Summary. We prove the previsible representation property for the filtration of the Brownian snake and give a representation of the martingales in the filtration associated to the historical Brownian motion. We deduce a representation of the martingale measure of the historical Brownian motion.

Key words: Super-Brownian motion, Brownian snake, previsible representation property, martingale measure

Representation of the martingales for the Brownian snake

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1 Introduction and statement of results

In this paper we deal with the Brownian snake introduced by Le Gall which can be seen as a continuous-time parametrization of a tree of branching trajectories. Abundant litterature has shown the interest of this process to prove results on super-processes, to express solutions of certain semi-linear pde or to describe the limit behaviour of important interacting particle systems. See [Lg] for a comprehensive treatement of the subject. The definitions of the basic objects and the terminology used in the present introduction are however recalled in the next section.

The Brownian snake (W_s) is a simple example of process taking its values in the space $\mathcal{W}$ of stopped paths in $\mathbf{R}^d$. In [DS] we have developped some tools of stochastic calculus for this process, in particular an Itô formula; a simplified form of this statement is as follows. Suppose $F^{(2)} : \mathcal{W} \to \mathbf{R}$ is a continuous function and $F^{(1)}, F : \mathcal{W} \to \mathbf{R}$ are defined by

$$\begin{aligned} F^{(1)}(w) &= \int_0^\zeta F^{(2)}(w_{\leq r})\, dr \\ F(w) &= \int_0^\zeta F^{(1)}(w_{\leq r})\, dr = \int_0^\zeta \int_0^r F^{(2)}(w_{\leq u})\, du \\ &= \int_0^\zeta (\zeta - u)\, F^{(2)}(w_{\leq u})\, du \end{aligned}$$

then, for $0 \leq r < t$:

$$F(W_t) = F(W_r) + \int_r^t F^{(1)}(W_s)\, d\zeta_s + \frac{1}{2} \int_r^t F^{(2)}(W_s)\, ds. \tag{1}$$

The present paper is devoted to some applications of this formula concerning the representation of the martingales associated to the Brownian snake.

First, the filtration $(\mathcal{F}_t^W)$ of the Brownian snake has a surprising previsible representation property :

Theorem 1 *For every random variable $X \in \mathrm{L}^2(\mathcal{F}_\infty^W)$, there exists a $(\mathcal{F}_s^W)$–previsible process (H_s) vanishing a.s. on $\{s,\ \zeta_s = 0\}$ such that*

$$\mathrm{E}\left(\int_0^\infty H_s^2\ ds\right) < +\infty \text{ and } X = \mathrm{E}(X) + \int_0^\infty H_s\ d\zeta_s. \tag{2}$$

The proof is given in section 3. By standard arguments (see [RY] V.3.4), we quickly deduce the following corollary

Proposition 2 *For every local martingale (M_s) with respect to the filtration $(\mathcal{F}_s^W)$, there exists a $(\mathcal{F}_s^W)$–previsible process (H_s) locally in L^2 such that*

$$M_s = M_0 + \int_0^s H_r\ d\zeta_r. \tag{3}$$

Note that the above stochastic integral is effectively a local martingale because the integrand H_r vanishes on $\{s,\ \zeta_s = 0\}$ and we could write the integral with respect to the martingale part of the reflecting Brownian motion (ζ_s).

One of the interests of the Brownian snake lies in its connections with super-Brownian motion. More precisely let us consider a Brownian snake starting from $\tilde{x}$ and set

$$\forall t \geq 0,\ \ X_t = \int_0^{\tau_1} d_{(s)} L_s^t(\zeta)\, \delta_{\hat{W}_s} \text{ and } H_t = \int_0^{\tau_1} d_{(s)} L_s^t(\zeta)\, \delta_{W_s}$$

where $L_s^t(\zeta)$ is the local time of the lifetime process $(\zeta_.)$ at level t and time s, and $\tau_1 = \inf\left\{s \geq 0; L_s^0(\zeta) > 1\right\}$ is the hitting time of 1 by the local time of ζ at level 0. The process (X_t) [resp. (H_t)] takes its values in the space $\mathcal{M}_F(\mathbf{R}^d)$ [resp. $\mathcal{M}_F(\mathcal{W})$] and is called super-Brownian motion starting from δ_x [resp. historical Brownian motion starting from $\delta_{\tilde{x}}$]. To be honest, the usual definitions include a factor 1/4 that we have dropped here to simplify notations, as we did in [DS]. From this definition emerges a new filtration to be considered. Let

