Representations of the Brownian snake with drift

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Abstract : We consider a path-valued process which is a generalization of the classical Brownian snake introduced by Le Gall. More precisely we add a drift term b to the lifetime process, which may depends on the spatial process. This consequently introduce a coupling between the lifetime process and the spatial motion. This process can be obtained from the standard Brownian snake by Girsanov's theorem or by killing of the spatial motion. It can also be viewed as the limit of discrete snakes or, in some special cases, as conditioned Brownian snakes. We also use this process to describe the solutions of the non-linear partial differential equation $\Delta u = 4u^2 + 4b\,u$.

AMS Subject Classifications: 60J25, 60G44, 60G57, 35J65

Keywords: path-valued process, Brownian snake, drift, killed process, Poisson measure, martingale problem, semi-linear p.d.e., Donsker theorem

1 Introduction

The Brownian snake –as named by Dynkin and Kuznetsov– is a Markov process which takes its values in the set of paths in $\mathbb{R}^d$. This process, introduced by Le Gall ([6, 7]), has been used successfully to investigate various properties of super-Brownian motion. It also gives a representation of the solutions of the semi-linear partial differential equation $\Delta u = u^2$ and leads to a classification of those solutions. Loosely speaking the Brownian snake $(W_s)_{s\geq 0}$ is a parametrization of a tree of Brownian trajectories. For each $s \geq 0$, W_s is a stopped path in $\mathbb{R}^d$. A stopped path w in $\mathbb{R}^d$ is a continuous function from $[0, +\infty)$ to $\mathbb{R}^d$ constant after a time $\zeta(w)$ that we call lifetime of the path. The process $(\zeta_s = \zeta(W_s))_{s\geq 0}$ is a reflecting Brownian motion on $[0, +\infty)$. For every $s \geq 0$, the path W_s is a Brownian trajectory in $\mathbb{R}^d$, stopped at time ζ_s. Moreover two trajectories W_s and $W_{s'}$ coincide up to time $\inf_{[s,s']} \zeta$ and behave independently after that time. More precisely the

conditional law of the process $(W_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$ is that of an inhomogeneous Markov process whose kernels are given by the previous description. In an intuitive dynamical point of view, if the lifetime decreases we erase the trajectory and if the lifetime increases we extend the trajectory by an independent Brownian trajectory up to the new lifetime. As spatial motion, Brownian motion in $\mathbb{R}^d$ can be replaced by a general Markov process ξ satisfying only a regularity assumption. We then call this process a *standard Brownian snake* associated to the spatial motion ξ. In this denomination the term "Brownian" refers to the evolution of the lifetime.

In [5], Dhersin and Serlet construct a "modified Brownian snake" which is a generalization of the standard Brownian snake. The generalization operates in two directions. The first one (inspired by [11]) will not be carried on in this paper and consist in a modification of the "branching rate" of the trajectories. The second one is the subject of this paper. It consists in the addition of a drift term to the lifetime of the snake. More precisely a path-valued process (W_s) is constructed such that its lifetime (ζ_s) satisfies on $\{\zeta_s > 0\}$ the equation

$$d\zeta_s = d\gamma_s - b(W_s)\, ds$$

where (γ_s) is a linear reflecting Brownian motion and b is a function on $\mathcal{W}$, the set of all stopped paths. We will call this process a b-snake. The existence of this process is not obvious as the evolution of the lifetime depends of the trajectories in $\mathbb{R}^d$. But it is easy to obtain the b-snake from the ordinary snake via Girsanov theorem. Moreover the b-snake is the solution of a well posed martingale problem. These results will be recalled in section 2.

The b-snake is related to super-Brownian motion with drift as the standard snake is related to ordinary super-Brownian motion. This is proved for instance in [5]. Super-Brownian motion with drift arises in applications. For example, Durrett and Perkins have shown that this process is the limit of rescaled contact process with long-range interaction (see [3]).

We want here to interpret the drift term of the b-snake in terms of killing. This can be done in several ways.

First we consider that b depends only on the "terminal point" $\hat{w} = w(\zeta)$ of the stopped path w that is $b(w) = \hat{b}(\hat{w})$ where $\hat{b}$ is a continuous nonnegative function on $\mathbb{R}^d$. In this case we consider a standard Brownian snake $(\tilde{W}_s)_{s\geq 0}$ with lifetime $(\tilde{\zeta}_s)_{s\geq 0}$ associated to a spatial motion which is a Markov process ξ in $\mathbb{R}^d$ having generator A, killed according to the function $2\hat{b}$. Thus the spatial motion actually takes place in $\mathbb{R}^d \cup \{\partial\}$ where ∂ denotes a cemetary point. Note that $(\tilde{\zeta}_s)$ is still supposed to be a reflecting

Brownian motion. Then we proceed to a time change in order to skip the periods where the spatial motion is killed. We set

$$A_s = \inf\{r;\ \int_0^r du\ \mathbf{1}_{\{\tilde{W}_u(\tilde{\zeta}_u)\neq\partial\}}\ > s\} \text{ and } W_s = \tilde{W}_{A_s}$$

Then $(W_s)_{s\geq 0}$ is simply a b-snake as it will be shown in section 3. Thus spatial killing is transformed after the suitable time change into a drift of the lifetime.

The second approach holds for a general continuous and nonnegative function b on $\mathcal{W}$. It consists of a "Poissonian pruning of the tree of trajectories". We place marks according to a random Poisson measure and cut all the corresponding branches. More precisely let us now consider a standard Brownian snake $(\tilde{W}_s)$ with lifetime $(\tilde{\zeta}_s)$ associated to a spatial motion which is a Markov process ξ in $\mathbb{R}^d$ with generator A (not killed!). We define the epigraph of $(\tilde{\zeta}_s)$ by

$$\Lambda = \{(s,t) \in [0,+\infty) \times [0,+\infty);\ t < \tilde{\zeta}_s\}.$$

Each $(s,t) \in \Lambda$ determines an excursion of the lifetime process $(\tilde{\zeta}_s)_{s\geq 0}$ above level t containing time s. The interval excursion is $[\alpha(s,t), \beta(s,t)]$ if we use the notations:

$$\alpha(s,t) = \sup\{s' \leq s;\ \tilde{\zeta}_{s'} = t\} \text{ and } \beta(s,t) = \inf\{s' \geq s;\ \tilde{\zeta}_{s'} = t\}.$$

We define a random point measure $\mathcal{N}$ whose conditional law knowing $(\tilde{W}_s)_{s\geq 0}$ is a Poisson measure

$$\mathcal{N}(ds\, dt) = \sum_i \delta_{(s_i,t_i)}(ds\, dt)$$

on $[0,+\infty) \times [0,+\infty)$ with intensity

$$\mathbf{1}_\Lambda(s,t)\ \frac{2\, b(\tilde{W}_s^{\leq t})}{\beta(s,t) - \alpha(s,t)}\ ds\, dt.$$

In this expression the notation $w^{\leq t}$ refers to the stopped path $w(\cdot \wedge t)$ with lifetime $\zeta(w) \wedge t$. Then we erase the branches corresponding to the excursion intervals $[\alpha(s_i,t_i), \beta(s_i,t_i)]$ associated to the atoms of the Poisson measure. This is a very redundant pruning since these intervals are far from being disjoint. Of course the erasure just consists in a time change. Denoting Leb

the Lebesgue measure and $(\cdot)^{\mathbf{c}}$ the complement of a set, we set

$$C_s = \mathrm{Leb}\left[[0,s] \cap \left(\bigcup_i [\alpha(s_i,t_i), \beta(s_i,t_i)]\right)^{\mathbf{c}}\right],$$
$$A_s = \inf\{r;\ C_r > s\};\quad W_s = \tilde{W}_{A_s}$$

Then $(W_s)_{s\geq 0}$ is a b-snake. This result is shown in section 4. Out of the snake context a straightforward corollary is the following result. This result is obtained when b depends only on the lifetime of the path i.e. $b(w) = \check{b}(\zeta(w))$ (In the case where $\check{b}$ is constant, this result can be found in [11]).

