THE HOCHSCHILD COHOMOLOGY RING OF A CLASS OF SPECIAL BISERIAL ALGEBRAS

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Abstract. We consider a class of self-injective special biserial algebras $\Lambda_N$ over a field $K$ and show that the Hochschild cohomology ring of $\Lambda_N$ is a finitely generated $K$-algebra. Moreover, the Hochschild cohomology ring of $\Lambda_N$ modulo nilpotence is a finitely generated commutative $K$-algebra of Krull dimension two. As a consequence the conjecture of [21], concerning the Hochschild cohomology ring modulo nilpotence, holds for this class of algebras.

Introduction

Let $K$ be a field. For $m \geq 1$, let $Q$ be the quiver with $m$ vertices, labeled 0, 1, \ldots, $m-1$, and $2m$ arrows as follows:

\[
\begin{array}{cccccccc}
& a & & a & & a & & a \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\end{array}
\]

Let $a_i$ denote the arrow that goes from vertex $i$ to vertex $i+1$, and let $\bar{a}_i$ denote the arrow that goes from vertex $i+1$ to vertex $i$, for each $i = 0, \ldots, m-1$ (with the obvious conventions modulo $m$). We denote the trivial path at the vertex $i$ by $e_i$. Paths are written from left to right.

In this paper, we study the Hochschild cohomology ring of a family of algebras $\Lambda_N$ given by this quiver $Q$ and certain relations. Let $N \geq 1$, and let $\Lambda_N = KQ/I_N$ where $I_N$ is the ideal of $KQ$ generated by $a_i a_{i+1}$, $\bar{a}_i \bar{a}_{i-1}$, and $a_i \bar{a}_i \bar{a}_{i-1} a_{i-1}$, for $i = 0, \ldots, m-1$. These algebras are all self-injective special biserial algebras and as such play an important role in various aspects of representation theory of algebras.

First, consider the case $N = 1$, so that $\Lambda_1 = KQ/I_1$ where $I_1$ is the ideal of $KQ$ generated by $a_i a_{i+1}$, $\bar{a}_i \bar{a}_{i-1}$ and $a_i \bar{a}_i \bar{a}_{i-1} a_{i-1}$, for $i = 0, \ldots, m-1$. In the case where $m$ is even, this algebra occurred in the presentation by quiver and relations of the Drinfeld double of the generalised Taft algebras studied in [6]. It also occurs in the study of the representation theory of $U_q(\mathfrak{sl}_2)$; see [20, 22, 23]; see also [5]. In the case $N = 1$, the Hochschild cohomology ring of the algebra $\Lambda_1$ with $m = 1$ is discussed in [1], where $q$-analogues of $\Lambda_1$ were used to answer negatively a question of Happel, in that they have finite dimensional Hochschild cohomology ring but are of infinite global
dimension when \( q \in \mathbb{K}^* \) is not a root of unity. The more general algebras \( \Lambda_N \) occur in \([10, 9]\), in which the authors determine the Hopf algebras associated to infinitesimal groups whose principal blocks are tame when \( \mathbb{K} \) is an algebraically closed field of characteristic \( p \geq 3 \); among the principal blocks that are obtained in this classification are the algebras \( \Lambda_{p^n} \) with \( m = p^r \) vertices, for some integers \( n \) and \( r \).

This paper describes explicitly the structure of the Hochschild cohomology ring of the algebras \( \Lambda_N \), for all \( N \geq 1 \) and \( m \geq 1 \), and in all characteristics. In particular, we determine the Hochschild cohomology ring of the algebras \( \Lambda_1 \) of \([6]\) and the algebras of \([10]\).

The main results show that the Hochschild cohomology ring, \( HH^*(\Lambda_N) \), is a finitely generated \( \mathbb{K} \)-algebra. We give an explicit basis for each cohomology group \( HH^p(\Lambda_N) \) together with generators of the algebra \( HH^*(\Lambda_N) \). We then determine explicitly the Hochschild cohomology ring modulo nilpotence, \( HH^*(\Lambda_N)/\mathcal{N} \), where \( \mathcal{N} \) is the ideal of \( HH^*(\Lambda_N) \) generated by all homogeneous nilpotent elements. Furthermore, we show that the conjecture of \([21]\) holds for all \( \Lambda_N \), that is, that \( HH^*(\Lambda_N)/\mathcal{N} \) is a finitely generated \( \mathbb{K} \)-algebra. Moreover, we show that \( HH^*(\Lambda_N)/\mathcal{N} \) is a commutative ring of Krull dimension 2.

It is to be expected that the results in this paper will give some insight into the more general problem of verifying the conjecture of \([21]\) for all self-injective special biserial algebras. It is known that the conjecture is not true in general (see \([24]\) in which the author gives a counter-example which is a 7-dimensional Koszul algebra that is not self-injective). It is therefore important to find out for which algebras the conjecture is true in order to apply the results of \([21, 7]\). For comparison, we recall that the conjecture has been verified for finite-dimensional monomial algebras \( A \) in \([13]\), where it was shown that \( HH^*(A)/\mathcal{N} \) is a finitely generated commutative algebra of Krull dimension at most 1. It has also been proved for many classes of finite-dimensional self-injective algebras, including self-injective algebras of finite representation type (see \([14]\)), cocommutative Hopf algebras (see \([10]\)), and, more recently, pointed Hopf algebras with abelian group of grouplike elements (see \([19]\)) and Hopf algebras of rank one (see \([2]\)). Note that in this last paper there are general results describing the structure of the Hochschild cohomology ring of a smash product of an algebra with a Hopf algebra. Although these results could, in principle, be applied to the algebras in the current paper, this would require further significant structural information to be determined.