$$\tau_s^t = \inf\left\{u;\ \int_0^{u\wedge\tau_1} \mathbf{1}_{\{\zeta_v \leq t\}}\ dv > s\right\}$$

be the inverse of the time spent by the lifetime (ζ_s) under level t with the convention $\inf\{\emptyset\} = \tau_1$. We see that at least the following σ–algebras naturally arise :

$$\begin{aligned}
\mathcal{G}_t &= \sigma(W_{\tau_s^t},\ s \geq 0)\\
\mathcal{G}_t^\zeta &= \sigma(\zeta_{\tau_s^t},\ s \geq 0)\\
\mathcal{G}_t^X &= \sigma(X_r,\ r \leq t)
\end{aligned}$$

$$\begin{aligned}\mathcal{G}_t^H &= \sigma(H_r,\ r \le t)\\ \mathcal{F}_s &= \sigma(W_{r\wedge\tau_1},\ r \le s)\\ \mathcal{F}_s^\zeta &= \sigma(\zeta_{r\wedge\tau_1},\ r \le s).\end{aligned}$$

We note the following obvious relations

$$\mathcal{G}_t^\zeta = \mathcal{G}_t \cap \mathcal{F}_\infty^\zeta,\ \ \mathcal{G}_t^X \subset \mathcal{G}_t^H \subset \mathcal{G}_t,\ \ \mathcal{G}_\infty = \mathcal{F}_\infty.$$

A question now arises concerning the representation of the martingales in the "vertical" filtration $(\mathcal{G}_t)$. Since super-Brownian motion (and historical Brownian motion) has often been studied via martingale problems we already know a class of $(\mathcal{G}_t)$–martingale : for every bounded ϕ in the domain of the generator A,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(A\phi)\, ds$$

defines a martingale and moreover the quadratic variation of this martingale is known to be

$$\langle M(\phi)\rangle_t = 4 \int_0^t X_s(\phi^2)\, ds. \tag{4}$$

An interpretation with the Brownian snake can be given.

Proposition 3 ([DS] Lemma 10 and Theorem 7) *We have*

$$M_t(\phi) = 2 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, \phi(\hat{W}_s)\, d\zeta_s \tag{5}$$

and this process is a $(\mathcal{G}_t)$-martingale with quadratic variation given by (4).

A natural question is then to ask if any $(\mathcal{G}_t)$-martingale can be expressed in a way that generalizes the above expression.

Theorem 4 *For every $(\mathcal{G}_t)$–martingale (M_t) which is bounded in L^2, there exists a $(\mathcal{F}_s)$–previsible process (H_s) such that*

$$\mathrm{E}\left(\int_0^{\tau_1} H_s^2\, ds\right) < +\infty \text{ and } \forall t \ge 0,\ \ M_t = M_0 + \int_0^{\tau_1} H_s\, \mathbf{1}_{(0,t]}(\zeta_s)\, d\zeta_s.$$

By [Je] this result is known when it is restricted to the filtration $(\mathcal{G}_t^\zeta)$ i.e. considering only the lifetime. In that case it is essentially seen on Tanaka's formula interpreted as a reflection equation. Our proof is given in section 4. We deduce the corollary :

Proposition 5 *For every $Y \in \mathrm{L}^2(\mathcal{G}_t)$, there exists a $(\mathcal{F}_s)$–previsible process (H_s) such that*

$$Y = \mathrm{E}(Y) + \int_0^{\tau_1} H_s\, \mathbf{1}_{(0,t]}(\zeta_s)\, d\zeta_s \text{ and } \mathrm{E}\left(\int_0^{\tau_1} H_s^2\, \mathbf{1}_{(0,t]}(\zeta_s)\, ds\right) < +\infty.$$

The representation given in (5) can be pushed a little further into the terminology of martingale measure as the notation already suggests. The notion of martingale measure of super-processes is explained for instance in [Da] Chapter 7; Example 7.1.3 covers the case of super-Brownian motion as it is defined here. In our setting the martingale measure of super-Brownian motion or even historical Brownian motion is easily described. It is stated in the following proposition where L denotes the generator of the so-called A–path process which is the process in $\mathcal{W}$ whose lifetime increases at constant speed 1 and consists in a trajectory of the diffusion governed by A.

