Summary. We prove the predictable representation property for the filtration of the Brownian snake and give a representation of the martingales in the filtration associated to the historical Brownian motion. We deduce a representation of the martingale measure of the historical Brownian motion.

Key words: Super-Brownian motion, Brownian snake, predictable representation property, martingale measure
Representation of the martingales for the Brownian snake

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1 Introduction and statement of results

In this paper we deal with the Brownian snake introduced by Le Gall which can be seen as a continuous-time parametrization of a tree of branching trajectories. Abundant literature has shown the interest of this process to prove results on super-processes, to express solutions of certain semi-linear pde or to describe the limit behaviour of important interacting particle systems. See [Lg] for a comprehensive treatment of the subject. The definitions of the basic objects and the terminology used in the present introduction are however recalled in the next section.

The Brownian snake \((W_t)\) is a simple example of process taking its values in the space \(\mathcal{W}\) of stopped paths in \(\mathbb{R}^d\). In [DS] we have developed some tools of stochastic calculus for this process, in particular an Itô formula; a simplified form of this statement is as follows. Suppose \(F^{(2)} : \mathcal{W} \to \mathbb{R}\) is a continuous function and \(F^{(1)}, F : \mathcal{W} \to \mathbb{R}\) are defined by

\[
F^{(1)}(u) = \int_0^t F^{(2)}(w_{\leq r}) \, dr
\]

\[
F(u) = \int_0^t F^{(1)}(w_{\leq r}) \, dr = \int_0^t \int_0^r F^{(2)}(w_{\leq u}) \, du
\]

then, for \(0 \leq r < t\):

\[
F(W_t) = F(W_r) + \int_r^t F^{(1)}(W_s) \, d\kappa_s + \frac{1}{2} \int_r^t F^{(2)}(W_s) \, ds. \quad (1)
\]

The present paper is devoted to some applications of this formula concerning the representation of the martingales associated to the Brownian snake.
First, the filtration \((\mathcal{F}_t^W)\) of the Brownian snake has a surprising previsible representation property:

**Theorem 1** For every random variable \(X \in L^2(\mathcal{F}_\infty^W)\), there exists a \((\mathcal{F}_s^W)\)-previsible process \((H_s)\) vanishing a.s. on \(\{s, \zeta_s = 0\}\) such that

\[
E \left( \int_0^\infty H_s^2 \, ds \right) < +\infty \quad \text{and} \quad X = E(X) + \int_0^\infty H_s \, d\zeta_s. \tag{2}
\]

The proof is given in section 3. By standard arguments (see [RY] V.3.4), we quickly deduce the following corollary

**Proposition 2** For every local martingale \((M_s)\) with respect to the filtration \((\mathcal{F}_s^W)\), there exists a \((\mathcal{F}_s^W)\)-previsible process \((H_s)\) locally in \(L^2\) such that

\[
M_s = M_0 + \int_0^s H_r \, d\zeta_r. \tag{3}
\]

Note that the above stochastic integral is effectively a local martingale because the integrand \(H_r\) vanishes on \(\{s, \zeta_s = 0\}\) and we could write the integral with respect to the martingale part of the reflecting Brownian motion \((\zeta_s)\).

One of the interests of the Brownian snake lies in its connections with super-Brownian motion. More precisely, let us consider a Brownian snake starting from \(\bar{x}\) and set

\[
\forall t \geq 0, \quad X_t = \int_0^{\tau_t} d_s L_s^x(\zeta) \, d\bar{W}_s \quad \text{and} \quad H_t = \int_0^{\tau_t} d_s L_s^x(\zeta) \, d\zeta,
\]

where \(L_s^x(\zeta)\) is the local time of the lifetime process \((\zeta)\) at level \(\bar{x}\) and time \(s\), and \(\tau_t = \inf\{s \geq 0; L_s^x(\zeta) > 1\}\) is the hitting time of \(1\) by the local time of \(\zeta\) at level \(0\). The process \((X_t)\) [resp. \((H_t)\)] takes its values in the space \(\mathcal{M}_F(\mathbb{R}^d)\) [resp. \(\mathcal{M}_F(W)\)] and is called super-Brownian motion starting from \(\delta_{\bar{x}}\) [resp. historical Brownian motion starting from \(\delta_{\bar{x}}\)]. To be honest, the usual definitions include a factor \(1/4\) that we have dropped here to simplify notations, as we did in [DS]. From this definition emerges a new filtration to be considered. Let