We now outline the structure of the paper. Section 1 describes the minimal projective resolution \((P^*, \partial^*)\) of the algebra \( \Lambda_N \) as a bimodule, together with an explanation of the construction of this resolution. This construction is of more general interest and applicability. Section 2 describes the methods used to find the dimensions of the kernel and image of the induced maps in the complex \( \text{Hom}(P^*, \Lambda_N) \) and hence gives the dimension of each Hochschild cohomology group \( HH^p(\Lambda_N) \) in the case \( m \geq 3 \). In calculating \( HH^*(\Lambda_N) \), we first consider the general case \( m \geq 3 \), and start with a description in section 3 of the centre of \( \Lambda_N \), that is, of \( HH^0(\Lambda_N) \), before giving the structure of the Hochschild cohomology ring for \( m \geq 3 \) in sections 4 and 5. Section 6 deals with the case \( m = 2 \) and section 7 with the case \( m = 1 \). The final section summarises the paper, and remarks that the finiteness conditions (Fg1) and (Fg2) of \([7]\) hold when \( N = 1 \), thus enabling one to describe the support varieties for finitely generated modules over the algebras \( \Lambda_1 \).

### 1. A Minimal Projective Bimodule Resolution

Let \( N \geq 1 \), and let \( \Lambda_N = \mathbb{K}Q/I_N \) where \( I_N \) is the ideal of \( \mathbb{K}Q \) generated by \( a_i a_{i+1}, \bar{a}_{i-1} a_{i-2} \) and \((a_i a_j)^N = (\bar{a}_{i-1} a_{i-1})^N \), for \( i = 0, 1, \ldots, m - 1 \). Where there is no confusion, we label the arrows of \( Q \) generically by \( a \) and \( \bar{a} \). We write \( \sigma(\alpha) \) for the trivial path corresponding to the origin of the arrow \( \alpha \), so that \( \sigma(a_i) = e_i \) and \( \sigma(\bar{a}_i) = e_{i+1} \). We write \( t(\alpha) \) for the trivial path corresponding to
The Hochschild cohomology ring of a $\mathbb{K}$-algebra $\Lambda$ is given by $HH^\ast(\Lambda) = \text{Ext}^\ast_{\Lambda^e}(\Lambda, \Lambda) = \bigoplus_{n \geq 0} \text{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$ with the Yoneda product, where $\Lambda^e = \Lambda^{\text{op}} \otimes_{\mathbb{K}} \Lambda$ is the enveloping algebra of $\Lambda$. Since all tensors are over the field $\mathbb{K}$ we write $\otimes$ for $\otimes_{\mathbb{K}}$ throughout.

We now describe a minimal projective bimodule resolution $(P^\ast, \partial^\ast)$ of $\Lambda_N$. By [16], we know that the multiplicity of $\Lambda e_i \otimes e_j \Lambda$ as a direct summand of $P^n$ is equal to the dimension of $\text{Ext}^n_{\Lambda}(S_i, S_j)$. Thus we have, for $n \geq 0$, that

$$P^n = \bigoplus_{i=0}^{m-1} \bigoplus_{r=0}^{n} \Lambda N e_i \otimes e_{i+n-2} \Lambda N.$$

We wish to label the summands of $P^n$ by certain elements in the path algebra $\mathbb{K}Q$. The description given here is motivated by [12] where the first terms of a minimal bimodule resolution of a finite-dimensional quotient of a path algebra were determined explicitly from the early terms of the minimal right $\Lambda$-resolution of $\Lambda/\tau$, where $\tau$ is the Jacobson radical of $\Lambda$, using [15].

We start by recalling briefly the theory of projective resolutions developed in [15] and [12]. In [15], the authors give an explicit inductive construction of a minimal projective resolution of $\Lambda/\tau$ as a right $\Lambda$-module, for a finite-dimensional algebra $\Lambda$ over a field $\mathbb{K}$. For $\Lambda = \mathbb{K}Q/I$ and finite-dimensional, they define $g^0$ to be the set of vertices of $Q$, $g^1$ to be the set of arrows of $Q$, and $g^2$ to be a minimal set of uniform relations in the generating set of $I$, and then show that there are subsets $g^n, n \geq 3$, of $\mathbb{K}Q$, where $x \in g^n$ are uniform elements satisfying $x = \sum_{y \in g^{n-1}} yr_y = \sum_{z \in g^{n-2}} z s_z$ for unique $r_y, s_z \in \mathbb{K}Q$, which can be chosen in such a way that there is a minimal projective $\Lambda$-resolution of the form

$$\cdots \rightarrow Q^4 \rightarrow Q^3 \rightarrow Q^2 \rightarrow Q^1 \rightarrow Q^0 \rightarrow \Lambda/\tau \rightarrow 0$$

having the following properties.

1. For each $n \geq 0$, $Q^n = \prod_{x \in g^n} t(x) \Lambda$.
2. For each $x \in g^n$, there are unique elements $r_j \in \mathbb{K}Q$ with $x = \sum r_j g^{n-1} r_j$.
3. For each $n \geq 1$, using the decomposition of (2), for $x \in g^n$, the map $Q^n \rightarrow Q^{n-1}$ is given by

$$t(x) \lambda \mapsto \sum_j r_j t(x) \lambda,$$

where the elements of the set $g^n$ are labeled by $g^n = \{ g^n_j \}$. Thus the maps in this minimal projective resolution of $\Lambda/\tau$ as a right $\Lambda$-module are determined by the elements $r_j$ which are uniquely determined by (2).