Proposition 6 *Let us set, for $t \geq 0$ and $\Omega \in \mathcal{B}(\mathcal{W})$,*

$$M_t(\Omega) = 2 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\ \mathbf{1}_{\Omega}(W_r)\ d\zeta_r. \tag{6}$$

Then, $(M_t(\Omega),\ t \geq 0,\ \Omega \in \mathcal{B}(\mathcal{W}))$ defines an L^2–martingale measure $M(ds\,dw)$. It is associated to the historical Brownian motion, that is, for every $\phi : \mathcal{W} \to \mathbf{R}$ in the domain of L and bounded,

$$H_t(\phi) = H_0(\phi) + \int_0^t H_s(L\phi)\ ds + \int_0^t \int_{\mathcal{W}} \phi(w)\ M(ds\,dw). \tag{7}$$

This martingale measure is orthogonal and its intensity is the random measure ν on $\mathbf{R}_+ \times \mathcal{W}$ given by

$$\int_{\mathbf{R}_+ \times \mathcal{W}} \psi(t,w)\ \nu(dt\,dw) = 4 \int_0^{\tau_1} \psi(\zeta_s, W_s)\ ds. \tag{8}$$

We recall that the intensity of a martingale measure is defined so that $\nu([0,t] \times \Omega)$, for Ω Borel subset of W, is the quadratic variation of the martingale $(M_t(\Omega))$.

2 Basic objects and notations

We will use the following common notations :
$\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbf{R}_+ = [0, +\infty)$.
$\mathcal{C}(X, Y)$: set of continuous functions from metric space X to metric space Y.
$\sigma(X_i, i \in I)$: σ-algebra generated by the random variables $X_i,\ i \in I$ in a fixed probability space, completed with all negligible sets.
$\mathcal{B}(X)$: Borel σ-algebra of the metric space X.
$\mathcal{M}_F(X)$: set of all finite measure on the metric space X equipped with the (metrizable) topology of weak convergence and its Borel σ-algebra $\mathcal{B}(\mathcal{M}_F(X))$.

A stopped path is a couple (w, ζ), where $\zeta \geq 0$ is called the lifetime of the path, and $w : \mathbf{R}_+ \to \mathbf{R}^d$ is a continuous mapping, which is constant on $[\zeta, +\infty)$. We denote by $\mathcal{W}$ the set of all stopped paths. We sometimes abbreviate (w, ζ) into w and denote $\zeta(w)$ the lifetime. The distance on $\mathcal{W}$ is $d(w, w') = \sup_{t\geq 0} |w(t) - w'(t)| + |\zeta(w) - \zeta(w')|$, making $\mathcal{W}$ a Polish space. We denote by $\hat{w} = w(\zeta)$ the endpoint of w, and $\tilde{x}$ the path of lifetime 0 started at $x \in \mathbf{R}^d$. Finally, we denote by $w_{\leq r}$ the path of lifetime $\zeta(w) \wedge r$ such that for $u \geq 0$, $w_{\leq r}(u) = w(u \wedge r)$.

Let us fix a diffusion in $\mathbf{R}^d$ with generator A. The Brownian snake started at x with spatial motion governed by A is the strong Markov continuous process $W = (W_s, s \geq 0)$ with values in $\mathcal{W}$ characterized by the following properties:

1. $W_s(0) = x$ for every s;
2. The lifetime process $\zeta_s = \zeta(W_s)$ is a reflecting Brownian motion in $\mathbf{R}_+$;
3. Conditionally on $(\zeta_s, s \geq 0)$, the distribution of $(W_s, s \geq 0)$ is that of an inhomogeneous Markov process whose transition kernels are described as follows: for every $s < s'$,
   - $W_{s'}^{\leq m} = W_s^{\leq m}$ where $m = \inf_{r\in[s,s']} \zeta_r$;
   - $(W_{s'}(m + t), 0 \leq t \leq \zeta_{s'} - m)$ is independent of W_s conditionally on $W_s(m)$ and has the law of a diffusion in $\mathbf{R}^d$ with generator A, starting from $W_s(m)$ and stopped at time $\zeta_{s'} - m$.