Let $(\tilde{\zeta}_s)_{s\geq 0}$ be a reflecting Brownian motion, $\Lambda, \alpha(\cdot,\cdot), \beta(\cdot,\cdot)$ be defined as previously, $\mathcal{N}$ be a Poisson measure with intensity

$$\mathbf{1}_\Lambda(s,t)\ \frac{2\check{b}(\zeta_t)}{\beta(s,t)-\alpha(s,t)}\ ds\,dt,$$

let C_s, A_s be defined as previously, then $\zeta_s = \tilde{\zeta}_{A_s}$ is a reflecting Brownian motion with drift $\check{b}$ that is $d\zeta_s = d\gamma_s - \check{b}(\zeta_s)\,ds$ on $\{\zeta_s > 0\}$ where (γ_s) is a Brownian motion.

The b-snake can also be constructed by approximation by discrete models. By Donsker's theorem we know that Brownian motion reflecting at 0 is the limit as $N \uparrow +\infty$ of a reflecting non-biased random walk scaled by factor $1/N$ in time variable and $1/\sqrt{N}$ in space. This applies to the lifetime of the Brownian snake. So it is not surprising that standard Brownian snake is the limit of discrete Brownian snakes $(W_t^N)_{t\geq 0}$ such that

- the lifetime $(\zeta_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ is a non-biased random walk on $\frac{1}{\sqrt{N}}\mathbb{N}$, reflecting at 0
- the lifetime is linear between consecutive times of $\frac{1}{N}\mathbb{N}$
- the conditional law of $(W_t^N)_{t\geq 0}$ knowing $(\zeta_t^N)_{t\geq 0}$ is the one of standard Brownian snake.

The b-snake $(\tilde{W}_t)_{t\geq 0}$ can be obtained similarly as limit of *discrete b-snakes* $(\tilde{W}_t^N)_{t\geq 0}$, replacing the non-biased random walk $(\zeta_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ by a random walk $(\tilde{\zeta}_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ biased according to b. More precisely the probability for the Markov chain $(\tilde{\zeta}_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ to go upward or downward are (approximately) equal to $\frac{1}{2}(1-\frac{1}{\sqrt{N}}b(W_t^N))$ and $\frac{1}{2}(1+\frac{1}{\sqrt{N}}b(W_t^N))$, respectively.

A proof of this result can be given recalling the fact that b-snake may be obtained by Poissonian pruning on a standard Brownian snake. In fact

if we operate a (discrete) Poissonian pruning on the discrete snake $(W_t^N)_{t\geq 0}$ we precisely obtain the law of $(\tilde{W}_t^N)_{t\geq 0}$. Let us give a brief interpretation of this result via Galton-Watson trees. The Markov chain $(W_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ is canonically associated to a Galton-Watson tree with a geometric progeny law of parameter 1/2. Indeed the height of the contour process of this tree is a non-biased random walk and gives the lifetime $(\zeta_t^N)_{t\in\frac{1}{N}\mathbb{N}}$. Spatial motions are then associated to the branches of the tree giving the paths $(W_t^N)_{t\in\frac{1}{N}\mathbb{N}}$. Two paths W_t^N and $W_{t'}^N$ then correspond to two sets of vertices on the tree which coincide up to a certain vertex. This vertex has a height which is the minimum of ζ^N between times t and t'. Poissonian pruning then consists in removing certain excursions of the snake that is removing certain edges in the tree. For simplicity let us suppose that b is constant (but the reasoning can be extended to general b). Then the pruning is a percolation on the tree and the connected component of the root is associated to the discrete b-snake. But we know that this component is a geometric Galton-Watson tree, this time with parameter different from 1/2. Thus the height of the contour process $(\tilde{\zeta}_t^N)_{t\in\frac{1}{N}\mathbb{N}}$ is now a biased random walk.

An application of the notion of b-snake is the representation of solutions of a semi-linear p.d.e. From [7] we know that standard Brownian snake gives a representation of the nonnegative solutions of $\Delta u = u^2$ with finite or infinite boundary conditions. We show in section 6 that the b-snake is associated similarly to the equation $\Delta u = 4u^2 + 4\hat{b}\,u$. For this result we suppose that b is a continuous nonnegative function of the form $b(w) = \hat{b}(\hat{w})$.

Finally we show that some b-snakes can be viewed as conditioned Brownian snake. More precisely we will show in section 7 that conditioning the standard Brownian snake to stay in a domain yields a b-snake. This result can be interpreted in terms of super-Brownian motion. The super-Brownian motion whose range is conditioned to be contained in a fixed domain is a super-Brownian motion with drift.

To conclude this introduction let us mention that a recent paper of Watanabe [12] deals with the same subject. However the aims and methods are different. Watanabe works mainly on the connections with the work of Warren [10].

2 Standard snake and b-snake

We consider a Polish space (E, d) and a continuous homogeneous Markov process ξ with values in E and generator A. We denote by $\mathcal{C}([0,+\infty), E)$ the space of all continuous functions from $[0,+\infty)$ to E. As annonced in

the introduction we call stopped path in E a pair $(w,\zeta) \in \mathcal{C}([0,+\infty),E) \times [0,+\infty)$ such that, for every $t \geq \zeta$, $w(t) = w(\zeta)$ and denote by $\mathcal{W}$ the space of all stopped paths in E. We call ζ the lifetime of the stopped path (w,ζ) and denote $\hat{w} = w(\zeta)$ the "terminal point" of the path w. We often abuse notation and write only w to designate the stopped path (w,ζ). In this case we use the notation $\zeta(w)$ or ζ_w to designate the lifetime. The space $\mathcal{W}$ is a complete metric space when equipped with the metric

$$\mathrm{dist}((w,\zeta);\ (w',\zeta')) = \sup_{t\geq 0} d(w(t),w'(t)) + |\zeta - \zeta'|.$$

For $x \in E$, we denote $\tilde{x}$ the path with lifetime 0 constantly equal to x. An easy example of a process with values in $\mathcal{W}$ is the so-called A–path process associated with ξ. This process is a homogeneous Markov process characterized under the probability Q_w $(w \in \mathcal{W})$ by:

- $W_0 = w$;
- if $s \geq 0$, then $\zeta_s = \zeta_w + s$, $W_s^{\leq \zeta_w} = w$ and the distribution of process $(W_s(\zeta_w+u), u \in [0,s))$ is that of the process ξ started at $\hat{w}$ and stopped at time s.

The generator of the A–path process will be denoted by L. For $F(w) = h(\zeta,\hat{w})$ where $h : [0,+\infty) \times E \to \mathbb{R}$ is a bounded continuous function such that $\left(\frac{\partial}{\partial t} + A\right) h$ exists and is also bounded, it is easy to show that

$$LF(w) = \left(\frac{\partial}{\partial t} + A\right) h(\zeta,\hat{w}).$$

We call standard Brownian snake with spatial motion ξ the continuous strong Markov process $((W_s,\zeta_s);\ s \in [0,+\infty))$ with values in $\mathcal{W}$ caracterized under the probability $\mathbb{P}_w$, by the following properties

- $W_0 = w$,
- $(\zeta_s;\ s \in [0,+\infty))$ has the law of a reflecting Brownian motion starting from $\zeta(w)$,
- the conditional law of $(W_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$ is the law of an inhomogeneous Markov process whose transition kernels are described as follows : for $0 \leq s < t$,
  - $W_t(u) = W_s(u)$ for all $u \leq m(s,t) = \inf_{s\leq v\leq t} \zeta_v$,

– conditionally given $W_s(m(s,t))$, $(W_t(m(s,t)+u),\ u \geq 0)$ is independent of W_s and distributed as the process ξ starting from $W_s(m(s,t))$ and stopped at time $\zeta_t - m(s,t)$.