\[
\tau_s^\wedge = \inf\left\{u; \int_0^{u \wedge \tau_s} 1_{\{\zeta_s \leq t\}} \, dv > s\right\}
\]

be the inverse of the time spent by the lifetime \((\zeta_s)\) under level \(t\) with the convention \(\inf\{\emptyset\} = \tau_s\). We see that at least the following \(\sigma\)-algebras naturally arise:

\[
\mathcal{G}_t = \sigma(W^t_s, s \geq 0) \\
\mathcal{G}_s^x = \sigma(\zeta^t_s, s \geq 0) \\
\mathcal{G}_r^x = \sigma(X_r, r < t)
\]
\[ G_t^H = \sigma(H_r, r \leq t) \]
\[ \mathcal{F}_s^H = \sigma(W_{r\wedge\tau_1}, r \leq s) \]
\[ \mathcal{F}_s^X = \sigma(\mathcal{G}_{r\wedge\tau_1}, r \leq s) \].

We note the following obvious relations
\[ G_t^N = \mathcal{G}_t \cap \mathcal{F}_{\infty}^N, \quad G_t^X \subset G_t^H \subset G_t, \quad \mathcal{G}_{\infty} = \mathcal{F}_{\infty}. \]

A question now arises concerning the representation of the martingales in the “vertical” filtration \( \mathcal{G}_t \). Since super-Brownian motion (and historical Brownian motion) has often been studied via martingale problems we already know a class of \( \mathcal{G}_t \)-martingales : for every bounded \( \phi \) in the domain of the generator \( A \),
\[ M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(A\phi) \, ds \]
defines a martingale and moreover the quadratic variation of this martingale is known to be
\[ \langle M(\phi) \rangle_t = 4 \int_0^t X_s(\phi^2) \, ds. \] (4)

An interpretation with the Brownian snake can be given.

**Proposition 3 ([DS] Lemma 10 and Theorem 7)** We have
\[ M_t(\phi) = 2 \int_0^{\tau_1} 1_{[s,t]}(\zeta_s) \phi(W_s) \, d\zeta_s \] (5)
and this process is a \( \mathcal{G}_t \)-martingale with quadratic variation given by (4).

A natural question is then to ask if any \( \mathcal{G}_t \)-martingale can be expressed in a way that generalizes the above expression.

**Theorem 4** For every \( \mathcal{G}_t \)-martingale \( (M_t) \) which is bounded in \( L^2 \), there exists a \( \mathcal{F}_s \)-previsible process \( (H_s) \) such that
\[ \mathbb{E} \left( \int_0^t H_s^2 \, ds \right) < +\infty \text{ and } \forall t \geq 0, \quad M_t = M_0 + \int_0^t H_s 1_{[0,t]}(\zeta_s) \, d\zeta_s. \]

By [Je] this result is known when it is restricted to the filtration \( \mathcal{G}_t^N \) i.e. considering only the lifetime. In that case it is essentially seen on Tanaka’s formula interpreted as a reflection equation. Our proof is given in section 4.

We deduce the corollary :

**Proposition 5** For every \( Y \in L^2(\mathcal{G}_t) \), there exists a \( \mathcal{F}_s \)-previsible process \( (H_s) \) such that
\[ Y = \mathbb{E}(Y) + \int_0^{\tau_1} H_s 1_{[s,t]}(\zeta_s) \, d\zeta_s \text{ and } \mathbb{E} \left( \int_0^{\tau_1} H_s^2 1_{[0,t]}(\zeta_s) \, ds \right) < +\infty. \]
The representation given in (5) can be pushed a little further into the terminology of martingale measure as the notation already suggests. The notion of martingale measure of super-processes is explained for instance in [Da] Chapter 7; Example 7.1.3 covers the case of super-Brownian motion as it is defined here. In our setting the martingale measure of super-Brownian motion or even historical Brownian motion is easily described. It is stated in the following proposition where \( L \) denotes the generator of the so-called \( A \)-path process which is the process in \( \mathcal{W} \) whose lifetime increases at constant speed 1 and consists in a trajectory of the diffusion governed by \( A \).

