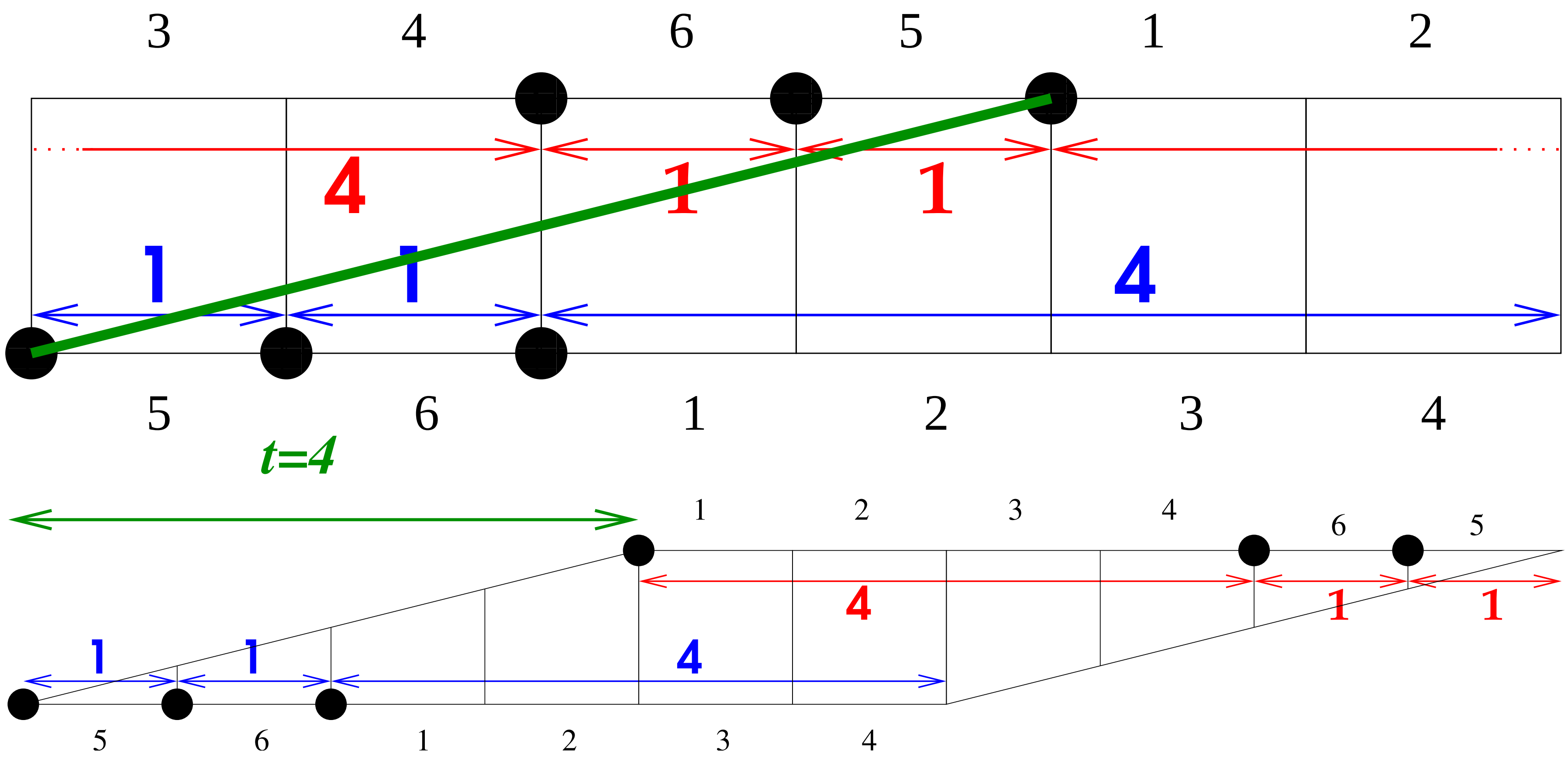
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Countings of one-cylinder surfaces in $\mathcal{H}(2)$