Le Gall has proved the existence of the standard Brownian snake with motion ξ under the following technical continuity assumption that we will suppose to hold from now on: there exist an integer $k \geq 3$ and two positive constants C, ε such that for every $x \in E$, $t > 0$

$$E_x \left(\sup_{0 \leq s \leq t} d(x, \xi_s)^k \right) \leq C\ t^{2+\varepsilon}. \tag{1}$$

For $x \in E$, the path $\tilde{x}$ is regular for the standard Brownian snake and it is possible to consider the Itô measure $\mathbb{N}_x$ of the excursions of the snake outside $\tilde{x}$. Under $\mathbb{N}_x$ the lifetime process is distributed as the Itô measure of positive excursion of Brownian motion and the conditional law knowing the lifetime is the same as the one under $\mathbb{P}_w$. See [7] for more details.

The above construction is easily extended to the case where spatial motion ξ in E is killed, that is, takes its values in $E^\partial = E \cup \{\partial\}$. In this case we denote by $\mathcal{W}^\partial$ the stopped paths in E^∂. We use also the notations $\mathbb{P}_w^\partial$ and $\mathbb{N}_x^\partial$ instead of $\mathbb{P}_w$ and $\mathbb{N}_x$, respectively. For a general construction of the Brownian snake with a discontinuous spatial motion, we refer to [1].

The notion of a b-snake is introduced in [5]. We consider a Borel measurable function b on $\mathcal{W}$. We define laws $\mathbb{P}_w^b$ $(w \in \mathcal{W})$ on the canonical space $\mathcal{C}([0,+\infty), \mathcal{W})$ by :

$$\frac{d\mathbb{P}_w^b}{d\mathbb{P}_w}\Big|_{\mathcal{F}_s} = \exp\left(-\int_0^s b(W_r)\ d\beta_r - \frac{1}{2}\int_0^s b^2(W_r)\ dr \right)$$

where $(W_s)_{s \geq 0}$ now denotes the canonical process, $\mathcal{F}_s = \sigma(W_u,\ u \leq s)$, and β is the Brownian motion arising from the Tanaka representation $\zeta_s = \beta_s - \frac{1}{2}\ell_s^0(\zeta)$. The above definition is licit if the exponential local martingale on the right-hand side is a martingale that is has mean 1 for every s. In the case where the spatial motion is Brownian motion in $\mathbb{R}^d$ a sufficient condition is :

$$|b(w)| \leq C \left(\sup_{s \in [0,\zeta]} |w(s)| \right)^\beta$$

with $\beta < 2/3$. This is a consequence of Novikov criterion and a large deviation property for the Brownian snake (see [9] Theorem 1). For simplicity *we will suppose throughout this paper that b is bounded.*

In [5], Dhersin and Serlet proved that $\mathbb{P}_w^b$ is the solution of a well posed martingale problem. For $x \in \mathbb{R}^d$ let us denote $\mathcal{D}_x$ the set of functions F that can be written

$$F(w) = \int_0^\zeta g(w_{\leq r})\, dr$$

where $g : \mathcal{W} \to \mathbb{R}$ is bounded and belongs to the domain of L and satisfies $g(\tilde{x}) = 0$. Let us fix $w \in \mathcal{W}$ with $w(0) = x$. Then $\mathbb{P}_w^b$ is the unique probability measure on $\mathcal{C}([0, +\infty), \mathcal{W})$ such that, under $\mathbb{P}_w^b$, the canonical process is the solution of the following martingale problem: for every $F \in \mathcal{D}_x$,

$$M(F)_s = F(W_s) - F(w) - \frac{1}{2}\int_0^s Lg(W_r)\, dr - \int_0^s g(W_r)\, b(W_r)\, dr$$

is a $(\mathcal{F}_s)$-martingale with quadratic variation

$$\langle M(F)\rangle_s = \int_0^s g^2(W_r)\, dr.$$

Moreover this martingale is in fact

$$M(F)_s = \int_0^s g(W_r)\, d\zeta_r$$

for a standard snake (i.e. when $b = 0$, that is, under $\mathbb{P}_w$). At this point, we must note that, although ζ is only a semi-martingale, the finite variation process of the former integral vanishes thanks to the condition $g(\tilde{x}) = 0$ and so, under $\mathbb{P}_w$, the former formula for $M(F)$ gives a martingale.

To make precise a point of the introduction we note that, under $\mathbb{P}_w^b$, the lifetime process $(\zeta_s)_{s\geq 0}$ is a diffusion which solves

$$d\zeta_s - \frac{1}{2}d\ell_s^0(\zeta) = d\gamma_s - b(W_s)\, ds,$$

where $(\gamma_s)_{s\geq 0}$ is a linear Brownian motion.

3 Killing the spatial motion

Let E be a Polish space, ξ be a Markov process with values in E and generator A. We suppose that $\hat{b}$ is a bounded nonnegative Borel function on E and we set $b(w) = \hat{b}(\hat{w})$. This assumption on the form of b is not a real restriction to the study of b-snakes. Indeed, if we consider as spatial motion

the A-path process (instead of the Markov process ξ of generator A), then the terminal point $\hat{w}$ is a path distributed as ξ (stopped at its lifetime). We will use this trick in the next section.

We add to E a cemetary point ∂. Then we denote ξ^∂ the Markov process with values in $E^\partial = E \cup \{\partial\}$ which has the law of ξ killed according to $2b$. Its generator A^∂ is given, for g belonging to the domain of A by $A^\partial g = Ag - 2\hat{b}g$. Moreover the A^∂-path process has a generator $\tilde{L}$ given in terms of the generator of the A-path by $\tilde{L}g(w) = Lg(w) - 2b(w)\,g(w)$. In this equality the function g on $\mathcal{W}$ belongs to the domain of L and is extended to $\mathcal{W}^\partial$ by setting $g(w) = 0$ if $w([0,\zeta)) \ni \partial$, and similarly for Lg.

Theorem 1 *Let $(\tilde{W}_s)_{s\geq 0}$ be a standard Brownian snake with spatial motion ξ^∂ in E^∂. Set*

$$C_s = \mathrm{Leb}[\{u \in [0,s];\ \tilde{W}_u(\tilde{\zeta}_u) \neq \partial\}];\ A_s = \inf\{r; C_r > s\} \text{ and } W_s = \tilde{W}_{A_s}$$

Then $(W_s)_{s\geq 0}$ is a b-snake.

Proof. We will identify the b-snake thanks to its martingale problem. We fix a path $w \in \mathcal{W}$. We know that for every bounded g in the domain of L that satisfies $g(w(0)) = 0$ and for $F(w) = \int_0^\zeta g(w_{\leq r})\,dr$, we can write

$$F(\tilde{W}_s) = F(\tilde{W}_0) + \tilde{M}_s + \frac{1}{2}\int_0^s \tilde{L}g(\tilde{W}_r)\,dr$$

where $(\tilde{M}_s)_{s\geq 0}$ is under $\mathbb{P}_w^\delta$ a martingale with respect to the filtration $(\tilde{\mathcal{F}}_s)_{s\geq 0}$ where $\tilde{\mathcal{F}}_s = \sigma(\tilde{W}_u; u \leq s)$. In this equality we substitute A_s for s. We remark that the random times A_s are $(\tilde{\mathcal{F}}_s)$-stopping times. We deduce that $M_s = \tilde{M}_{A_s}$ is a martingale with respect to the filtration $\mathcal{F}_s = \sigma(\tilde{W}_{A_u};\ u \leq s)$. Moreover

$$\begin{aligned}
\int_0^{A_s} \tilde{L}g(\tilde{W}_r)\,dr &= \int_0^{A_s} \Big((Lg - 2bg)(\tilde{W}_r)\Big)\,dr \\
&= \int_0^{A_s} \Big((Lg - 2bg)(\tilde{W}_r)\Big)\,\mathbf{1}_{\{\tilde{W}_r(\tilde{\zeta}_r)\neq\partial\}}\,dr \\
&= \int_0^{A_s} \Big((Lg - 2bg)(\tilde{W}_r)\Big)\,dC_r \\
&= \int_0^{s} \Big((Lg - 2bg)(\tilde{W}_{A_r})\Big)\,dr \\
&= \int_0^{s} ((Lg - 2bg)(W_r))\,dr
\end{aligned}$$

We get finally

$$F(W_s) = F(W_0) + M_s + \frac{1}{2}\int_0^s (Lg - 2bg)(W_r)\, dr$$

and we recognize the martingale problem of the b-snake.