In [12], these same sets $g^n$ are used to give an explicit description of the first three maps in a minimal projective resolution of $\Lambda$ as a $\Lambda, \Lambda$-bimodule, thus connecting the minimal projective resolution of $\Lambda/\tau$ as a right $\Lambda$-module with a minimal projective resolution of $\Lambda$ as a $\Lambda, \Lambda$-bimodule. We use the same ideas here to give a minimal projective bimodule resolution for the algebra $\Lambda_N$, for all $N \geq 1$. In the case where $N = 1$, the algebra $\Lambda_1$ is Koszul, and we refer to [11] which uses this approach and gives a minimal projective bimodule resolution for any Koszul algebra.

We start with the case $N = 1$ and define sets $g^n$ in the path algebra $\mathbb{K}Q$ which we will use to label the generators of $P^n$.

1.1. **The bimodule resolution for $\Lambda_1$.** Consider the algebra $\Lambda_1$. For each vertex $i = 0, \ldots, m-1$ and for each $r = 0, \ldots, n$, we define elements $g^n_{r,i}$ in $\mathbb{K}Q$ as follows. Let

$$g^n_{r,i} = \sum_{p} (-1)^p a_p$$
simplify to

Using our conventions that

\[ p \in e_i \mathbb{K} Q, \]

(i) \( p \) contains \( r \) arrows of the form \( \bar{a} \) and \( n - r \) arrows of the form \( a \), and

(ii) \( s = \sum \alpha_i = 0 \).

It follows that \( g^n_{r,i} \in e_i (\mathbb{K} Q) e_{i+n-2} \), for \( i = 0, \ldots, m - 1 \) and \( r = 0, \ldots, n \). Since the elements \( g^n_{r,i} \) are uniform elements, we may define \( \sigma (g^n_{r,i}) = e_i \) and \( t (g^n_{r,i}) = e_{i+n-2} \). Then

\[ P^n = \bigoplus_{i=0}^{m-1} \bigoplus_{r=0}^{n-1} \sigma (g^n_{r,i}) \otimes t (g^n_{r,i}) A_1. \]

We set

\[ g^n = \bigcup_{i=0}^{m-1} \{ g^n_{r,i} \mid r = 0, \ldots, n \}. \]

It is easy to see, for the cases \( n = 0, 1, 2 \), that we have \( g^n_{0,i} = e_i \), \( g^n_{1,i} = a_i \), and \( g^n_{i,i} = -a_i \). Thus

\[ g^n_r = \{ e_i \mid i = 0, \ldots, m - 1 \}, \]
\[ g^n_{1,i} = \{ a_i, -a_i \mid i = 0, \ldots, m - 1 \}, \]
\[ g^n = \{ a_i a_{i+1}, a_i a_i - a_{i-1} a_i, -a_i a_i - 2 \text{ for all } i \}, \]

so that \( g^2 \) is a minimal set of uniform relations in the generating set of \( I_1 \).

To describe the map \( \partial^n : P^n \rightarrow P^{n-1} \), we need to write the elements of the set \( g^n \) in terms of the elements of the set \( g^{n-1} \). The proof of the next lemma is straightforward, and is left to the reader.

**Lemma 1.1.** For the algebra \( A_1 \), for \( i = 0, 1, \ldots, m - 1 \) and \( r = 0, 1, \ldots, n \), we have:

\[ g^n_{r,i} = g^{n-1}_{r-1,i} a_{i+n-2} + (-1)^r g^{n-1}_{r-1,i} a_{i+n-2} = (-1)^r a_i g^{n-1}_{r,i} + (-1)^r a_i g^{n-1}_{r-1,i}, \]

with the conventions that \( g^{n-1}_{r-1,i} = 0 \) and \( g^{n-1}_{r,i} = 0 \) for all \( n, i \). Thus

\[ g^n_{0,i} = g^{n-1}_{0,i} a_{i+n-1} = a_i g^{n-1}_{0,i} \text{ and } g^n_{i,i} = (-1)^n g^{n-1}_{n-1,i} a_{i-n+1} = (-1)^n a_i g^{n-1}_{n-1,i}. \]

Since \( A_1 \) is Koszul, we now use [11] to give a minimal projective bimodule resolution \( (P^n, \partial^n) \) of \( A_1 \). We define the map \( \partial^n : P^n \rightarrow A_1 \) to be the multiplication map. However, to define \( \partial^n \) for \( n \geq 1 \), we first need to introduce some notation. In describing the image of \( \sigma (g^n_{r,i}) \otimes t (g^n_{r,i}) \) under \( \partial^n \) in the projective module \( P^{n-1} \), we use subscripts under \( \otimes \) to indicate the appropriate summands of the projective module \( P^{n-1} \). Specifically, let \( \otimes_r \) denote a term in the summand of \( P^{n-1} \) corresponding to \( g^{n-1}_{r-1,i} \), and \( \otimes_r \) denote a term in the summand of \( P^{n-1} \) corresponding to \( g^{n-1}_{r-1,i} \), where the appropriate index is of the vertex and may always be uniquely determined from the context. Indeed, since the relations are uniform along the quiver, we can also take labeling elements defined by a formula independent of \( i \), and hence we omit the index \( i \) when it is clear from the context. Recall that nonetheless all tensors are over \( \mathbb{K} \).