The filtration $(\mathcal{F}_t^W)$ used in the introduction is the filtration associated to (W_s), completed the usual way (see [RY] p. 45 and 93 for precisions on completion) and

$$\mathcal{F}_\infty^W = \sigma\Big(\bigcup_{t\geq 0} \mathcal{F}_t^W\Big).$$

3 Proof of the previsible representation property (Theorem 1)

This proof is inspired by Exercise 3.15 of [RY], dealing with the classical Brownian filtration. In that case a step of the proof is to solve a linear differential equation. This is replaced in our path space setting by an integral equation that we first discuss.

Lemma 7 *Let $\alpha > 0$ and*

$$\mathcal{E}_\alpha = \left\{ \psi \in \mathcal{C}(\mathcal{W}, \mathbf{R});\ \sup_{w\in\mathcal{W}} e^{-\alpha\,\zeta}\, |\psi(w)| < +\infty \right\}.$$

For all $\lambda > 0$, if $\alpha > \sqrt{2\lambda}$ and $f \in \mathcal{E}_\alpha$ then there exists $\varphi \in \mathcal{E}_\alpha$ such that

$$\forall w \in \mathcal{W},\ \ \frac{1}{2}\varphi(w) - \lambda \int_0^\zeta (\zeta - u)\, \varphi(w_{\leq u})\, du = f(w).$$

Proof. It is easy to see that the formula $||\psi||_\alpha = \sup_{w \in \mathcal{W}} e^{-\alpha\,\zeta}\,|\psi(w)|$ defines a norm on the vector space $\mathcal{E}_\alpha$ which makes this space complete. The result of the lemma consists in finding a fixed point for the map

$$\theta \;:\; \varphi \to \left(w \to 2f(w) + 2\lambda \int_0^\zeta (\zeta - u)\,\varphi(w_{\leq u})\;du \right).$$

It is easy to verify that θ maps $\mathcal{E}_\alpha$ into itself. In order to apply the classical Lipschitz fixed point theorem to θ in the Banach space $\mathcal{E}_\alpha$, it remains to check that θ satisfies the Lipschitz condition. For $\varphi_1, \varphi_2 \in \mathcal{E}_\alpha$,

$$\begin{aligned}
||\theta(\varphi_1) - \theta(\varphi_2)||_\alpha &= \sup_{w \in \mathcal{W}} e^{-\alpha\,\zeta} \left| 2\lambda \int_0^\zeta (\zeta - u)\,(\varphi_1 - \varphi_2)(w_{\leq u})\;du \right| \\
&\leq \sup_{w \in \mathcal{W}} e^{-\alpha\,\zeta} \left| 2\lambda \int_0^\zeta (\zeta - u)\,e^{\alpha u}||\varphi_1 - \varphi_2||_\alpha\;du \right| \\
&\leq 2\lambda\;||\varphi_1 - \varphi_2||_\alpha \;\sup_{\zeta > 0} \left\{ e^{-\alpha\,\zeta} \int_0^\zeta (\zeta - u)\,e^{\alpha\,u}\;du \right\} \\
&\leq 2\lambda\;||\varphi_1 - \varphi_2||_\alpha \int_0^{+\infty} v\,e^{\alpha\,v}dv = \frac{2\lambda}{\alpha^2}\;||\varphi_1 - \varphi_2||_\alpha.
\end{aligned}$$

Since the last ratio is by assumption smaller than 1, the proof of the lemma is complete.

We now denote $\mathcal{E}$ the increasing limit of the sets $\mathcal{E}_\alpha$, $\alpha > 0$, that is $\mathcal{E} = \bigcup_{\alpha > 0} \mathcal{E}_\alpha$. We are now ready to give a previsible representation for certain variables, namely the type on the left-hand side of the following equality.

Lemma 8 *For every $f \in \mathcal{E}$ and every $\lambda > 0$, there exist $g_0, g_1 \in \mathcal{E}$ such that, for every $r > 0$,*

$$\int_r^\infty e^{-\lambda s}\;f(W_s)\;ds = e^{-\lambda r}\;g_0(W_r) + \int_r^\infty e^{-\lambda s}\;g_1(W_s)\;d\zeta_s. \tag{9}$$

Moreover g_0 vanishes at 0 in the following sense : $g_0(w) = 0$ if $\zeta(w) = 0$ and identically for g_1.