It remains to compute the quadratic variation process of the martingale M. But we know that

$$\tilde{M}_s = \int_0^s g(\tilde{W}_r) d\zeta_r$$

and by the definition of A and the fact that $g(w) = 0$ if $\partial \in w([0,\zeta))$, we have

$$\langle M \rangle_s = \langle \tilde{M} \rangle_{A_s} = \int_0^{A_s} g(\tilde{W}_r)^2 dr.$$

Then a similar computation to the former one leads to

$$\langle M \rangle_s = \int_0^s g(W_r)^2 dr$$

which is the expected quadratic variation process. □

4 Poissonian pruning

In this section b is a bounded nonnegative Borel function on $\mathcal{W}$. As in the introduction we consider a standard Brownian snake $\tilde{W}$ associated to a spatial motion of generator A and we suppose here that it starts a.s. from a trivial path $\tilde{x}$. We recall the notation Λ for the epigraph of the lifetime $(\tilde{\zeta}_s)_{s\geq 0}$ and $[\alpha(s,t), \beta(s,t)]$ for the excursion interval above level t and containing time s. Let $\mathcal{N}(ds\, dt) = \sum_i \delta_{(s_i,t_i)}(ds\, dt)$ be, conditionally on $(\tilde{W}_s, \tilde{\zeta}_s)$, a Poisson measure with intensity

$$n(ds\, dt) = \mathbf{1}_\Lambda(s,t)\, \frac{2\, b(\tilde{W}_s^{\leq t})}{\beta(s,t) - \alpha(s,t)}\, ds\, dt$$

We introduce the time change

$$C_s = \mathrm{Leb}\left[[0,s] \cap \left(\bigcup_i [\alpha(s_i,t_i), \beta(s_i,t_i)]\right)^{\mathbf{c}}\right], \quad A_s = \inf\{r;\ C_r > s\}$$

and set $W_s = \tilde{W}_{A_s}$.

Theorem 2 *The process $(W_s)_{s\geq 0}$ defined by the above procedure is a b-snake.*

Proof. Let us first set $\zeta^1_s = \tilde{\zeta}_s$ and $W^1_s(t) = \tilde{W}^{\leq t}_s$. Then $(W^1_s)_{s\geq 0}$ is a standard Brownian snake with spatial motion the A-path in $\mathcal{W}$. We define

$$\Gamma = \{(s,t) \in \Lambda;\ \exists i,\ s \in [\alpha(s_i,t_i), \beta(s_i,t_i)],\ t \geq t_i\}.$$

We set $\zeta^\partial_s = \zeta^1_s$ for every $s \geq 0$ and $W^\partial_s(t) = W^1_s(t)$ if $(s,t) \notin \Gamma$ and $W^\partial_s(t) = \partial$ if $(s,t) \in \Gamma$. Then $(W^\partial_s)_{s\geq 0}$ is a standard Brownian snake with spatial motion the A-path in $\mathcal{W} \cup \{\partial\}$ killed according to $2b$. Indeed it is easy to see that $(W^\partial_s)_{s\geq 0}$ is still a Markov process. Thus it is enough to check that $(W^\partial_s, W^\partial_{s'})$ has the desired joint law. We can even work conditionally on $(\zeta^\partial_s)_{s\geq 0}$ since its law is clear. It follows from the construction of $(W^\partial_s)_{s\geq 0}$ and from the independance property of Poisson measures that the paths W^∂_s and $W^\partial_{s'}$ coincide up to time $\inf_{[s,s']} \zeta^\partial$ and are independent after that time. Finally it suffices to check that one-dimensional marginal W^∂_s has the desired law. But it is the law of the A-path process killed at a time τ and the probability that $\tau > T$ conditionally on $(W^1_s(u);\ u \leq T)$ is

$$\begin{aligned}
\exp -n\left(\bigcup_{u\leq T} [\alpha(s,u), \beta(s,u)] \times [0,u]\right) &= \exp -\int_0^T du \int_{\alpha(s,u)}^{\beta(s,u)} dr \frac{2b(\tilde{W}^{\leq u}_r)}{\beta(r,u) - \alpha(r,u)} \\
&= \exp -\int_0^T du\ 2b(W^\partial_s(u))
\end{aligned}$$

which is the sought-after expression. □

As a consequence we may apply the result of section 3 and conclude that $W_s(r) = W^\partial_{A_s}(r)(r)$ defines a b-snake.

5 Approximation by discrete models

In this section we suppose that b is a nonnegative bounded *continuous* function on $\mathcal{W}$. Let us first recall the classical discrete approximation of the Brownian snake. We consider a Markov chain $(\zeta^N_{k/N})_{k\geq 0}$ which is a non-biased random walk on $\frac{1}{\sqrt{N}}\mathbb{N}$ reflecting at 0. That is for every $t \in \frac{1}{N}\mathbb{N}$, $x \in \frac{1}{\sqrt{N}}\mathbb{N} \setminus \{0\}$,

$$\mathbb{P}\left(\zeta^N_{t+\frac{1}{N}} = x + \frac{1}{\sqrt{N}} | \zeta^N_t = x\right) = \mathbb{P}\left(\zeta^N_{t+\frac{1}{N}} = x - \frac{1}{\sqrt{N}} | \zeta^N_t = x\right) = \frac{1}{2}$$

We define (ζ_t^N) for $t \in [0,+\infty)$ by making this process linear on every $[k/N,(k+1)/N]$. Then we obtain a $\mathcal{W}$-valued process (W_t^N) with lifetime (ζ_t^N) by assuming that the conditional law knowing the lifetime is the same as the one of standard Brownian snake. This process is called simple snake by Dynkin. The following result is well known (see [8]).

Proposition 3 *The process $(W_t^N)_{t\geq 0}$ converges in law to the standard Brownian snake.*

We are now going to see that b-snake can be obtained as a modification of the previous construction, changing non-biased random walk into biased random walk. More precisely we consider a $\mathcal{W}$-valued Markov chain $(\tilde{W}_{k/N}^N)_{k\geq 0}$ whose lifetime $(\tilde{\zeta}_{k/N}^N)_{k\geq 0}$ is a biased random walk on $\frac{1}{\sqrt{N}}\mathbb{N}$ reflecting at 0 such that, for every $t \in \frac{1}{N}\mathbb{N}$, $x \in \frac{1}{\sqrt{N}}\mathbb{N}\setminus\{0\}$,

$$\mathbb{P}\left(\zeta_{t+\frac{1}{N}}^N = x+\frac{1}{\sqrt{N}}\big|\zeta_t^N = x\right) = \frac{\exp -\frac{2}{\sqrt{N}}b(W_t^N)}{1+\exp -\frac{2}{\sqrt{N}}b(W_t^N)} \sim \frac{1}{2}(1-\frac{1}{\sqrt{N}}b(W_t^N))$$

$$\mathbb{P}\left(\zeta_{t+\frac{1}{N}}^N = x-\frac{1}{\sqrt{N}}\big|\zeta_t^N = x\right) = \frac{1}{1+\exp -\frac{2}{\sqrt{N}}b(W_t^N)} \sim \frac{1}{2}(1+\frac{1}{\sqrt{N}}b(W_t^N))$$

As usual, the path $\tilde{W}_{(k+1)/N}^N$ is defined conditionally to $\tilde{W}_{k/N}^N$ and to $\tilde{\zeta}_{(k+1)/N}^N$ by erasing $\tilde{W}_{k/N}^N$ in the case $\tilde{\zeta}_{(k+1)/N}^N < \tilde{\zeta}_{k/N}^N$ and by prolonging $\tilde{W}_{k/N}^N$ in the case $\tilde{\zeta}_{(k+1)/N}^N > \tilde{\zeta}_{k/N}^N$ by an independant piece of trajectory of generator A. Linear interpolation is then carried out to get $(\tilde{W}_t^N)_{t\in[0,+\infty)}$

Proposition 4 *The process $(\tilde{W}_t^N)$ converges in law to the b-snake.*

Proof. Our strategy is as follows. We start with standard approximation (W_t^N) and proceed to a (discrete) Poissonian pruning according to b, as in section 4. We show that the resulting process $(W_{A_t^N}^N)$ has the law of $(\tilde{W}_t^N)$. But the limit law as $N\uparrow+\infty$ of (W_t^N) is that of standard Brownian snake (W_t). We show that $(W_{A_t^N}^N)$ converges in law to (W_{A_t}) a standard Brownian snake "pruned according to b". The result follows since (W_{A_t}) was shown to have the law of the b-snake.