**Proposition 6** Let us set, for \( t \geq 0 \) and \( \Omega \in \mathcal{B}(\mathcal{W}) \),

\[
M_t(\Omega) = 2 \int_0^{T_t} 1_{(0,t]}(\zeta_r) \, 1_\Omega(W_r) \, d\zeta_r.
\]

Then, \( (M_t(\Omega), t \geq 0, \Omega \in \mathcal{B}(\mathcal{W})) \) defines an \( L^2 \)-martingale measure \( M(ds \, dw) \). It is associated to the historical Brownian motion, that is, for every \( \phi : \mathcal{W} \to \mathbb{R} \) in the domain of \( L \) and bounded,

\[
H_\phi(\phi) = H_\phi(\phi) + \int_0^t H_\phi(L \phi) \, ds + \int_0^t \int_\mathcal{W} \phi(w) \, M(ds \, dw).
\]

This martingale measure is orthogonal and its intensity is the random measure \( \nu \) on \( \mathbb{R}_+ \times \mathcal{W} \) given by

\[
\int_{\mathbb{R}_+ \times \mathcal{W}} \psi(t, w) \, \nu(dt \, dw) = 4 \int_0^{T_t} \psi(\zeta_s, W_s) \, ds.
\]

We recall that the intensity of a martingale measure is defined so that \( \nu([0,t] \times \Omega) \), for \( \Omega \) Borel subset of \( \mathcal{W} \), is the quadratic variation of the martingale \( (M_t(\Omega)) \).

2 Basic objects and notations

We will use the following common notations:

- \( \mathbb{N} = \{1, 2, 3, \ldots \} \), \( \mathbb{Z}^+ = \{0, 1, 2, \ldots \} \), \( \mathbb{R}^+ = [0, +\infty) \).
- \( \mathcal{C}(X, Y) \): set of continuous functions from metric space \( X \) to metric space \( Y \).
- \( \sigma(X_i, i \in I) \): \( \sigma \)-algebra generated by the random variables \( X_i \), \( i \in I \) in a fixed probability space, completed with all negligible sets.
- \( \mathcal{B}(X) \): Borel \( \sigma \)-algebra of the metric space \( X \).
- \( \mathcal{M}_F(X) \): set of all finite measure on the metric space \( X \) equipped with the (metrizable) topology of weak convergence and its Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{M}_F(X)) \).

A stopped path is a couple \( (w, \zeta) \), where \( \zeta \geq 0 \) is called the lifetime of the path, and \( w : \mathbb{R}^+ \to \mathbb{R}^d \) is a continuous mapping, which is constant.
on \([\zeta, +\infty)\). We denote by \(\mathcal{W}\) the set of all stopped paths. We sometimes abbreviate \((w, \zeta]\) into \(w\) and denote \(\zeta(w)\) the lifetime. The distance on \(\mathcal{W}\) is
\[
d(u, v) = \sup_{t \geq 0} |u(t) - v(t)| + |\zeta(u) - \zeta(v)|,
\]
making \(\mathcal{W}\) a Polish space. We denote by \(w = w(\zeta)\) the endpoint of \(w\), and \(x\) the path of lifetime 0 started at \(x \in \mathbb{R}^d\). Finally, we denote by \(w \leq_r\) the path of lifetime \(\zeta(w) \wedge r\) such that for \(u \geq 0\), \(w \leq_r(u) = w(u \wedge r)\).

Let us fix a diffusion in \(\mathbb{R}^d\) with generator \(A\). The Brownian snake started at \(x\) with spatial motion governed by \(A\) is the strong Markov continuous process \(W = (W_s, s \geq 0)\) with values in \(\mathcal{W}\) characterized by the following properties:

1. \(W_0(0) = x\) for every \(s\);
2. The lifetime process \(\zeta_s = \zeta(W_s)\) is a reflecting Brownian motion in \(\mathbb{R}_+\);
3. Conditionally on \((\zeta_s, s \geq 0)\), the distribution of \((W_s, s \geq 0)\) is that of an inhomogeneous Markov process whose transition kernels are described as follows: for every \(s < s'\),
   - \(W_{s'}^{\leq m} = W_{s'}^{\leq m}\) where \(m = \inf_{t \leq s'} \zeta_t\);
   - \((W_s(m + t), 0 \leq t \leq \zeta_s - m)\) is independent of \(W_s\) conditionally on \(W_s(m)\) and has the law of a diffusion in \(\mathbb{R}^d\) with generator \(A\), starting from \(W_s(m)\) and stopped at time \(\zeta_s - m\).