Proof. Let us first remark that, since f belongs to a certain $\mathcal{E}_\alpha$, we have

$$\int_r^\infty e^{-\lambda s}\;|f(W_s)|\;ds \leq ||f||_\alpha \;\int_r^\infty e^{-\lambda s}\;e^{\alpha\zeta_s}\;ds$$

and the integral on the right-hand side is finite because the reflecting Brownian motion (ζ_s) satisfies the law of the iterated logarithm. Hence the integral appearing on the left-hand side of equation (9) is defined almost surely. So is the integral on the right-hand side using a similar argument and [RY] IV.1.26.

By increasing α if necessary, we may suppose that $\alpha > \sqrt{2\lambda}$. By Lemma 7, we can associate to $f \in \mathcal{E}_\alpha$ a continuous function $\varphi \in \mathcal{E}_\alpha$ as specified. Let $F^{(2)} = \varphi$ and $F^{(1)}, F$ be defined as in the assumptions of formula (1). Note that $F^{(1)}$ and F vanish at 0, in the sense defined in the statement of the lemma. It is easy to check that $F^{(1)}, F \in \mathcal{E}_\alpha$ and more precisely,

$$|F(w)| \leq ||\varphi||_\alpha \, \frac{e^{\alpha\,\zeta}}{\alpha^2}. \tag{10}$$

We obtain, by formula (1) and the classical Itô formula for a product, for $0 \leq r < t$:

$$e^{-\lambda t}\, F(W_t) - e^{-\lambda r}\, F(W_r) = \int_r^t e^{-\lambda s} \left(\frac{1}{2}\varphi - \lambda\, F\right)(W_s)\; ds + \int_r^t e^{-\lambda s} F^{(1)}(W_s)\; d\zeta_s.$$

We recall that $(1/2)\,\varphi - \lambda F = f$. Using the bound (10), the law of the iterated logarithm for (ζ_s) entails that $\lim_{t\to+\infty} e^{-\lambda\, t}\; F(W_t) = 0$, almost surely. Therefore we get

$$\int_r^\infty e^{-\lambda s}\; f(W_s)\; ds = -e^{-\lambda r}\; F(W_r) - \int_0^\infty e^{-\lambda s}\; F^{(1)}(W_s)\; d\zeta_s$$

We obtain the sought after representation, up to a change of notations.

Lemma 9 *For all* $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n > 0$, $f_1, \ldots, f_n \in \mathcal{E}$, *there exist* $\mu_0, \mu_1, \mu_j^k > 0$, $(1 \leq k \leq n-1, 0 \leq j \leq k)$, $g_0, g_1, g_j^k \in \mathcal{E}$ $(1 \leq k \leq n-1, 0 \leq j \leq k)$ *with* g_0, g_1, g_0^k *vanishing at 0, such that, for all* $r \geq 0$,

$$\begin{aligned}
&\int_{\{r<s_1<\cdots<s_n\}} \left(\prod_{i=1}^n e^{-\lambda_i\, s_i}\, f_i(W_{s_i})\right)\; ds_1 \ldots ds_n \\
&= e^{-\mu_0\, r}\; g_0(W_r) + \int_r^{+\infty} \left\{ e^{-\mu_1\, s}\; g_1(W_s) + \sum_{k=1}^{n-1} e^{-\mu_0^k\, s}\; g_0^k(W_s) \right. \\
&\qquad \left. \int_{\{r<s_1<\cdots<s_k<s\}} \left(\prod_{j=1}^k e^{-\mu_j^k\, s_j}\; g_j^k(W_{s_j})\right)\; ds_1 \ldots ds_k \right\}\; d\zeta_s
\end{aligned}$$

with the convention that the sum over k *disappears if* $n = 1$.