Now, for \( n \geq 1 \), keeping the above notation and using [11], we define the map \( \partial^n : P^n \rightarrow P^{n-1} \) for the algebra \( A_1 \) as follows:

\[ \partial^n : \sigma (g^n_{r,i}) \otimes t (g^n_{r,i}) \mapsto \begin{cases} e_i \otimes_r a_{i+n-2} + (-1)^n e_i \otimes_{r-1} a_{i+n-2} \\ + (-1)^n a_i \otimes_r e_{i+n-2} + (-1)^n a_i \otimes_{r-1} e_{i+n-2} \end{cases}. \]

Using our conventions that \( g^{n-1}_{0,i} = 0 \) and \( g^{n-1}_{r,i} = 0 \) for all \( n, i \), the degenerate cases \( (r = 0, r = n) \) simplify to

\[ \partial^n : \sigma (g^{n-1}_{0,i}) \otimes t (g^{n-1}_{0,i}) \mapsto e_i \otimes_0 a_{i+n-1} + (-1)^n a_i \otimes_0 e_{i+n} \]
where the first term is in the summand corresponding to $g_{0,i}^{n-1}$ and the second term is in the summand corresponding to $g_{1,i+1}^{n-1}$, whilst

$$\partial^n: o(g_{n,i}^a) \otimes t(g_{n,i}^a) \mapsto (-1)^n e_i \otimes_{n-1} \bar{a}_{i-n} + \bar{a}_{i-1} \otimes_{n-1} e_{i-n},$$

with the first term in the summand corresponding to $g_{n-1,i}^{n-1}$, and the second term in the summand corresponding to $g_{n-1,i-1}^{n-1}$.

The following result is now immediate from [11, Theorem 2.1].

**Theorem 1.2.** With the above notation, $(P^n, \partial^n)$ is a minimal projective resolution of $A_1$ as a $A_1, A_1$-bimodule.

1.2. The bimodule resolution for $A_N$ with $N \geq 1$. We now consider the general case $N \geq 1$. In this case we use the approach of [11, 12] to construct a minimal projective resolution for $A_N$. We keep the conventions that $g_{n+1,i}^{n-1} = 0$ and $g_{n,i}^{n-1} = 0$, for all $n, i$, throughout the paper.

**Definition 1.3.** For the algebra $A_N$ ($N \geq 1$), for $i = 0, \ldots, m-1$ and $r = 0, \ldots, n$, we define $g_{0,i}^0 = e_i$, and, inductively for $n \geq 1$,

$$g_{r,i}^n = \begin{cases} g_{r-1,i}^{n-1} a + (-1)^n g_{r-1,i}^{n-1} \bar{a}(a\bar{a})^{N-1} & \text{if } n - 2r > 0; \\
\quad g_{r,i}^n a(\bar{a}a) + (-1)^n g_{r-1,i}^{n-1} \bar{a} & \text{if } n - 2r < 0; \\
\quad g_{r,i}^n a(\bar{a}a)^{N-1} + g_{r-1,i}^{n-1} \bar{a}(a\bar{a})^{N-1} & \text{if } n = 2r. \end{cases}$$

**Remark.** (1) If $N = 1$, the above definition reduces to that given for $A_1$.

(2) We have $g_{0,i}^0 = a_i$ and $g_{1,i}^1 = -\bar{a}_{i-1}$, for all $i$. Also $g_{0,i}^1 = a_i a_{i+1}$, $g_{1,i}^1 = (a_i \bar{a}_i)^N - (a_{i-1} \bar{a}_{i-1})^N$ and $g_{2,i}^2 = -a_{i-2} \bar{a}_{i-2}$, for all $i$. Hence $g^2$ is a minimal set of uniform relations in the generating set of $I_N$.

(3) The projectives in a minimal projective bimodule resolution of $A_N$ are given by

$$P^n = \bigoplus_{i=0}^{m-1} [\bigoplus_{r=0}^{n} A_N o(g_{r,i}^a) \otimes t(g_{r,i}^a) A_N].$$

The next result is an analogue of Lemma 1.1 for the cases $n - 2r > 0$ and $n - 2r < 0$. The proof of the Lemma is easy to verify and is omitted.

**Lemma 1.4.** For the algebra $A_N$ ($N \geq 1$), for $i = 0, \ldots, m-1$, $n \geq 1$, and $r = 0, \ldots, n$, we have:

$$g_{r,i}^n = \begin{cases} g_{r-1,i}^{n-1} a + (-1)^n g_{r-1,i}^{n-1} \bar{a}(a\bar{a})^{N-1} = (-1)^r a g_{r-1,i}^{n-1} + (-1)^r \bar{a}(a\bar{a})^{N-1} g_{r-1,i}^{n-1} & \text{if } n - 2r > 0; \\
\quad g_{r,i}^n a(\bar{a}a)^{N-1} + (-1)^n g_{r-1,i}^{n-1} \bar{a} = (-1)^r a(\bar{a}a)^{N-1} g_{r-1,i}^{n-1} + (-1)^r \bar{a} g_{r-1,i}^{n-1} & \text{if } n - 2r < 0. \end{cases}$$

We now define the maps $\partial^n$.