The filtration \(\mathcal{F}_t^W\) used in the introduction is the filtration associated to \((W_s)\), completed the usual way (see [RY] p. 45 and 93 for precisions on completion) and
\[
\mathcal{F}_t^W = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t^W).
\]

3 Proof of the previsible representation property
(Theorem 1)

This proof is inspired by Exercise 3.15 of [RY], dealing with the classical Brownian filtration. In that case a step of the proof is to solve a linear differential equation. This is replaced in our path space setting by an integral equation that we first discuss.

**Lemma 7** Let \(\alpha > 0\) and
\[
\mathcal{E}_\alpha = \left\{ \psi \in \mathcal{C}(\mathcal{W}, \mathbb{R}); \sup_{w \in \mathcal{W}} e^{-\alpha \zeta} |\psi(w)| < +\infty \right\}.
\]
For all \(\lambda > 0\), if \(\alpha > \sqrt{2\lambda}\) and \(f \in \mathcal{E}_\alpha\) then there exists \(\varphi \in \mathcal{E}_\alpha\) such that
\[
\forall w \in \mathcal{W}, \quad \frac{1}{2} \varphi(w) - \lambda \int_{\zeta - u}^\zeta \varphi(w \leq u) \, du = f(w).
\]
Proof. It is easy to see that the formula \( \|\varphi\|_{c} = \sup_{w \in \mathcal{W}} e^{-\alpha \xi} |\varphi(w)| \) defines a norm on the vector space \( \mathcal{E}_{\alpha} \) which makes this space complete. The result of the lemma consists in finding a fixed point for the map

\[
\theta : \varphi \rightarrow \left( w \rightarrow 2 f(w) + 2\int_{0}^{\xi} (\zeta - u) \varphi(w_u) \, du \right).
\]

It is easy to verify that \( \theta \) maps \( \mathcal{E}_{\alpha} \) into itself. In order to apply the classical Lipschitz fixed point theorem to \( \theta \) in the Banach space \( \mathcal{E}_{\alpha} \), it remains to check that \( \theta \) satisfies the Lipschitz condition. For \( \varphi_1, \varphi_2 \in \mathcal{E}_{\alpha} \),

\[
\| \theta(\varphi_1) - \theta(\varphi_2) \|_{\alpha} = \sup_{w \in \mathcal{W}} e^{-\alpha \xi} \left( 2 \int_{0}^{\xi} (\zeta - u) (\varphi_1(w_u) - \varphi_2(w_u)) \, du \right)
\leq \sup_{w \in \mathcal{W}} e^{-\alpha \xi} \left( 2 \int_{0}^{\xi} (\zeta - u) e^{\alpha u} \| \varphi_1 - \varphi_2 \|_{\alpha} \, du \right)
\leq 2\lambda \| \varphi_1 - \varphi_2 \|_{\alpha} \sup_{\zeta > 0} \left\{ e^{-\alpha \xi} \int_{0}^{\xi} (\zeta - u) e^{\alpha u} \, du \right\}
\leq 2\lambda \| \varphi_1 - \varphi_2 \|_{\alpha} \int_{0}^{+\infty} v e^{\alpha v} \, dv = \frac{2\lambda}{\alpha} \| \varphi_1 - \varphi_2 \|_{\alpha}.
\]

Since the last ratio is by assumption smaller than 1, the proof of the lemma is complete.

We now denote \( \bar{\mathcal{E}} \) the increasing limit of the sets \( \mathcal{E}_{\alpha}, \alpha > 0 \), that is \( \bar{\mathcal{E}} = \bigcup_{\alpha > 0} \mathcal{E}_{\alpha} \). We are now ready to give a previsible representation for certain variables, namely the type on the left-hand side of the following equality.

Lemma 8 For every \( f \in \mathcal{E} \) and every \( \lambda > 0 \), there exist \( g_{\bar{0}}, g_{\bar{1}} \in \mathcal{E} \) such that, for every \( r > 0 \),

\[
\int_{-r}^{r} e^{-\lambda s} f(W_s) \, ds = e^{-\lambda r} g_{\bar{0}}(W_r) + \int_{-r}^{r} e^{-\lambda s} g_{\bar{1}}(W_s) \, d\zeta_s. \quad (9)
\]

Moreover \( g_{\bar{0}} \) vanishes at 0 in the following sense : \( g_{\bar{0}}(w) = 0 \) if \( \zeta(w) = 0 \) and identically for \( g_{\bar{1}} \).