Proof. By equation (9), we know that the lemma is true for $n = 1$. Then we proceed by induction. Admitting the result at rank $n \geq 1$, we examine the case of rank $n+1$:

$$\begin{aligned}
&\int_{\{r<s_1<\cdots<s_{n+1}\}} \left(\prod_{i=1}^{n+1} e^{-\lambda_i\, s_i}\, f_i(W_{s_i})\right)\; ds_1 \ldots ds_{n+1} \\
&= \int_r^{+\infty} ds_1\;\; e^{-\lambda_1\, s_1}\, f_1(W_{s_1}) \int_{\{s_1<s_2<\cdots<s_{n+1}\}} \left(\prod_{i=2}^{n+1} e^{-\lambda_i\, s_i}\, f_i(W_{s_i})\right)\; ds_2 \ldots ds_{n+1}
\end{aligned}$$

$$= \int_r^{+\infty} ds_1 \; e^{-\lambda_1 s_1} f_1(W_{s_1}) \; \Bigg[e^{-\mu_0 s_1} g_0(W_{s_1}) + \int_{s_1}^{+\infty} \Big\{ e^{-\mu_1 s} g_1(W_s) +$$

$$\sum_{k=2}^{n} e^{-\mu_0^k s} g_0^k(W_s) \int_{\{s_1<s_2<\cdots<s_k<s\}} \prod_{j=2}^{k} e^{-\mu_j^k s_j} g_j^k(W_{s_j}) \, ds_2 \ldots ds_k \Big\} \; d\zeta_s \Bigg]$$

$$= \int_r^{+\infty} e^{-(\lambda_1+\mu_0) s_1} (f_1 \, g_0)(W_{s_1}) ds_1$$

$$+ \int_r^{+\infty} e^{-\mu_1 s} g_1(W_s) \left(\int_r^s e^{-\lambda_1 s_1} f_1(W_{s_1}) ds_1 \right) d\zeta_s$$

$$+ \sum_{k=2}^{n} \int_r^{+\infty} e^{-\mu_0^k s} g_0^k(W_s) \left(\int_r^s ds_1 e^{-\lambda_1 s_1} f_1(W_{s_1}) \right.$$

$$\left. \int_{\{s_1<s_2<\cdots<s_k<s\}} \left(\prod_{j=2}^{k} e^{-\mu_j^k s_j} g_j^k(W_{s_j}) \right) ds_2 \ldots ds_k \right) d\zeta_s .$$

The first equality is simply Fubini's formula; then we use the induction hypothesis for the integral with respect to $s_2, \ldots, s_{n+1}$; and for the last equality a stochastic version of Fubini's theorem. To the first term obtained at the last equality we can apply the result at rank 1 i.e. Equation (9); the second and third term are the desired quantities to obtain the sought-after formula, up to a change of notations of course.

Lemma 10 *For every $\hat{s} > 0$, there exist, for every $i \in \mathbf{N}$, coefficients $m_i \in \mathbf{N}$, $\alpha_j^i \in \mathbf{R}$, $\lambda_j^i > 0$ for $1 \le j \le m_i$ such that, for every continuous function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ (absolutely) integrable over $\mathbf{R}_+$,*

$$\int_0^{+\infty} \left(\sum_{j=1}^{m_i} \alpha_j^i \, e^{-\lambda_j^i s} \right) \gamma(s) \, ds \overset{i \to +\infty}{\longrightarrow} \gamma(\hat{s}).$$

Proof. We first consider an approximation $p_i(s)\, ds$ of the Dirac measure $\delta_{\hat{s}}$ with continuous density whose support is contained in $(0, 2\hat{s})$ so that we have

$$\int_0^{+\infty} p_i(s) \, \gamma(s) \, ds \overset{i \to +\infty}{\longrightarrow} \gamma(\hat{s})$$

for every continuous γ. The set of functions over $\mathbf{R}_+$

$$\Lambda = \left\{ \psi : s \to \sum_{j=1}^{m} \alpha_j \, e^{-\lambda_j s}; \;\; m \in \mathbf{N}, \; \alpha_j \in \mathbf{R}, \; \lambda_j > 0, \;\; \psi(0) = 0 \right\}$$

is a linear subspace, closed under multiplication. Let $\arg z \in [-\pi, \pi)$ denote the value of the argument of $z \in \mathbf{U} = \{z \in \mathbf{C}; \;\; |z| = 1\}$. On the compact