**Definition 1.5.** For the algebra $A_N$ ($N \geq 1$), and for $n \geq 0$, we define maps $\partial^n$ as follows. For $n = 0$, the map $\partial^0: P^0 \to A_N$ is the multiplication map. For $n \geq 1$, and $i = 0, 1, \ldots, m-1$, the map $\partial^n: P^n \to P^{n-1}$ is given by $\partial^n: e_i \otimes_{n} e_{i+n-2r} \mapsto$
\[
\begin{aligned}
e_i \otimes_r e_{i+(n-1)-2r} a + (-1)^{n+r} & a e_{i+1} \otimes_r e_{i+1+(n-1)-2r} \\
& + (-1)^n a_{i-1} e_{i-1} \otimes_r e_{i-1+(n-1)-2r} \\
& + (-1)^n a_{i} e_{i} \otimes_r e_{i+(n-1)-2r} \\
& + (-1)^n a_{i+1} e_{i+1} \otimes_r e_{i+1+(n-1)-2r} \\
& + (-1)^n a_{i+2} e_{i+2} \otimes_r e_{i+2+(n-1)-2r} \\
\end{aligned}
\]

if \(n - 2r > 0\),

\[
\begin{aligned}
e_i \otimes_r e_{i+(n-1)-2r} a + (-1)^{n+r} & a e_{i+1} \otimes_r e_{i+1+(n-1)-2r} \\
& + (-1)^n a_{i-1} e_{i-1} \otimes_r e_{i-1+(n-1)-2r} \\
& + (-1)^n a_{i} e_{i} \otimes_r e_{i+(n-1)-2r} \\
& + (-1)^n a_{i+1} e_{i+1} \otimes_r e_{i+1+(n-1)-2r} \\
& + (-1)^n a_{i+2} e_{i+2} \otimes_r e_{i+2+(n-1)-2r} \\
\end{aligned}
\]

if \(n - 2r < 0\),

\[
\sum_{k=0}^{N-1} (a)_{k} \left[ (e_{i} \otimes \frac{1}{z} e_{i-1} + (-1)^{n} \otimes \right. \left. \frac{1}{z} \right] \otimes \frac{1}{z} \left( \frac{1}{z} \right) \otimes e_{i} \left( a a \right)_{N-k-1} \]

\[
+ (a a) \left[ (e_{i} \otimes \frac{1}{z} e_{i-1} + (-1)^{n} \otimes \right. \left. \frac{1}{z} \right] \otimes \frac{1}{z} \left( \frac{1}{z} \right) \otimes e_{i} \left( a a \right)_{N-k-1} \]

if \(n - 2r = 0\).

In order to prove that \((P^n, \partial^n)\) is a minimal projective resolution of \(\Lambda_N\) as a \(\Lambda_N, \Lambda_N\)-bimodule, we use an argument which was given in [12]. To do this, we note that

\[
\Lambda_N / r \otimes \Lambda_N P^n \cong \bigoplus_{i=0}^{m-1} \otimes_{v=0}^{n} t(g_{r,v})N
\]

as right \(\Lambda_N\)-modules and that the map \(\text{id} \otimes \partial^n : \Lambda_N / r \otimes \Lambda_N P^n \rightarrow \Lambda_N / r \otimes \Lambda_N P^{n-1}\) is equivalent to the map \(\bigoplus_{i=0}^{m-1} \otimes_{v=0}^{n} t(g_{r,v})N\) given by

\[
t(g_{r,v}) \mapsto \begin{cases} 
\text{id}(g_{r,v}) & \text{if } n - 2r > 0, \\
(-1)^{n} \text{id}(g_{r,v}) & \text{if } n - 2r < 0, \\
0 & \text{if } n - 2r = 0.
\end{cases}
\]

It is then straightforward (with Lemma 1.4) to see that \((\Lambda_N / r \otimes \Lambda_N P^n, \text{id} \otimes \partial^n)\) is a minimal projective resolution of \(\Lambda_N / r\) as a right \(\Lambda_N\)-module, which satisfies the conditions of [15] as explained above, for all \(n \geq 1\). For completeness in the proof of Theorem 1.6, we explain in full the strategy used in [12, Proposition 2.8].

**Theorem 1.6.** With the above notation, \((P^n, \partial^n)\) is a minimal projective resolution of \(\Lambda_N\) as a \(\Lambda_N, \Lambda_N\)-bimodule, for all \(n \geq 1\).

**Proof.** The first step is to verify that we have a complex by showing that \(\partial^2 = 0\); this is straightforward and the details are left to the reader.

Now suppose for contradiction that \(\text{Ker} \partial^{n-1} \not\subseteq \text{Im} \partial^n\) for some \(n \geq 1\). Then there is some non-zero map \(\text{Ker} \partial^{n-1} \rightarrow \text{Ker} \partial^{n-1} / \text{Im} \partial^n\). Hence there is a simple \(\Lambda_N, \Lambda_N\)-bimodule \(U \otimes V\) and epimorphism \(\text{Ker} \partial^{n-1} / \text{Im} \partial^n \rightarrow U \otimes V\) such that the composition \(f : \text{Ker} \partial^{n-1} \rightarrow \text{Ker} \partial^{n-1} / \text{Im} \partial^n \rightarrow U \otimes V\) is a non-zero epimorphism.