Proof. Let us first remark that, since \( f \) belongs to a certain \( \mathcal{E}_{\alpha} \), we have

\[
\int_{-r}^{r} e^{-\lambda s} \| f(W_s) \| \, ds \leq \| f \|_{\alpha} \int_{-r}^{r} e^{-\lambda s} e^{\alpha \zeta_s} \, ds
\]

and the integral on the right-hand side is finite because the reflecting Brownian motion \( (\zeta_s) \) satisfies the law of the iterated logarithm. Hence the integral appearing on the left-hand side of equation (9) is defined almost surely. So is the integral on the right-hand side using a similar argument and [RY] IV.1.26.
By increasing \( \alpha \) if necessary, we may suppose that \( \alpha > \sqrt{2\lambda} \). By Lemma 7, we can associate to \( f \in \mathcal{E}_\alpha \) a continuous function \( \varphi \in \mathcal{E}_\alpha \) as specified. Let \( F^{(2)} = \varphi \) and \( F^{(1)}, F \) be defined as in the assumptions of formula (1). Note that \( F^{(1)} \) and \( F \) vanish at 0, in the sense defined in the statement of the lemma. It is easy to check that \( F^{(1)}, F \in \mathcal{E}_\alpha \) and more precisely,

\[
|F(w)| \leq ||\varphi||_\alpha e^{\alpha \zeta}.
\]  

(10)

We obtain, by formula (1) and the classical Itô formula for a product, for \( 0 \leq r < t \):

\[
e^{-\lambda t} F(W_t) - e^{-\lambda r} F(W_r) = \int_r^t e^{-\lambda s} \left( \frac{1}{2} \left( \varphi - \lambda F \right)(W_s) \right) ds + \int_r^t e^{-\lambda s} F^{(1)}(W_s) d\zeta_s.
\]

We recall that \( (1/2) \varphi - \lambda F = f \). Using the bound (10), the law of the iterated logarithm for \( (\zeta_s) \) entails that \( \lim_{s \to +\infty} e^{-\lambda s} F(W_s) = 0 \), almost surely. Therefore we get

\[
\int_r^\infty e^{-\lambda s} f(W_s) ds = -e^{-\lambda r} F(W_r) - \left[ \int_r^\infty e^{-\lambda s} F^{(1)}(W_s) d\zeta_s \right]
\]

We obtain the sought after representation, up to a change of notations.

**Lemma 9** For all \( n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n > 0, f_1, \ldots, f_n \in \mathcal{E} \), there exist \( h_0, h_1, h_2 > 0 \), \( (1 \leq k \leq n - 1, 0 \leq j \leq k), g_0, g_1, g_2 \in \mathcal{E} \) \( (1 \leq k \leq n - 1, 0 \leq j \leq k) \) with \( g_0, g_1, g_2 \) vanishing at 0, such that, for all \( r \geq 0 \),

\[
\int_{[r < s_1 < \ldots < s_n]} \left( \prod_{i=1}^n e^{-\lambda_i s_i} f_i(W_{s_i}) \right) ds_1 \ldots ds_n
\]

\[
\quad = e^{-h r} g_0(W_r) + \int_r^\infty \left\{ e^{-h s} g_1(W_s) + \sum_{k=1}^{n-1} e^{-h s} g_2(W_s) \right\} \int_{[r < s_1 < \ldots < s_k < s]} \left( \prod_{j=1}^k e^{-\lambda_j s_j} g_2(W_{s_j}) \right) ds_1 \ldots ds_k d\zeta_s
\]

with the convention that the sum over \( k \) disappears if \( n = 1 \).