As annonced we consider first the process (W_t^N) defined previously. We consider a Poisson measure $\mathcal{N}^N$ on $\frac{1}{N}\mathbb{N}\times\frac{1}{\sqrt{N}}\mathbb{N}$ with intensity

$$n^N = \sum_{k,p} \mathbf{1}_{\frac{p}{\sqrt{N}}<\zeta_{k/N}^N} \quad \frac{\frac{1}{N\sqrt{N}}2b(W_{k/N}^{N\ \leq p/\sqrt{N}})}{\beta^N(\frac{k}{N},\frac{p}{\sqrt{N}})-\alpha^N(\frac{k}{N},\frac{p}{\sqrt{N}})-\frac{1}{N}} \quad \delta_{(\frac{k}{N},\frac{p}{\sqrt{N}})}$$

In this expression $[\alpha^N(\frac{k}{N},\frac{p}{\sqrt{N}}),\beta^N(\frac{k}{N},\frac{p}{\sqrt{N}})]$ is the excursion interval of (ζ_t^N) above level $\frac{p}{\sqrt{N}}$ containing time $\frac{k}{N}$. We denote Λ^N the union of all intervals $[\alpha^N(\frac{k}{N},\frac{p}{\sqrt{N}}),\beta^N(\frac{k}{N},\frac{p}{\sqrt{N}})]$ for $(\frac{k}{N},\frac{p}{\sqrt{N}})$ atom of the Poisson point measure $\mathcal{N}^N$. We set

$$A_t^N = \inf\{t';\ \int_0^{t'} du\ \mathbf{1}_{\Lambda^N}(u)\ > t\}$$

We have to show that $(W^N_{A_t^N})$ has the same law as $(\tilde{W}_t^N)$. Thus we have to check that $(W^N_{A^N_{k/N}})_{k\in\mathbb{N}}$ is a Markov chain with a lifetime governed by the above transition probabilities. Let us verify the latter point. For $x \in \frac{1}{\sqrt{N}}\mathbb{N}\backslash\{0\}$ and $t \in \frac{1}{N}\mathbb{N}$ let us estimate the conditional probability of $\zeta^N_{A^N_{t+\frac{1}{N}}} = x + \frac{1}{\sqrt{N}}$ knowing $\zeta^N_{A_t^N} = x$. We look at the original trajectory $\zeta^N_{A_t^N+u}$. Some –let us say a number k of – excursions above level x may be present in the original trajectory and then erased by the change of time. The probability that such an excursion appears in the original trajectory and is erased is $\frac{1}{2}(1-q)$ with

$$q = \exp - \left(\frac{1}{\sqrt{N}}\, 2b(W_t^N)\right).$$

This is easily seen using the definition of the intensity n^N. As a consequence,

$$\mathbb{P}\left[\zeta^N_{A^N_{t+\frac{1}{N}}} = x + \frac{1}{\sqrt{N}} \mid \zeta^N_{A_t^N} = x\right] = \sum_{k\geq 0}\left(\frac{1}{2}(1-q)\right)^k\ \frac{1}{2}q = \frac{q}{1+q}$$

as desired.

But we know that (W_t^N) converges in law to the standard Brownian snake (W_t). By the Dudley-Skorokhod representation theorem we may suppose that the convergence is uniform for t varying in a compact set, almost surely. As in section 4 we define a Poisson measure $\mathcal{N} = \sum_i \delta_{(s_i,t_i)}$ of intensity n and a change of time (A_t) by the formulas

$$n(ds\,dt) = \mathbf{1}_\Lambda(s,t)\ \frac{2b(W_s^{<t})}{\beta(s,t)-\alpha(s,t)}\ ds\,dt,$$

$$A_s = \inf\left\{r;\ \mathrm{Leb}\left[[0,r]\cap\left(\bigcup_i[\alpha(s_i,t_i),\beta(s_i,t_i)]\right)^{\mathbf{c}}\right] > s\right\}$$

We note that for every closed rectangle $[a,b]\times[c,d]$ included in the open set $\{(s,t);\ s<\zeta_t\}$ we have:

$$
\begin{aligned}
&n^N([a,b]\times[c,d])\\
&\quad=\frac{1}{N\sqrt{N}}\sum_{\substack{a\le\frac{k}{N}\le b\\ c\le\frac{p}{\sqrt{N}}\le d}}\frac{2b(W^{N\ \le p/\sqrt{N}}_{k/N})}{\beta^N(\frac{k}{N},\frac{p}{\sqrt{N}})-\alpha^N(\frac{k}{N},\frac{p}{\sqrt{N}})-\frac{1}{N}}\mathbf{1}_{\frac{p}{\sqrt{n}}<\zeta^N_{k/N}}\\
&\quad\to\int_a^b ds\int_c^d dt\ \mathbf{1}_\Lambda(s,t)\ \frac{2b(W_s^{\le t})}{\beta(s,t)-\alpha(s,t)}=n([a,b]\times[c,d])
\end{aligned}
$$

The last convergence is obtained by the dominated convergence theorem using lemma 9 of the appendix to get almost everywhere convergence. As a consequence we may assume that $\mathcal{N}^N([a,b]\times[c,d])$ converges to $\mathcal{N}([a,b]\times[c,d])$ for every rectangle included in Λ with rational bounds, almost surely. Then it is not difficult to see that $(W^N_{A^N_t})$ converges to (W_{A_t}) uniformly for t varying in a compact set, in probability. Indeed we fix $T>0$ and set $\Lambda=\{(s,t)\in[0,T]\times\mathbb{R}^*_+;\ s<\zeta_t\}$, $\Lambda_N=\{(s,t)\in[0,T]\times\mathbb{R}^*_+;\ s<\zeta^N_t\}$. We choose U contained in Λ and all Λ_N for N large on which $\beta-\alpha$ is bounded from below by a positive constant and such that all $\Lambda_N\setminus U$ for N large and $\Lambda\setminus U$ have a small Lebesgue measure. We can even choose U as a finite union of rectangles with rational bounds. We note that $\mathcal{N}(U)<+\infty$ and that the weak convergence of $\mathcal{N}^N$ to $\mathcal{N}$ on U implies the convergence of the corresponding atoms. Using lemma 9 of the appendix we can deal with the contribution to $|A^N_t-A_t|$ of the atoms of $\mathcal{N}^N$ and $\mathcal{N}$ in U. To control the contributions to A^N_t and A_t of the atoms outside U we recall that $\int_{\Lambda\setminus U}(\beta-\alpha)\,d\mathcal{N}$ has –thanks to the expression of the intensity n of $\mathcal{N}$– a mean bounded from above by $||b||_\infty$ Leb$(\Lambda\setminus U)$. Thus this variable can be controlled in probability and similarly for $\int_{\Lambda\setminus U}(\beta^N-\alpha^N)\,d\mathcal{N}^N$.