We can verify directly that \((\Lambda_N / r \otimes \Lambda_N P^n, \text{id} \otimes \partial^n)\) is a minimal projective resolution of \(\Lambda_N / r\) as a right \(\Lambda_N\)-module, so we have that \(\Lambda_N / r \otimes \Lambda_N \text{Im} \partial^n \cong \text{Im} (\text{id} \otimes \partial^n) = \text{Ker} (\text{id} \otimes \partial^{n-1}) = \Lambda_N / r \otimes \Lambda_N \text{Ker} \partial^{n-1}\). Applying the functor \(\Lambda_N / r \otimes \Lambda_N -\) to the map \(\partial^n : P^n \rightarrow P^{n-1}\) gives:

\[
\begin{array}{ccc}
\Lambda_N / r \otimes \Lambda_N P^n & \xrightarrow{\text{id} \otimes \partial^n} & \Lambda_N / r \otimes \Lambda_N P^{n-1} \\
\downarrow & & \downarrow \\
\text{Im} \partial^n & \xrightarrow{\text{Ker} \partial^{n-1}} & \Lambda_N / r \otimes \Lambda_N U \otimes V.
\end{array}
\]

Now, \(U\) is a simple left \(\Lambda_N\)-module (and \(V\) is a simple right \(\Lambda_N\)-module) so \(\Lambda_N / r \otimes \Lambda_N U \otimes V\) is non-zero. Since the functor \(\Lambda_N / r \otimes \Lambda_N -\) preserves epis, the map \(\text{id} \otimes \partial^n\) is non-zero, so the composition \(\Lambda_N / r \otimes \Lambda_N P^n \rightarrow \Lambda_N / r \otimes \Lambda_N U \otimes V\) is non-zero. However, this is simply the functor \(\Lambda_N / r \otimes \Lambda_N -\) applied to the composition \(P^n \xrightarrow{\partial^n} \text{Ker} \partial^{n-1} \xrightarrow{f} U \otimes V\). Now the composition \(P^n \xrightarrow{\partial^n} \text{Ker} \partial^{n-1} \xrightarrow{f} U \otimes V\).
Ker $\partial^{n-1} \xrightarrow{f} U \otimes V$ is zero since $\text{Im} \partial^n \subseteq \text{Ker} f$. This gives the required contradiction, so the complex is exact.

Finally, minimality follows since we know that the projectives are those of a minimal projective resolution of $\Lambda_N$ as a $\Lambda_N, \Lambda_N$-bimodule from [16].

We note that the degenerate cases ($r = 0, r = n$) in the maps $\partial^n$ may be written as:

$$\partial^n: e_i \otimes_0 e_{i+n} \mapsto e_i \otimes_0 e_{i+n-1} a + (-1)^n a e_{i+1} \otimes_0 e_{i+n}$$

and

$$\partial^n: e_i \otimes_n e_{i-n} \mapsto ae_{i-1} \otimes_n e_{i-n} + (-1)^n e_i \otimes_{n-1} e_{i-n+1}. a.$$  

1.3. Conventions on notation. So far, we have tried to simplify notation by denoting the idempotent $\sigma(g^n_{r,i}) \otimes (g^n_{r,i})$ of the summand $\Lambda_N \otimes (\Lambda_N)$ of $\Lambda_N$ uniquely by $e_i \otimes e_{i+n-2r}$, where $0 \leq i \leq m - 1$. However, even this notation with subscripts under the tensor product symbol becomes cumbersome in computations and in describing the elements of the Hochschild cohomology ring. Thus we make some conventions (by abuse of notation) which we keep throughout the paper.

First, we note that, in the language of [3], the $n$-th projective module $\Lambda_N$ in a projective bimodule resolution of $\Lambda_N$ is denoted $S \otimes_S V_n \otimes_S S$ where $S$ is the semisimple $k$-algebra generated by the idempotents $\{e_0, e_1, \ldots, e_m-1\}$ and $V_n$ is the $S, S$-bimodule generated by the set

$$\{g^n_{r,i} \mid i = 0, 1, \ldots, m - 1, \ r = 0, 1, \ldots, n\}.$$  

(Noe that here $\otimes_S$ denotes a tensor over $S$, that is, over a finite sum of copies of $k$; we continue to reserve the notation $\otimes$ exclusively for tensors over $k$.) There is a bijective correspondence between the idempotent $\sigma(g^n_{r,i}) \otimes (g^n_{r,i})$ and the term $e_i \otimes_S g^n_{r,i} \otimes_S e_{i+n-2r}$ in $V_n$, for $i = 0, 1, \ldots, m - 1$ and $r = 0, 1, \ldots, n$.

Since $e_{i+n-2r} \in \{e_0, e_1, \ldots, e_m-1\}$, it would be usual to reduce the subscript $i + n - 2r$ modulo $m$. However, to make it explicitly clear to which summand of the projective module $\Lambda_N$ we are referring and thus to avoid confusion, whenever we write $e_i \otimes e_{i+k}$ for an element of $\Lambda_N$, we will always have $i \in \{0, 1, \ldots, m - 1\}$ and consider $i + k$ as an element of $\mathbb{Z}$, in that $r = (n - k)/2$ and $e_i \otimes e_{i+k} = e_i \otimes r \cdot e_{i+k}$ and thus lies in the $\frac{n-k}{2}$-th summand of $\Lambda_N$. We do not reduce $i + k$ modulo $m$ in any of our computations. In this way, when considering elements in $\Lambda_N$, our element $e_i \otimes e_{i+k}$ corresponds uniquely to the idempotent $\sigma(g^n_{r,i}) \otimes (g^n_{r,i})$ of $\Lambda_N$ (or, equivalently, to $e_i \otimes_S g^n_{r,i} \otimes_S e_{i+n-2r}$) with $r = (n - k)/2$, for each $i = 0, 1, \ldots, m - 1$.