$\mathbf{U}$, equipped with uniform topology, we can apply the the classical Stone-Weierstrass approximation theorem to the set of continuous functions :

$$\left\{ z \to a + \psi\left(\tan \frac{1}{4}(\arg z + \pi)\right) \; ; \; a \in \mathbf{R}, \; \psi \in \Lambda \right\}$$

in order to approximate by functions of this set, the continuous function $z \to p_i(\tan[(\arg z + \pi)/4])$. It is thus possible to find $a_i \in \mathbf{R}, m_i \in \mathbf{N}, \; \alpha_j^i \in \mathbf{R}, \; \lambda_j^i > 0$ such that

$$\sup_{s \in \mathbf{R}_+} |p_i(s) - \psi_i(s)| \overset{i \to +\infty}{\longrightarrow} 0 \text{ with } \psi_i \; : \; s \to a_i + \sum_{j=1}^{m_i} \alpha_j^i \; e^{-\lambda_j^i s} \quad \in \Lambda.$$

By considering the value at 0 we may suppose that $a_i = 0$. We have found the desired sequence of functions.

Proof of Theorem 1. We denote by R the linear subspace of $\mathrm{L}^2(\mathcal{F}_\infty^W)$ consisting of variables X admitting the specified representation (2) with (H_s) a previsible process such that $H_s = 0$ a.s. on $\{s; \; \zeta_s = 0\}$. With such a representation we obtain

$$\mathrm{E}(X^2) = (\mathrm{E}X)^2 + \mathrm{E}\left(\int_0^{+\infty} H_s^2 \; ds\right).$$

By a classical argument (cf [RY] p.199) we deduce that R is complete hence closed in $\mathrm{L}^2(\mathcal{F}_\infty^W)$.

Lemma 9 implies that for all $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n > 0$, $f_1, \ldots, f_n$ bounded and continuous on $\mathcal{W}$ (hence in $\mathcal{E}$), the set R contains the variable

$$\int_{\{0 < s_1 < \cdots < s_n\}} \prod_{i=1}^n e^{-\lambda_i s_i} \; f_i(W_{s_i}) \; ds_1 \ldots ds_n$$

hence also the variable

$$\prod_{i=1}^n \int_0^\infty e^{-\lambda_i s} \; f_i(W_s) \; ds$$

and, by linear combination, R contains also the variable :

$$\prod_{i=1}^n \int_0^\infty \left(\sum_{j=1}^{m_i} \alpha_j^i \; e^{-\lambda_j^i s} \right) \; e^{-s} \; f_i(W_s) \; ds$$

where the coefficients $\alpha_j^i \in \mathbf{R}, \lambda_j^i > 0, m_i \in \mathbf{N}$ are arbitrary.

We deduce from Lemma 10 that, for every $n \in \mathbf{N}$, for all $f_1, \ldots, f_n$ bounded and continuous and all $\hat{s}_1, \ldots, \hat{s}_n > 0$, the set R contains the variable

$$\prod_{i=1}^{n} f_i(W_{\hat{s}_i})$$

(dropping useless constant exponential factors) and this is clearly sufficient to claim that $R = \mathrm{L}^2(\mathcal{F}_\infty^W)$.

4 Representation in filtration $(\mathcal{G}_t)$

4.1 Proof of Theorem 4

We first establish that a process (M_t) given as in the statement of the theorem is effectively a martingale, that is , for all $t, h > 0$, for every $\mathcal{G}_t$-mesurable U,

$$\mathrm{E}\left[(M_{t+h} - M_t)\, U\right] = \mathrm{E}\left[\left(\int_0^{\tau_1} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s\right) U\right] = 0 \qquad (11)$$

We fix $\varepsilon > 0$ and introduce the successive time intervals of descent from $t+\varepsilon$ down to t, that is we consider the $(\mathcal{F}_s)$–stopping times $(S_k, k \ge 0)$ and $(T_k, k \ge 0)$ defined by $S_0 = T_0 = 0$, and if $k \ge 1$,

$$\begin{aligned} S_k &= \inf\left\{s \in (T_{k-1}, \tau_1); \zeta_s = t+\varepsilon\right\}, \\ T_k &= \inf\left\{s \in (S_k, \tau_1); \zeta_s = t\right\}, \end{aligned}$$

with the convention $\inf \emptyset = \tau_1$. Equation (11) will be proved, by letting $\varepsilon \downarrow 0$, as soon as we can show that, for every $k \in \mathbf{N}$,