**Proof.** By equation (9), we know that the lemma is true for \( n = 1 \). Then we proceed by induction. Admitting the result at rank \( n \geq 1 \), we examine the case of rank \( n + 1 \):

\[
\int_{[r < s_1 < \ldots < s_{n+1}]} \left( \prod_{i=1}^{n+1} e^{-\lambda_i s_i} f_i(W_{s_i}) \right) ds_1 \ldots ds_{n+1}
\]

\[
\quad = \int_r^\infty ds_1 e^{-\lambda_1 s_1} f_1(W_{s_1}) \int_{[s_1 < s_2 < \ldots < s_{n+1}]} \left( \prod_{i=2}^{n+1} e^{-\lambda_i s_i} f_i(W_{s_i}) \right) ds_2 \ldots ds_{n+1}
\]
\[ \int_0^{\infty} ds \quad e^{-\lambda_1 s} f_1(W_{s_1}) \left[ e^{-\mu_0 s_1} g_0(W_{s_1}) + \int_0^{\infty} \left\{ e^{-\mu s} g_1(W_s) + \sum_{k=2}^{n} e^{-\nu_k s} g_k(W_s) \int_{s_1 < s_2 < \cdots < s_k < s} \prod_{j=2}^{k} e^{-\mu_j s_j} g_j(W_{s_j}) \, ds_2 \ldots ds_k \right\} \, ds \right] \]

\[ = \int_0^{\infty} ds \quad e^{-\lambda_1 s} f_1(W_{s_1}) \left( \int_0^{s} e^{-\lambda_1 s_1} f_1(W_{s_1}) \, ds_1 \right) \, ds \]

\[ + \sum_{k=2}^{n} \int_0^{\infty} ds \quad e^{-\nu_k s} g_k(W_s) \left( \int_0^{s} ds_1 e^{-\lambda_1 s_1} f_1(W_{s_1}) \left( \prod_{j=2}^{k} e^{-\mu_j s_j} g_j(W_{s_j}) \right) \, ds_2 \ldots ds_k \right) \, ds \]

The first equality is simply Fubini’s formula; then we use the induction hypothesis for the integral with respect to \( s_2, \ldots, s_{n+1} \); and for the last equality a stochastic version of Fubini’s theorem. To the first term obtained at the last equality we can apply the result at rank 1 i.e. Equation (9); the second and third term are the desired quantities to obtain the sought-after formula, up to a change of notations of course.

**Lemma 10** For every \( s > 0 \), there exist, for every \( i \in \mathbb{N} \), coefficients \( m_i \in \mathbb{N} \), \( \alpha_j^i \in \mathbb{R} \), \( \lambda_j^i > 0 \) for \( 1 \leq j \leq m_i \) such that, for every continuous function \( \gamma : \mathbb{R}_+ \to \mathbb{R} \) (absolutely) integrable over \( \mathbb{R}_+ \),

\[ \int_0^{+\infty} \left( \sum_{j=1}^{m_i} \alpha_j^i e^{-\lambda_j^i s} \right) \gamma(s) \, ds \xrightarrow{i \to +\infty} \gamma(s). \]

**Proof.** We first consider an approximation \( p_i(s) \, ds \) of the Dirac measure \( \delta_1 \) with continuous density whose support is contained in \((0, 2s)\) so that we have

\[ \int_0^{+\infty} p_i(s) \, ds \xrightarrow{i \to +\infty} \gamma(s) \]

for every continuous \( \gamma \). The set of functions over \( \mathbb{R}_+ \)

\[ A = \left\{ \psi : s \to \sum_{j=1}^{m_i} \alpha_j^i e^{-\lambda_j^i s}; \, m_i \in \mathbb{N}, \, \alpha_j^i \in \mathbb{R}, \, \lambda_j^i > 0, \, \psi(0) = 0 \right\} \]

is a linear subspace, closed under multiplication. Let \( \arg z \in [-\pi, \pi] \) denote the value of the argument of \( z \in \mathbb{U} = \{ z \in \mathbb{C}; \, |z| = 1 \} \). On the compact
U, equipped with uniform topology, we can apply the classical Stone-Weierstrass approximation theorem to the set of continuous functions:

\[
\left\{ z \rightarrow a + \psi \left( \tan \frac{1}{4} (\arg z + \pi) \right) ; a \in \mathbb{R}, \ \psi \in A \right\}
\]

in order to approximate by functions of this set, the continuous function \( z \rightarrow \rho_i(\tan(\arg z + \pi)/4) \). It is thus possible to find \( a_i \in \mathbb{R}, m_i \in \mathbb{N}, \ \alpha_j \in \mathbb{R}, \ \lambda_j > 0 \) such that

\[
\sup_{s \in \mathbb{R}_+} |\rho_i(s) - \psi_j(s)| \xrightarrow{s \to +\infty} 0 \text{ with } \psi_j : s \rightarrow a_i + \sum_{j=1}^{m_i} \alpha_j e^{-\lambda_j s} \in A.
\]

By considering the value at 0 we may suppose that \( a_i = 0 \). We have found the desired sequence of functions.