Since we take $0 \leq i \leq m - 1$ we clarify our conventions in Definition 1.5 in the cases $i = 0$ and $i = m - 1$, to ensure that we always have $0 \leq j \leq m - 1$ in the first idempotent entry of each tensor $(e_j \otimes \cdot)$. Specifically, we have:

$$\partial^n: e_0 \otimes e_{n-2r} \mapsto$$

$$\begin{cases}
\left(\begin{array}{l}
e_0 \otimes e_{(n-1)-2r} a + (-1)^{n-r} ae_1 \otimes e_{1+(n-1)-2r} \\
+ (-1)^{n-r} a \otimes a \otimes e_{1+(n-1)-2r} (a a)^{-1} e_{m-1} \otimes e_{m-1+(n-1)-2(r-1)} \\
+ (-1)^n e_0 \otimes e_{(n-1)-2(r-1) \cdot a} (a a)^{-1} & \text{if } n - 2r > 0,
\end{array}\right)
\end{cases}$$

$$\begin{cases}
\left(\begin{array}{l}
e_0 \otimes e_{(n-1)-2r} a \otimes a (a a)^{-1} + (-1)^{n+r} a \otimes a (a a)^{-1} e_1 \otimes e_{1+(n-1)-2r} \\
+ (-1)^{n+r} a \otimes a \otimes e_{1+(n-1)-2(r-1)} (a a)^{-1} e_{m-1} \otimes e_{m-1+(n-1)-2(r-1)} \\
+ (-1)^n e_0 \otimes e_{(n-1)-2(r-1) \cdot a} a & \text{if } n - 2r < 0,
\end{array}\right)
\end{cases}$$

$$\sum_{k=0}^{N-1} (a a)^k [e_0 \otimes e_{-1} a + (-1)^{k^2} a e_{m-1} \otimes a] (a a)^{N-k-1}$$

$$+ (a a)^k [(-1)^{k^2} ae_1 \otimes e_0 + e_0 \otimes e_1 a] (a a)^{N-k-1}$$

if $n - 2r = 0,$
and $\partial^n: e_{m-1} \otimes e_{m-1+n-2r} \mapsto$

\[
\begin{align*}
e_{m-1} \otimes e_{m-1+(n-1)-2r} &+ (-1)^{n+r}ae_0 \otimes e_{(n-1)-2r} \\
+(-1)^{n+r}a(\bar{a}a)N^{-1}e_{m-2} \otimes e_{m-2+(n-1)-2(r-1)} \\
+(-1)^ne_{m-1} \otimes e_{m-1+(n-1)-2(r-1)}\bar{a}(\bar{a}a)^{N-1} & \quad \text{if } n - 2r > 0, \\
e_{m-1} \otimes e_{m-1+(n-1)-2r}a(\bar{a}a)^N &+ (-1)^{n+r}a(\bar{a}a)^N \otimes e_{(n-1)-2r} \\
+(-1)^{n+r}ae_{m-2} \otimes e_{m-2+(n-1)-2(r-1)} \\
+(-1)^ne_{m-1} \otimes e_{m-1+(n-1)-2(r-1)}\bar{a} & \quad \text{if } n - 2r < 0, \\
\sum_{k=0}^{N-1}(\bar{a}a)^k[e_{m-1} \otimes e_{m-2}a + (-1)^{\bar{a}}ae_{m-2} \otimes e_{m-1}] & \quad (\bar{a}a)^N \otimes 1 \\
+(-1)^{\bar{a}m}\bar{a}e_0 \otimes e_{-1} + e_{m-1} \otimes e_m\bar{a} & \quad (\bar{a}a)^N \otimes 1 \\
\end{align*}
\]

if $n - 2r = 0$.

Our notation $e_i \otimes e_{i+k}$ is used without further comment throughout the rest of the paper.

We end this section by determining the dimension of each space $\text{Hom}_{\Lambda_N^n}(P^n, \Lambda_N)$ for each $n \geq 0$.

1.4. The complex $\text{Hom}_{\Lambda_N^n}(P^n, \Lambda_N)$. All our homomorphisms are $\Lambda_N^n$-homomorphisms and so we write $\text{Hom}(\cdot, \cdot)$ for $\text{Hom}_{\Lambda_N^n}(\cdot, \cdot)$.

Lemma 1.7. If $m \geq 3$, write $n = pm + t$ where $p \geq 0$ and $0 \leq t \leq m - 1$. Then

\[\dim \text{Hom}(P^n, \Lambda_N) = \begin{cases} (4p+2)mN & \text{if } t \neq m - 1 \\ (4p+4)mN & \text{if } t = m - 1. \end{cases}\]

If $m = 1$ or $m = 2$ then

\[\dim \text{Hom}(P^n, \Lambda_N) = 4N(n+1).\]

Proof. An element $f$ in $\text{Hom}(P^n, \Lambda_N)$ is determined by the elements $f(e_i \otimes e_{i+n-2r})$, which can be arbitrary elements of $e_{i+N}e_{i+n-2r}$.