As a corollary we obtain the desired convergence in law and the proof is complete. □

6 Obtaining a p.d.e.

We will use here the construction of the b-snake via a standard Brownian snake with a killed spatial motion described in section 3. We now consider snakes with values in $E=\mathbb{R}^d$. So $\hat{b}$ now denotes a continuous nonnegative function on $\mathbb{R}^d$. For simplicity we write b instead of $\hat{b}$. We recall that the spatial motion ξ^∂, killed according to $2b$, is a Markov process with generator

$A^\partial g = Ag - 2bg$. It can also be defined by a density formula: for every bounded measurable function φ on $\mathcal{W}^\partial$ such that $\varphi(w) = 0$ if $w([0,\zeta]) \ni \partial$, we have, for every $t > 0$ and every $x \in E$,

$$E_x^\partial\Big[\varphi(w^{\leq t})\Big] = E_x\left[\varphi(w^{\leq t})\exp\left(-\int_0^t 2b(w_s)ds\right)\right]. \tag{2}$$

We write ℓ_s^a for the bicontinuous version of the local time of the lifetime process ζ at time s and level a. We denote by P_x^a the law of the spatial motion starting at x and stopped at time a, by $P_x^{\partial,a}$ the law of the killed spatial motion starting at x and stopped at time a and by $\mathbb{N}_x^\partial$ the Ito measure of the excursions of the Brownian snake (with killed spatial motion ξ^∂) out of trivial path $\tilde{x}$.

Let D be a bounded domain of E. For every $w \in \mathcal{W}^\partial$, we set

$$\tau(w) = \inf\{t \geq 0,\ w_t \notin D\}$$

the exit time of w from D, with the convention $\inf\emptyset = +\infty$ (this happens in particular if w is killed before exiting D).

We construct the exit measure of W out of D as in [7]. First, we define the local time on the boundary by approximation: we set

$$L_s^D = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{\tau(W_u)\leq\zeta_u<\tau(W_u)+\varepsilon\}}du.$$

Proposition 5 *For every bounded continuous function φ on $\mathcal{W}$ and every $x \in D$, we have*

$$\mathbb{N}_x^\partial\left[\int_0^\infty \varphi(W_s)dL_s^D\right] = E_x\left[\mathbf{1}_{\{\tau(w)<\infty\}}\varphi(w^{\leq\tau})\exp\left(-\int_0^\tau 2b(w_s)ds\right)\right].$$

Proof. Let φ be a bounded continuous function on $\mathcal{W}$. Using the law of the snake conditionally on the lifetime, we have

$$\begin{aligned}
&\mathbb{N}_x^\partial\left[\int_0^\infty \varphi(W_s)dL_s^D\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{N}_x^\partial\left[\int_0^\infty \mathbf{1}_{\{\tau(W_u)\leq\zeta_u<\tau(W_u)+\varepsilon\}}\varphi(W_u)du\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{N}_x^\partial\left[\int_0^\infty du\int P_x^{\partial,\zeta_u}(dw)\mathbf{1}_{\{\tau(w)\leq\zeta_u<\tau(w)+\varepsilon\}}\varphi(w)\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{N}_x^\partial\left[\int_0^\infty du\int P_x^{\zeta_u}(dw)\mathbf{1}_{\{\tau(w)\leq\zeta_u<\tau(w)+\varepsilon\}}\varphi(w)\right.\\
&\qquad\qquad\left.\exp\left(-\int_0^{\zeta_u} 2b(w_s)ds\right)\right]
\end{aligned}$$

by formula (2).
Now, the occupation time formula and Fubini's theorem gives

$$\begin{aligned}
&\mathbb{N}_x^\partial\left[\int_0^\infty \varphi(W_s)dL_s^D\right] \\
&\quad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{N}_x^\partial\left[\int_0^\infty da\,\ell_\infty^a \int P_x^a(dw)\mathbf{1}_{\{\tau(w)\le a<\tau(w)+\varepsilon\}}\varphi(w)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\exp\left(-\int_0^a 2b(w_s)ds\right)\right] \\
&\quad = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\infty da\mathbb{N}_x^\partial[\ell_\infty^a]\int P_x^a(dw)\mathbf{1}_{\{\tau(w)\le a<\tau(w)+\varepsilon\}}\varphi(w) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \exp\left(-\int_0^a 2b(w_s)ds\right)
\end{aligned}$$

We must then note that the "law" of the lifetime process under $\mathbb{N}_x^\partial$ is Itô measure of positive Brownian excursions and consequently, for every a, $\mathbb{N}_x^\partial[\ell_\infty^a]=1$. We then finish our computation as follows

$$\begin{aligned}
&\mathbb{N}_x^\partial\left[\int_0^\infty \varphi(W_s)dL_s^D\right] \\
&\quad = \lim_{\varepsilon\to 0}\int P_x(dw)\mathbf{1}_{\{\tau(w)<\infty\}}\frac{1}{\varepsilon}\int_{\tau(w)}^{\tau(w)+\varepsilon} da\,\varphi(w^{\le a})\exp\left(-\int_0^a 2b(w_s)ds\right) \\
&\quad = E_x\left[\mathbf{1}_{\{\tau(w)<\infty\}}\varphi(w^{\le\tau})\exp\left(-\int_0^\tau 2b(w_s)ds\right)\right]
\end{aligned}$$

by continuity and dominated convergence. □

We then define the exit measure X^D out of D by

$$\langle X^D,\varphi\rangle = \int_0^\infty \varphi(\widehat{W}_s)dL_s^D.$$

Theorem 6 *For every continuous function f on ∂D, the function*

$$u(x) = \mathbb{N}_x^\partial\left[1-\exp\left(-\langle X^D,f\rangle\right)\right]$$

is solution of the integral equation:

$$\begin{aligned}
&u(x) + E_x\left[\mathbf{1}_{\{\tau(w)<\infty\}}\int_0^\tau u(w_r)\left(2\,u(w_r)+2b(w_r)\right)dr\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = E_x\left[\mathbf{1}_{\{\tau(w)<\infty\}}f(w_\tau)\right].
\end{aligned}$$

Proof. We first re-express $u(x)$ using the definition of X^D :

$$\begin{aligned} u(x) &= \mathbb{N}_x^\partial \left[1 - \exp - \langle X^D, f\rangle\right] \\ &= \mathbb{N}_x^\partial \left[1 - \exp - \int_0^\infty dL_s^D f(\widehat{W}_s)\right] \\ &= \mathbb{N}_x^\partial \left[\int_0^\infty dL_s^D f(\widehat{W}_s) \exp - \int_s^\infty dL_r^D f(\widehat{W}_r)\right]. \end{aligned}$$

We then replace $\exp - \int_s^\infty dL_r^D f(\widehat{W}_r)$ by its predictable projection which is, thanks to the Markov property, $\mathbb{E}_{W_s}^{\partial,*}\left[\exp - \int_0^\infty dL_r^D f(\widehat{W}_r)\right]$ where $\mathbb{P}_w^{\partial,*}$ is the law of the snake (with killed spatial motion) starting from w and killed when the lifetime reaches 0. We then use the Poisson representation of $\mathbb{P}_w^*$ given in [7], Prop 2.5:

$$\begin{aligned} u(x) &= \mathbb{N}_x^\partial \left[\int_0^\infty dL_s^D f(\widehat{W}_s) \mathbb{E}_{W_s}^{\partial,*}\left[\exp - \int_0^\infty dL_r^D f(\widehat{W}_r)\right]\right] \\ &= \mathbb{N}_x^\partial \left[\int_0^\infty dL_s^D f(\widehat{W}_s) \exp -2 \int_0^{\zeta_s} dr\, \mathbb{N}_{W_s(r)}^\partial \Big(1 - \exp - \langle X^D, f\rangle\Big)\right] \\ &= \mathbb{N}_x^\partial \left[\int_0^\infty dL_s^D f(\widehat{W}_s) \exp -2 \int_0^{\zeta_s} dr\, u\big(W_s(r)\big)\right] \\ &= E_x \left[\mathbf{1}_{\{\tau<\infty\}} f(w_\tau) \exp\left(-2\int_0^\tau dr\, u(w_r)\right) \exp\left(-\int_0^\tau 2b(w_r)dr\right)\right] \qquad (*) \end{aligned}$$