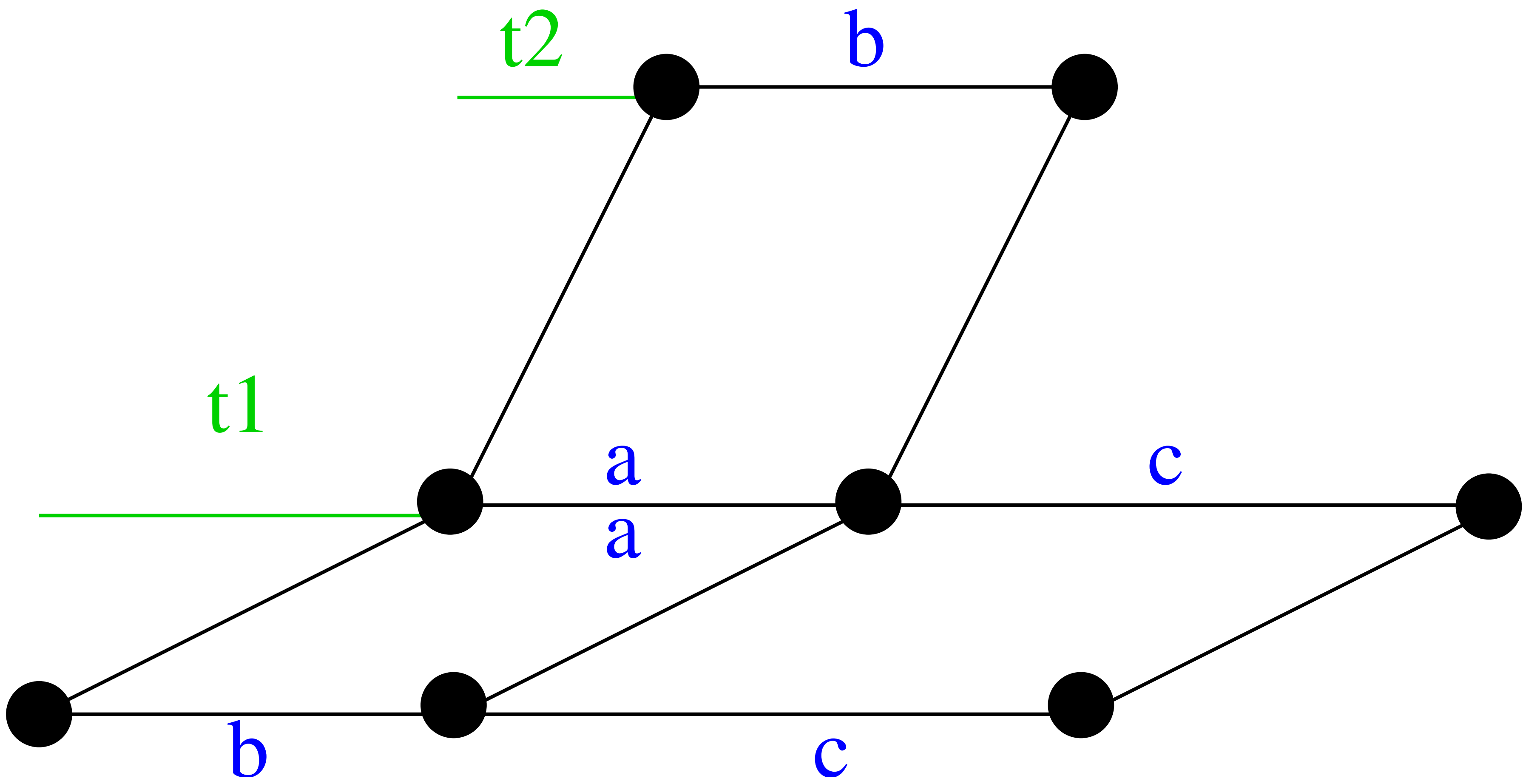
- The length of the cylinder, ℓ divides n,
- the twist can take ℓ values,
- taking acount of the order of ℓ_1, ℓ_2 and ℓ_3,

we get

$$\frac{1}{3}\sum_{\ell | n} \sum_{\substack{(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^{*3} \\ \ell_1+\ell_2+\ell_3=\ell}} \ell = \frac{1}{6}\sigma_3(n) - \frac{1}{2}\sigma_2(n) + \frac{1}{3}\sigma_1(n).$$