**Proof of Theorem 1.** We denote by \( R \) the linear subspace of \( L^2(F_{\infty}^W) \) consisting of variables \( X \) admitting the specified representation (2) with \( (H_s) \) a predictable process such that \( H_s = 0 \) a.s. on \( \{s ; \zeta_s = 0\} \). With such a representation we obtain

\[
E(X^2) = (EX)^2 + E \left( \int_0^{+\infty} H_s^2 \ ds \right).
\]

By a classical argument (cf [RY] p.199) we deduce that \( R \) is complete hence closed in \( L^2(F_{\infty}^W) \).

Lemma 9 implies that for all \( n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n > 0, f_1, \ldots, f_n \) bounded and continuous on \( W \) (hence in \( E \)), the set \( R \) contains the variable

\[
\int \{0 < s_1 < \cdots < s_n \} \prod_{i=1}^{n} e^{-\lambda_i s_i} f_i(W_{s_i}) \ ds_1 \ldots ds_n
\]

hence also the variable

\[
\prod_{i=1}^{n} \int_{0}^{\infty} e^{-\lambda_i s} f_i(W_s) \ ds
\]

and, by linear combination, \( R \) contains also the variable:

\[
\prod_{i=1}^{n} \int_{0}^{\infty} \left( \sum_{j=1}^{m_i} \alpha_j e^{-\lambda_j s} \right) e^{-s} f_i(W_s) \ ds
\]

where the coefficients \( \alpha_j \in \mathbb{R}, \lambda_j > 0, m_i \in \mathbb{N} \) are arbitrary.

We deduce from Lemma 10 that, for every \( n \in \mathbb{N}, \) for all \( f_1, \ldots, f_n \) bounded and continuous and all \( s_1, \ldots, s_n > 0, \) the set \( R \) contains the variable
\[ \prod_{i=1}^{n} f_i(W_i) \]

(dropping useless constant exponential factors) and this is clearly sufficient to claim that \( R = L^2(\mathcal{F}_\infty^W) \).

4 Representation in filtration (\( \mathcal{G}_t \))

4.1 Proof of Theorem 4

We first establish that a process \( (M_t) \) given as in the statement of the theorem is effectively a martingale, that is, for all \( t, h > 0 \), for every \( \mathcal{G}_t \)-measurable \( U \),

\[
E[(M_{t+h} - M_t)\ U] = E \left( \left( \int_0^{T_1} H_s \ 1_{(t < \zeta_s \leq t+h)} \ d\zeta_s \right) U \right) = 0 \quad (11)
\]

We fix \( \varepsilon > 0 \) and introduce the successive time intervals of descent from \( t + \varepsilon \) down to \( t \), that is we consider the \( (\mathcal{F}_s) \)-stopping times \( (S_k, k \geq 0) \) and \( (T_k, k \geq 0) \) defined by \( S_0 = T_0 = 0 \), and if \( k \geq 1 \),

\[
S_k = \inf \{ s \in (T_{k-1}, \tau_1) ; \zeta_s = t + \varepsilon \}, \\
T_k = \inf \{ s \in (S_k, \tau_1) ; \zeta_s = t \},
\]

with the convention \( \inf \emptyset = \tau_1 \). Equation (11) will be proved, by letting \( \varepsilon \downarrow 0 \), as soon as we can show that, for every \( k \in \mathbb{N} \),

\[
E \left( \left( \int_{S_k}^{T_k} H_s \ 1_{(t < \zeta_s \leq t+h)} \ d\zeta_s \right) U \right) = 0.
\]

By the definition of \( \mathcal{G}_t \), it is sufficient to prove that

\[
E \left[ \left( \int_{S_k}^{T_k} H_s \ 1_{(t < \zeta_s \leq t+h)} \ d\zeta_s \right) X \ G(W_{T_k+\varepsilon}) \right] = 0
\]

where \( X \) is \( \mathcal{F}_{S_k} \)-measurable and bounded and \( G \) is a bounded measurable function. By applying the Markov property at time \( T_k \) the left hand side of the above expression reduces to

\[
E \left[ X \left( \int_{S_k}^{T_k} H_s \ 1_{(t < \zeta_s \leq t+h)} \ d\zeta_s \right) E_{W_{T_k}}[G] \right].
\]