The last equality follows from proposition 5. We then compute

$$
\begin{aligned}
&E_x\left[\mathbf{1}_{\{\tau<\infty\}}f(w_\tau)\left(1-\exp-\int_0^\tau\Big(2u(w_r)+2b(w_r)\Big)dr\right)\right]\\
&\quad=E_x\left[\mathbf{1}_{\{\tau<\infty\}}f(w_\tau)\int_0^\tau dt\Big(2u(w_t)+b(w_t)\Big)\right.\\
&\qquad\qquad\left.\exp-\int_t^\tau\Big(2u(w_r)+2b(w_r)\Big)dr\right]\\
&\quad=\int_0^\infty dtE_x\left[\mathbf{1}_{\{t<\tau<\infty\}}f(w_\tau)\Big(2u(w_t)+b(w_t)\Big)\right.\\
&\qquad\qquad\left.\exp-\int_t^\tau\Big(2u(w_r)+2b(w_r)\Big)dr\right]\\
&\quad=\int_0^\infty dtE_x\left[\mathbf{1}_{\{t<\tau<\infty\}}\Big(2u(w_t)+b(w_t)\Big)\right.\\
&\qquad\qquad\left.E_{w_t}\left[f(w_\tau)\exp-\int_0^\tau\Big(2u(w_r)+2b(w_r)\Big)dr\right]\right]\\
&\quad=\int_0^\infty dtE_x\left[\mathbf{1}_{\{t<\tau<\infty\}}\Big(2u(w_t)+2b(w_t)\Big)u(w_t)\right]\\
&\quad=E_x\left[\mathbf{1}_{\{\tau<\infty\}}\int_0^\tau\Big(2u(w_t)+2b(w_t)\Big)u(w_t)dt\right]
\end{aligned}
$$

by formula $(*)$. $\square$

If we suppose (just for the following proposition) that the spatial motion is a Brownian motion, we can deduce from the integral equation of Theorem 6 a probabilistic representation of a semi-linear Dirichlet problem.

Proposition 7 *Let D be a bounded regular domain of $\mathbb{R}^d$ and f be a continuous function ∂D. We set, for every $x\in D$,*

$$
\begin{aligned}
u_1(x)&=\mathbb{N}_x^\partial\left[1-\exp\left(-\langle X^D,f\rangle\right)\right],\\
u_2(x)&=\mathbb{N}_x^\partial\left[X^D\neq 0\right],\\
u_3(x)&=\mathbb{N}_x^\partial\left[\{\hat{W}_s;\ s\geq 0\}\cap D^{\mathbf{c}}\neq\emptyset\right].
\end{aligned}
$$

Then the functions u_1,u_2,u_3 are nonnegative solutions of $\Delta u=4\,u^2+4b\,u$ in D. The function u_1 satisfies the boundary condition

$$
\lim_{x\to y,\,x\in D}u_1(x)=f(y),\quad y\in\partial D
$$

The functions u_2 and u_3 satisfy the infinite boundary condition

$$\lim_{x \to y,\, x \in D} u_i(x) = +\infty, \quad y \in \partial D, \; i \in \{2, 3\}$$

and they are respectively the minimal and the maximal nonnegative solution of $\Delta u = 4\,u^2 + 4b\,u$ with infinite boundary condition.

Proof. The p.d.e. which follows from the integral equation obtained in the previous theorem and the boundary conditions can be obtained as in [7]. □

7 Drift by conditioning

Let us consider a non-empty open subset D of $\mathbb{R}^d$. We denote $\mathcal{R} = \{\hat{W}_s, s \geq 0\}$ the range of the standard Brownian snake. We now study the b-snake with drift $b(x) = 2\mathbb{N}_x(\mathcal{R} \cap D^c \neq \emptyset)$. This object is not yet well defined since we have supposed up to now that b is bounded which is not the case of the function $x \to \mathbb{N}_x(\mathcal{R} \cap D^c \neq \emptyset)$ on D. But this function is bounded on every open set U whose closure $\overline{U}$ is contained in D. Thus the process can be defined up to the exit time from U, for every U. This forms a consistent family of laws and hence the existence of the desired b-snake is obtained with the usual martingale representation.

In what follows, we denote by ℓ_s the local time at zero of the lifetime process ζ, τ_α the time when ℓ reaches α and $\mathbb{P}_w^\alpha$ the law of the standard Brownian snake with spatial motion of generator A, starting from w and stopped at time τ_α. Under this probability, we have $\mathbb{P}_w^1(\mathcal{R} \subset D) > 0$ for every initial path w included in D thanks to the continuity condition (1). Thus we can condition the standard Brownian snake not to exit D under $\mathbb{P}_w^1$.

Theorem 8 *Let D be a non-empty open subset of $\mathbb{R}^d$.*

Then the standard Brownian snake conditioned not to exit D is, under $\mathbb{P}_w^1$, a b-snake with drift $b(x) = 2\mathbb{N}_x(\mathcal{R} \cap D^c \neq \emptyset)$.

Proof. Let us denote by $\mathbb{P}'_w$ the law of the standard Brownian snake starting from w, stopped when the local time at 0 of the lifetime process reaches 1 and conditioned not to exit D. We have

$$\mathbb{E}'_w\left[H(W_s, s \geq 0)\right] = \frac{1}{\mathbb{P}_w^1(\mathcal{R} \subset D)}\, \mathbb{E}_w^1\left[H(W_s, s \geq 0)\; \mathbf{1}_{\mathcal{R} \subset D}\right]$$

Here (W_s) denotes the canonical process and H is a bounded measurable functional on $\mathcal{C}([0, +\infty), \mathcal{W})$.

Let us use the notation $\mathcal{R}^T = \{\hat{W}_s, s \leq T\}$. We will show that $\mathbb{P}'_w$ satisfies the martingale problem of the b-snake. Let $h > 0$ and F be a bounded measurable functional on $\mathcal{C}([0,h), \mathcal{W})$. We compute,

$$\begin{aligned}
&\mathbb{E}'_w(F(W_s, s \leq h)) \\
&= \frac{1}{\mathbb{P}_w(\mathcal{R}^{\tau_1} \subset D)} \mathbb{E}_w\left(F(W_{s\wedge\tau_1}, s \leq h)\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\right) \\
&= \frac{1}{\mathbb{P}_w(\mathcal{R}^{\tau_1} \subset D)} \mathbb{E}_w\left[F(W_{s\wedge\tau_1}, s \leq h)\mathbf{1}_{h>\tau_1}\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\right] \\
&\qquad + \frac{1}{\mathbb{P}_w(\mathcal{R}^{\tau_1} \subset D)} \mathbb{E}_w\left[F(W_s, s \leq h)\mathbf{1}_{h\leq\tau_1}\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\right].
\end{aligned}$$

Let us use the Markov property for computing the second term. We have

$$\begin{aligned}
&\mathbb{E}_w\left[F(W_s, s \leq h)\mathbf{1}_{h\leq\tau_1}\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\right] = \\
&\qquad\qquad \mathbb{E}_w\left[F(W_s, s \leq h)\mathbf{1}_{h\leq\tau_1}\mathbb{P}^{1-\ell_h}_{W_h}(\mathcal{R} \subset D)\mathbf{1}_{\mathcal{R}^h\subset D}\right]
\end{aligned}$$

If we now decompose the snake before and after the first 0 of the lifetime process, we may write

$$\mathbb{P}^{1-\ell_h}_{W_h}(\mathcal{R} \subset D) = \mathbb{P}^*_{W_h}(\mathcal{R} \subset D)\mathbb{P}^{1-\ell_h}_{\tilde{w(0)}}(\mathcal{R} \subset D)$$

where $\mathbb{P}^*_w$ denotes the law of the standard Brownian snake starting from w and stopped when its lifetime process first reaches 0.