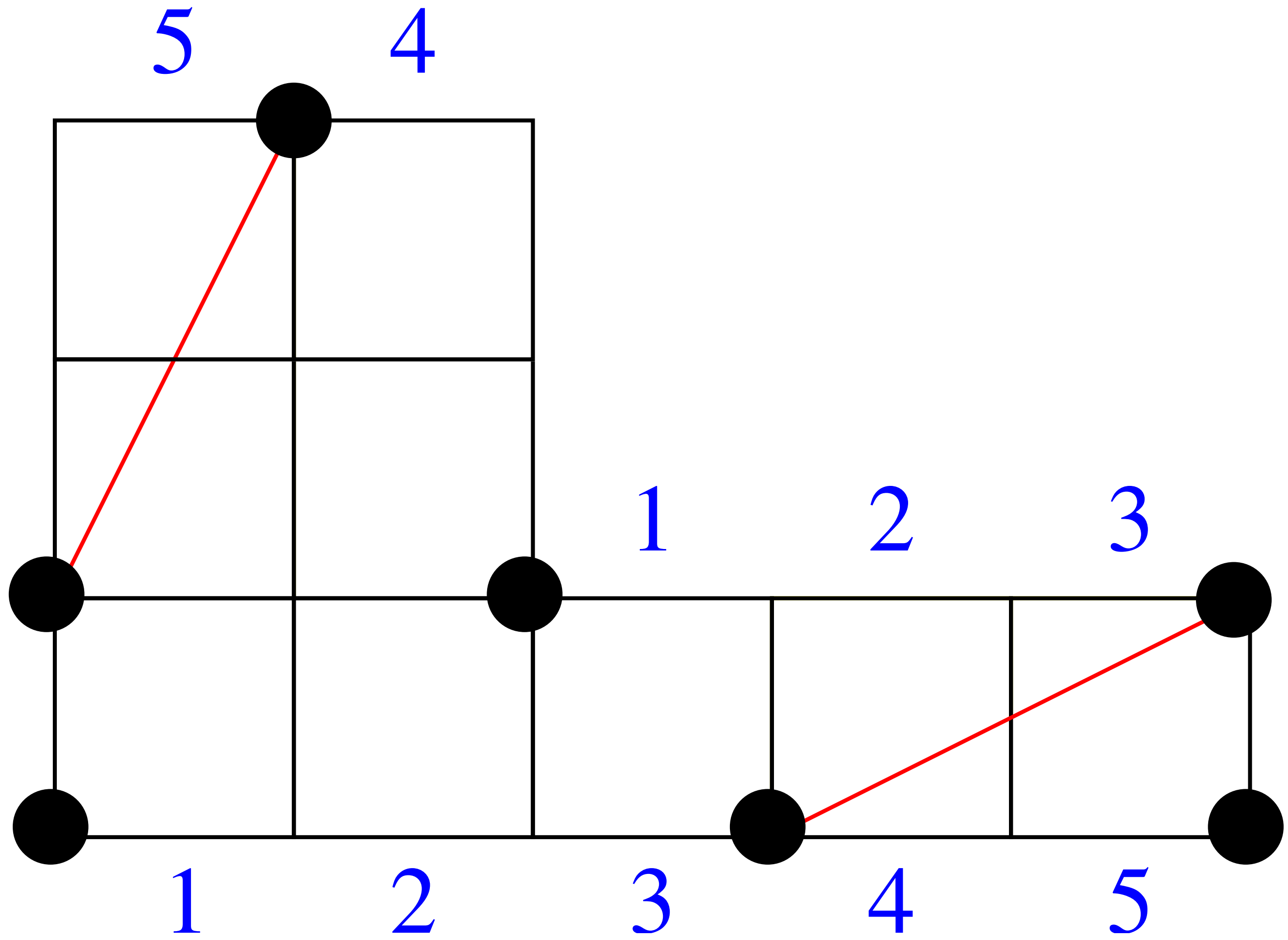
Zorich coordinates in $\mathcal{H}(2)$: torsion for two cylinder surfaces

By cutting & gluing, one can transform any 2-cylinder surface of $\mathcal{H}(2)$ in a surface having the following shape:

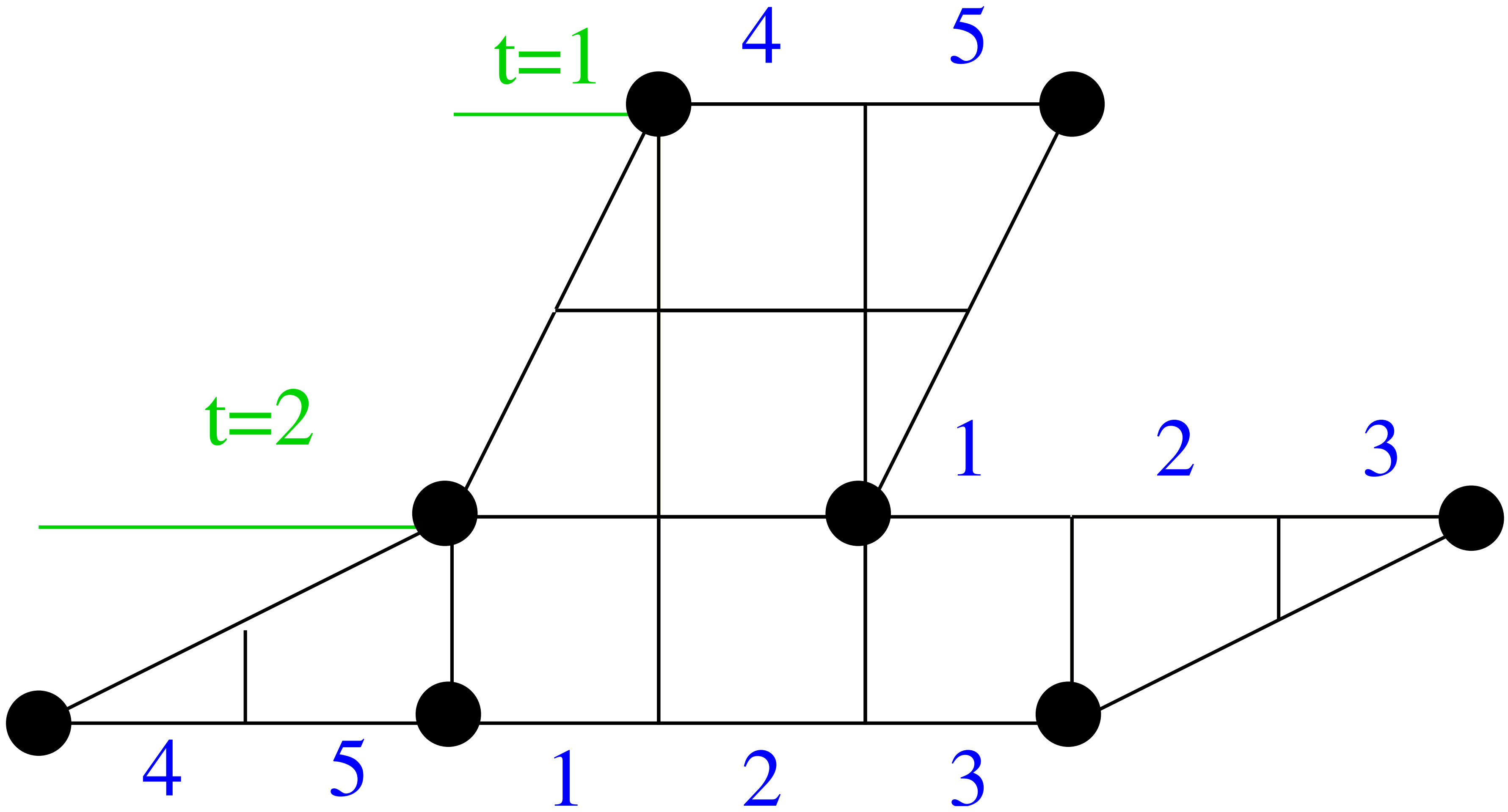


obtaining two torsion parameters.