But \( W_{T_k} = W_{S_k}^{T_k} \). The variable \( X \ E_{W_{S_k}}[G] \) is \( \mathcal{F}_{S_k} \)-measurable and bounded and we can represent it under the following form:

\[
 X \ E_{W_{S_k}}[G] = c + \int_{S_k}^{T_k} K_s \ d\zeta_s
\]
with a \((\mathcal{F}_s)\)-previsible process \((K_s)\). Therefore we have finally to prove that
\[
E \left[ \left( \int_{S_0}^{T_0} H_s \, 1_{[t < \zeta_s \leq t+h]} \, d\zeta_s \right) \left( c + \int_{0}^{S_0} K_s \, d\zeta_s \right) \right] = 0.
\]
The contribution coming from the multiplication by \(c\) is null, by applying the stopping Theorem for martingale \(\int_0^s H_s \, 1_{[t < \zeta_s \leq t+h]} \, d\zeta_s\). The remaining term is equal to
\[
E \left[ \int_0^{T_1} 1_{[S_0, T_1]}(s) \, H_s \, 1_{[t < \zeta_s \leq t+h]} \, 1_{[0, S_0]}(s) \, K_s \, d\zeta_s \right]
\]
which is clearly zero.

Now we denote by \((M_t)\) any \((\mathcal{G}_t)\)-martingale bounded in \(L^2\). Let \(M_\infty\) be the almost sure and \(L^2\) limit of \(M_t\). This variable of \(\mathcal{G}_\infty = \mathcal{F}_\infty\) can be represented as
\[
M_\infty = E(M_\infty) + \int_0^{T_1} H_s \, d\zeta_s
\]
with a \((\mathcal{F}_s)\)-previsible process \((H_s)\). Then
\[
M_t = E[M_\infty | \mathcal{G}_t] = E(M_\infty) + \int_0^{T_1} H_s \, 1_{[0 < \zeta_s \leq t]} \, d\zeta_s,
\]
the last equality resulting from the first part of the proof.

4.2 Comments on Proposition 6

It is straightforward that Formula (6) defines for every \(t > 0\), an \(L^2\)-valued finite measure. Firstly, it is finitely additive. Secondly we have, for every \(\Omega \in \mathcal{B}(\mathcal{W})\),
\[
\left\| M_t(\Omega) \right\|^2 = 4 \ E \left[ \int_0^{T_1} 1_{[\mathcal{W}]}(\zeta_s) \, 1_{\mathcal{G}_0}(W_r) \, dr \right]
\]
\[
\leq 4 \ E \left[ \int_0^{T_1} 1_{[\mathcal{W}]}(\zeta_s) \, d\zeta_s \right]
\]
\[
= 4 \ E \left[ \int_0^t L^2_\zeta(\zeta) \, d\alpha \right] = 4 \int_0^t E \left[ L^2_\zeta(\zeta) \right] \, d\alpha = 4t.
\]
The last equality follows from the Ray-Knight Theorem (or can be seen as the first moment of super-Brownian motion). Moreover it is clear by the dominated convergence Theorem that \(\left\| M_T(\Omega) \right\|_2\) converges to 0 if \(\Omega\) decreases to \(\emptyset\) and this proves the \(L^2\) countable additivity. Thus we are in the classical setting of martingale measures as described in [Da] Chapter 7. We have
\[
\int_0^t \int_{\mathcal{W}} \phi(w) \, M(ds \, dw) = 2 \int_0^{T_1} 1_{[\mathcal{W}]}(\zeta_s) \, \phi(W_r) \, d\zeta_r.
\]
Then Formulas (7) and (8) are essentially a reformulation of Proposition 3 for historical Brownian motion instead of super-Brownian motion, but this extension was also covered by [DS]. In particular, the quadratic variation of 
\[ \int_0^t \int_W \phi(w) M(ds, dw) \] is:

\[ 4 \int_0^t H_s(\phi^2) \, ds = 4 \int_0^t ds \int_0^{t_1} \phi^2(W_{t_r}) \, d[t_r] \, L^*_s(\zeta) \]

\[ = 4 \int_0^{t_1} 1_{\{B \geq s \}} \phi^2(W_{t_r}) \, dr \]

and this leads to (8).

References