From the Poisson representation given in proposition 2.5 of [7], we get

$$\mathbb{P}^*_w(\mathcal{R} \subset D) = \exp -2\int_0^{\zeta(w)} \mathbb{N}_{w(r)}(\mathcal{R} \cap D \neq \emptyset)dr.$$

For g belonging to the domain of L, we set $F^g(w) = \displaystyle\int_0^{\zeta(w)} g(w^{\leq r})dr$. We know that, under $\mathbb{P}_w$, if $g(\tilde{w(0)}) = 0$,

$$F^g(W_s) = F^g(w) + M^g_s + U^g_s$$

where $M^g = \int_0^{\cdot} g(W_r)d\zeta_r$ is a martingale with quadratic variation process $\langle M^g\rangle_s = \displaystyle\int_0^s g(W_r)^2 dr$ and $U^g_s = \dfrac{1}{2}\displaystyle\int_0^s Lg(W_r)dr$. If $g(\tilde{w(0)}) \neq 0$, we obtain

$$F^g(W_s) = F^g(w) + M^g_s + U^g_s + \frac{1}{2}g(\tilde{w(0)})\ell_s.$$

where $M^g = \int_0^\cdot g(W_r)d\gamma_r$ and $\gamma_r = \zeta_r - \frac{1}{2}\ell_r$ defines a Brownian motion, and U^g is as before.

To simplify the notations we identify b with $w \mapsto b(\hat{w})$. We proved that

$$\mathbb{P}^*_w(\mathcal{R} \subset D) = \exp -F^b(w).$$

So, we have under $\mathbb{P}_w$,

$$\begin{aligned}\mathbb{P}^*_{W_h}(\mathcal{R} \subset D) &= \exp -F^b(W_h) \\ &= \exp -\left(F^b(w) + M^b_h + U^b_h + \frac{1}{2}b(w(0))\ell_h\right).\end{aligned}$$

But, using that $Ab = b^2$,

$$U^b_h = \frac{1}{2}\int_0^h Ab(W_r)dr = \frac{1}{2}\int_0^h b^2(W_r)dr = \frac{1}{2}\langle M^b\rangle_h.$$

Thus, we obtain

$$\frac{\mathbb{P}^*_{W_h}(\mathcal{R} \subset D)}{\mathbb{P}^*_w(\mathcal{R} \subset D)} = \exp(-M^b_h - \frac{1}{2}\langle M^b\rangle_h - \frac{1}{2}b(\tilde{w(0)})\ell_h) = \mathcal{E}_h \exp -\frac{1}{2}b(w(0))\ell_h.$$

Moreover, the exponential formula for the Poisson point process of the snake excursions process gives, for $\alpha > 0$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}^\alpha_{\tilde{x}}(\mathcal{R} \subset D) = \exp -\frac{1}{2}\alpha b(x).$$

In particular,

$$\mathbb{P}^1_w(\mathcal{R} \subset D) = \mathbb{P}^*_w(\mathcal{R} \subset D)\mathbb{P}^1_{\tilde{w(0)}}(\mathcal{R} \subset D) = \exp -(F^b(w) + \frac{1}{2}b(w(0))).$$

We eventually get

$$\begin{aligned}\mathbb{E}'_w(F(W_s, s \le h)) = \mathbb{E}_w\left[F(W_{s\wedge\tau_1}, s \le h)\mathbf{1}_{h>\tau_1}\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\right]\exp(F^b(w)+\frac{1}{2}b(w(0))) \\ + \mathbb{E}_w\left[F(W_s, s \le h)\mathbf{1}_{h\le\tau_1}\mathbf{1}_{\mathcal{R}^h\subset D}\mathcal{E}_h\right]\end{aligned}$$

Thus,

$$Z_h = \left.\frac{d\mathbb{P}'_w}{d\mathbb{P}^1_w}\right|_{\mathcal{F}_h} = \mathbf{1}_{h\le\tau_1}\mathbf{1}_{\mathcal{R}^h\subset D}\mathcal{E}_h + \mathbf{1}_{h>\tau_1}\mathbf{1}_{\mathcal{R}^{\tau_1}\subset D}\exp(F^b(w) + \frac{1}{2}b(w(0)))$$

In particular, $\mathbb{P}'_w$-a.s., on $\{h < \tau_1\}$, $Z_h = \mathcal{E}_h$. Thanks to Girsanov theorem, if M is a martingale under $\mathbb{P}_w$, then $M' = M + V$ is a martingale under $\mathbb{P}'_w$ with same quadradic variation process, if

$$V = \frac{1}{Z} \cdot \langle M, Z \rangle$$

where the dot represents the integral operator and $\langle M, Z \rangle$ the bracket of M and N. We apply this theorem to $M = M^g$ with g in the domain of L that satisfies $g(\tilde{w(0)}) = 0$. We remind that $\mathcal{E} = 1 + \mathcal{E} \cdot M$. Then, on $\{h < \tau_1\}$,

$$\langle M^g, Z \rangle = \langle M^g, \mathcal{E} \rangle = \langle M^g, \mathcal{E} \cdot M^b \rangle = \mathcal{E} \cdot \langle M^g, M^b \rangle$$

But, as $\langle M^g, M^b \rangle_h = \int_0^h g(W_r) b(W_r) dr$, we get that

$$M'_h = M^g_h + \int_0^h g(W_r) b(W_r) dr$$

is a martingale under $\mathbb{P}'_w$ with quadratic variation process

$$\langle M' \rangle_h = \int_0^h g(W_r)^2 dr$$

and, under $\mathbb{P}'_w$,

$$F^g(W_h) = F^g(w) + M'_h - \int_0^h g(W_r) b(W_r) dr + \frac{1}{2} U^g_h$$

which is the martingale problem which characterizes the law of the b-snake. $\square$

Appendix: A convergence lemma.

Lemma 9 *Let $\mathcal{H}$ denotes the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $f^{-1}(\{0\})$ is an unbounded set containing 0. For $f \in \mathcal{H}$ we set*

$$\Lambda(f) = \{(s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+;\ t < f(s)\}$$

and, for $(s,t) \in \Lambda(f)$,

$$\alpha(f,s,t) = \sup\{s' \leq s;\ f(s') = t\},\ \beta(f,s,t) = \inf\{s' \geq s;\ f(s') = t\}.$$

We suppose that (f_n) is a sequence of functions of $\mathcal{H}$ converging to $f \in \mathcal{H}$, uniformly on every compact set of $\mathbb{R}_+$. We fix $(s,t) \in \Lambda(f)$ such that t is not a level of local extremum of f. We suppose that $s_n \to s$ and $t_n \to t$.

Then we have

$$\alpha(f_n, s_n, t_n) \to \alpha(f,s,t),\ \text{ and } \beta(f_n, s_n, t_n) \to \beta(f,s,t)$$

Proof. Let us suppose that a subsequence of $\alpha_n = \alpha(f_n, s_n, t_n)$ converges to a. In the inequalities $f_n(\alpha_n) = t_n$, $\alpha_n \leq s_n$, we pass to the limit along the subsequence and get $f(a) = t$, $a \leq s$ whence $\alpha(f,s,t) \geq a$. So $\limsup_n \alpha(f_n, s_n.t_n) \leq \alpha(f,s,t)$.

Now set $\alpha = \alpha(f,s,t)$ and let $\varepsilon > 0$ such that $\alpha + \varepsilon \leq s$. We have $f(\alpha) = t$. Since t is not a level of local extremum there exist s_1, s_2 such that $\alpha - \varepsilon \leq s_1 < \alpha < s_2 \leq \alpha + \varepsilon$ and $f(s_1) < t < f(s_2)$. For n large enough we have also $f_n(s_1) < t_n < f_n(s_2)$. Hence there exists s'_n such that $\alpha - \varepsilon \leq s_1 \leq s'_n \leq s_2 \leq s_n$ (the last inequality holds for n large enough). Thus $\alpha(f_n, s_n, t_n) \geq \alpha - \varepsilon$ and $\liminf_n \alpha(f_n, s_n, t_n) \geq \alpha(f,s,t) - \varepsilon$. This complete the proof for $\alpha(\cdot,\cdot,\cdot)$. The case of $\beta(\cdot,\cdot,\cdot)$ is similar. $\square$

Remark. We recall that the set of levels of local extrema of the continuous function f is finite or countable.

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