Example

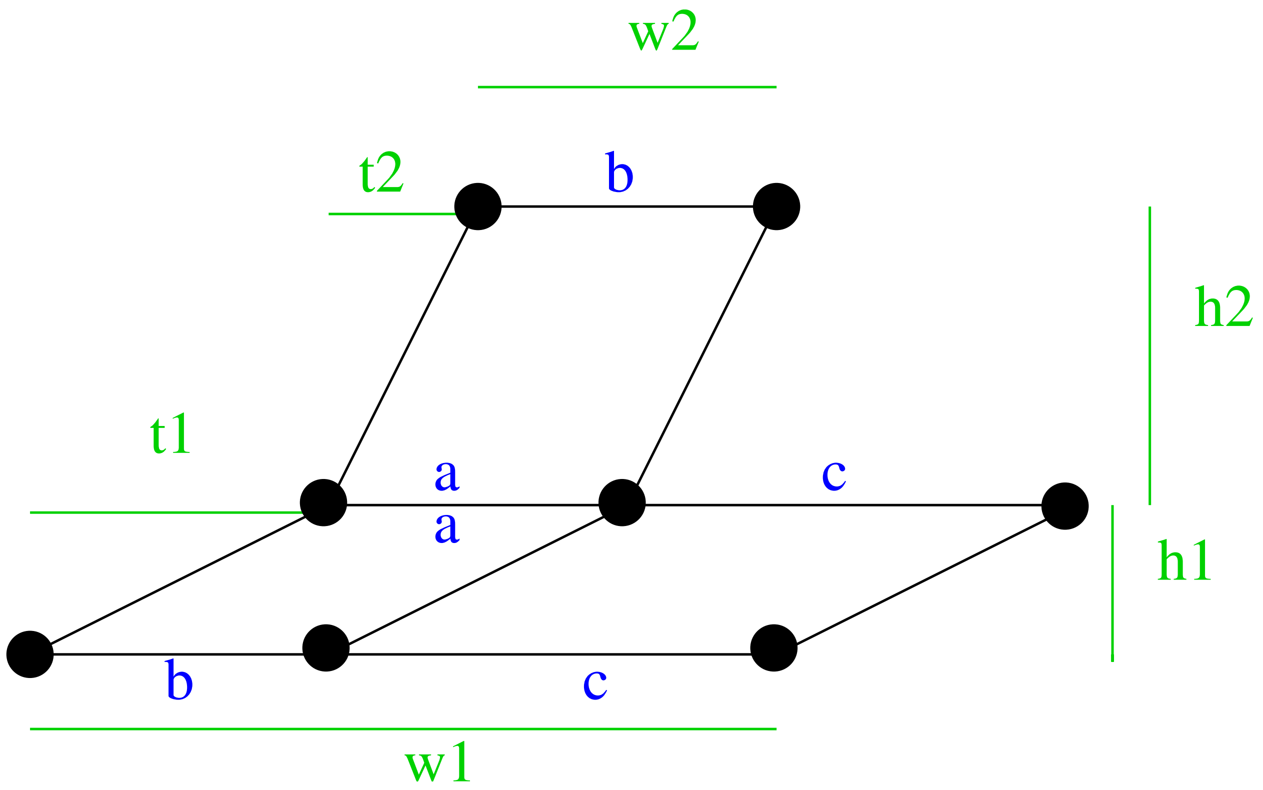


Example



Zorich coordinates in $\mathcal{H}(2)$

- two heights, h_1 et h_2
- two lengths $w_1 > w_2$
- two torsions $t_1 \in [0, w_1[$ and $t_2 \in [0, w_2[$.



The number of unit squares is $n = h_1 w_1 + h_2 w_2$.

Countings of two-cylinder surfaces in $\mathcal{H}(2)$

$$\sum_{\substack{(h_1,h_2,w_1,w_2)\in\mathbb{N}^{*4}\\ w_1>w_2\\ h_1w_1+h_2w_2=n}} w_1w_2 = \frac{1}{2}\sum_{s=1}^{n-1}\sigma_1(s)\sigma_1(n-s) - \frac{1}{2}n\sigma_1(n) + \frac{1}{2}\sigma_2(n).$$

Since E_2 is a quasimodular form of weight 4 and depth 2, we have $E_2 \in \mathbb{C}E_4 + DE_2$ where $D = 1/(2\pi i)d/dz$ and

$$E_4(z) = 1 + 240\sum_{n=1}^{+\infty}\sigma_3(n)e^{2i\pi nz}.$$

It follows that

$$E_2^2 = E_4 + 12DE_2$$

and

$$\sum_{s=1}^{n-1}\sigma_1(s)\sigma_1(n-s) = \frac{5}{12}\sigma_3(n) - \frac{1}{2}n\sigma_1(n) + \frac{1}{12}\sigma_1(n).$$

Countings surfaces in $\mathcal{H}(2)$

Putting all together, the number of inequivalent square-tiled surfaces in $\mathcal{H}(2)$ with n squares is

$$h(n) = \frac{3}{8}[\sigma_3(n) - (2n-1)\sigma_1(n)].$$

The generating series is a linear combination of quasimodular form:

$$\sum_{n=0}^{+\infty} h(n) e^{2i\pi nz} = \frac{1}{640}(9 - 10E_2 + 20DE_2 + E_4).$$

We recover (alternative method) a result of Eskin, Masur & Schmoll. This is an explicit version, in the case of $\mathcal{H}(2)$, of the general result of Bloch & Okounkov et Eskin & Okounkov.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$h(n)$	3	9	27	45	90	135	201	297	405	525	693	918	1062

$$\liminf_{n\to+\infty} \frac{h(n)}{n^3} = \frac{3}{8} = 0,375$$

$$\limsup_{n\to+\infty} \frac{h(n)}{n^3} = \frac{3}{8}\zeta(3) \approx 0,451.$$

What's next?

- Definition of an action of $SL(2, \mathbb{Z})$
- Determination of the orbits (Hubert & Lelièvre, McMullen)
- Countings by orbits, recover quasimodular form (Lelièvre & R.).