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s\right) U\right] = 0.$$

By the definition of $\mathcal{G}_t$, it is sufficent to prove that

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s\right) X\ G(W_{T_k+\cdot})\right] = 0$$

where X is $\mathcal{F}_{S_k}$-mesurable and bounded and G is a bounded mesurable function. By applying the Markov property at time T_k the left hand side of the above expression reduces to

$$\mathrm{E}\left[X\left(\int_{S_k}^{T_k} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s\right) \mathrm{E}_{W_{T_k}}[G]\right].$$

But $W_{T_k} = W_{S_k}^{\le t}$. The variable $X\, \mathrm{E}_{W_{S_k}^{\le t}}[G]$ is $\mathcal{F}_{S_k}$-mesurable and bounded and we can represent it under the following form :

$$X\, \mathrm{E}_{W_{S_k}^{\le t}}[G] = c + \int_0^{S_k} K_s\ d\zeta_s$$

with a $(\mathcal{F}_s)$-previsible process (K_s). Therefore we have finally to prove that

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s\right)\left(c+\int_0^{S_k} K_s\ d\zeta_s\right)\right]=0.$$

The contribution coming from the multiplication by c is null, by applying the stopping Theorem for martingale $\int_0^{\cdot} H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\ d\zeta_s$. The remaining term is equal to

$$\mathrm{E}\left[\int_0^{\tau_1} \mathbf{1}_{[S_k,T_k]}(s)\ H_s\ \mathbf{1}_{\{t<\zeta_s\le t+h\}}\mathbf{1}_{[0,S_k]}(s)\ K_s\ \ ds\right]$$

which is clearly zero.

Now we denote by (M_t) any $(\mathcal{G}_t)$–martingale bounded in L^2. Let M_∞ be the almost sure and L^2 limit of M_t. This variable of $\mathcal{G}_\infty=\mathcal{F}_\infty$ can be repesented as

$$M_\infty=\mathrm{E}(M_\infty)+\int_0^{\tau_1} H_s\ d\zeta_s$$

with a $(\mathcal{F}_s)$-previsible process (H_s). Then

$$M_t=\mathrm{E}[M_\infty|\mathcal{G}_t]=\mathrm{E}(M_\infty)+\int_0^{\tau_1} H_s\ \mathbf{1}_{\{0<\zeta_s\le t\}}\ d\zeta_s,$$

the last equality resulting from the first part of the proof.

4.2 Comments on Proposition 6

It is straightforward that Formula (6) defines for every $t>0$, an L^2–valued finite measure. Firstly, it is finitely additive. Secondly we have, for every $\Omega\in\mathcal{B}(\mathcal{W})$,

$$\begin{aligned}||M_t(\Omega)||_2^2&=4\ \mathrm{E}\left[\int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)\ \mathbf{1}_\Omega(W_r)\ dr\right]\\&\le 4\ \mathrm{E}\left[\int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)dr\right]\\&=4\ \mathrm{E}\left[\int_0^t L^a_{\tau_1}(\zeta)da\right]=4\int_0^t \mathrm{E}\left[L^a_{\tau_1}(\zeta)\right]da=4t.\end{aligned}$$

The last equality follows from the Ray-Knight Theorem (or can been seen as the first moment of super-Brownian motion). Moreover it is clear by the dominated convergence Theorem that $||M_t(\Omega)||_2$ converges to 0 if Ω decreases to $\emptyset$ and this proves the L^2 countable additivity. Thus we are in the classical setting of martingale measures as described in [Da] Chapter 7. We have

$$\int_0^t\int_{\mathcal{W}}\phi(w)\ M(ds\,dw)=2\int_0^{\tau_1}\mathbf{1}_{(0,t]}(\zeta_r)\ \phi(W_r)\ d\zeta_r.$$

Then Formulas (7) and (8) are essentially a reformulation of Proposition 3 for historical Brownian motion instead of super-Brownian motion, but this extension was also covered by [DS]. In particular, the quadratic variation of $\int_0^t \int_{\mathcal{W}} \phi(w)\, M(ds\, dw)$ is :

$$4 \int_0^t H_s(\phi^2)\, ds = 4 \int_0^t ds \int_0^{\tau_1} \phi^2(W_r)\, d_{(r)} L_r^s(\zeta)$$
$$= 4 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\, \phi^2(W_r)\, dr$$

and this leads to (8).

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