Suppose first that $m \geq 3$. Then, for vertices $i, j$, we have

\[\dim e_i \Lambda_N e_j = \begin{cases} 2N & \text{if } i = j, \\ N & \text{if } i - j \equiv \pm 1 \pmod{m}, \\ 0 & \text{otherwise}. \end{cases}\]

For a finite subset $X \subset Z$ and $0 \leq s \leq m - 1$, we define $\nu_s(X) = |\{x \in X : x \equiv s \pmod{m}\}|$. With this we have

\[\dim \text{Hom}(P^n, \Lambda_N) = mN(2\nu_0(I_n) + \nu_1(I_n) + \nu_{-1}(I_n))\]

where $I_n = \{n - 2r : 0 \leq r \leq n\}$. We observe that the disjoint union $I_n \cup I_{n-1}$ is equal to $[-n, n]$, the set of all integers $x$ with $-n \leq x \leq n$. Clearly $\nu_0([-n, n]) = 2p + 1$ and

\[\nu_1([-n, n]) = \begin{cases} 2p & \text{if } t = 0, \\ 2p + 1 & \text{if } 1 \leq t \leq m - 2, \\ 2p + 2 & \text{if } t = m - 1. \end{cases}\]

Moreover $\nu_{-1}([-n, n]) = \nu_1([-n, n])$, by symmetry.

The lemma now follows by induction on $n$, with the case $n = 0$ being clear. For the inductive step, note that

\[\dim \text{Hom}(P^n, \Lambda_N) = mN(2\nu_0(I_n) + \nu_1(I_n) + \nu_{-1}(I_n)) = mN(2\nu_0([-n, n]) + 2\nu_1([-n, n])) - \dim \text{Hom}(P^{n-1}, \Lambda_N).\]

The statement for $\dim \text{Hom}(P^n, \Lambda_N)$ now follows for $m \geq 3$. 

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In the case \( m = 1 \), we have \( P^n = \bigoplus_{pm=0}^n \Lambda_N \otimes \Lambda_N \) and \( \dim \Lambda_N = 4N \). Thus \( \dim \text{Hom}(P^n, \Lambda_N) = 4N(n+1) \).

Finally, for \( m = 2 \), suppose first that \( n \) is even. Then \( P^n \) has \( n+1 \) summands equal to \( \Lambda_N e_0 \otimes e_0 \) and \( n+1 \) summands equal to \( \Lambda_N e_1 \otimes e_1 \). By symmetry, \( \dim \text{Hom}(P^n, \Lambda_N) = 2(n+1) \dim e_1 \Lambda_N e_1 = 4N(n+1) \). For \( n \) odd, \( P^n \) has \( n+1 \) summands equal to \( \Lambda_N e_0 \otimes e_1 \Lambda_N \) and \( n+1 \) summands equal to \( \Lambda_N e_1 \otimes e_0 \). Hence, again using symmetry, we have \( \dim \text{Hom}(P^n, \Lambda_N) = 2(n+1) \dim e_0 \Lambda_N e_1 = 4N(n+1) \). This completes the proof.

\[ \square \]

2. The dimensions of the Hochschild cohomology spaces for \( m \geq 3 \).

We give here the main arguments we used to compute the kernel and the image of the differentials, and leave the details to the reader. We denote both the map \( P^n \to P^{n-1} \) and its induced map \( \text{Hom}(P^{n-1}, \Lambda_N) \to \text{Hom}(P^n, \Lambda_N) \) by \( \partial^n \). Then we have that \( \text{HH}^n(\Lambda_N) = \ker(\partial^{n+1}) / \text{im}(\partial^n) \). We assume without further comment that \( m \geq 3 \) throughout this section.

2.1. Explicit description of maps. In order to compute the Hochschild cohomology groups, we need an explicit description of the image of each \( \text{Hom}(\mathcal{P}(\Lambda_N), n) \) for \( i = 0, \ldots, m-1 \). We write \( n = pm + t \) with \( 0 \leq t \leq m-1 \), \( p \geq 0 \).

The image \( f(e_i \otimes e_{i+m-n-2r}) \) lies in \( \Lambda_N e_j \) where \( j \in \mathbb{Z} \) with \( j \equiv i + n - 2r \mod m \). Since \( e_i \Lambda_N e_j = 0 \) if the vertices \( i \) and \( j \) are sufficiently far apart in the quiver of \( \Lambda_N \) (at least for \( m \geq 3 \)), for a non-zero image \( f(e_i \otimes e_{i+m-n-2r}) \) we are only interested in examining the cases where \( i + n - 2r \equiv i, i-1, i+1 \mod m \). This leads to the consideration of terms of the form \( f(e_i \otimes e_{i+sm}) \), \( f(e_i \otimes e_{i+sm-1}) \), and \( f(e_i \otimes e_{i+sm+1}) \) where \( s \in \mathbb{Z} \), \( s \geq 0 \).

For any \( s \geq 0 \) and \( 0 \leq i \leq m-1 \), we have that \( f(e_i \otimes e_{i+sm}) \) is a linear combination of the elements \( (a\alpha)^k \), \( 0 \leq k \leq N-1 \), and \( (aa)^k \), \( 1 \leq k \leq N \); \( f(e_i \otimes e_{i+sm-1}) \) is a linear combination of the \( a(\alpha\alpha)^k \), \( 0 \leq k \leq N-1 \), and \( f(e_i \otimes e_{i+sm+1}) \) is a linear combination of the \( a(\alpha\alpha)^k \), \( 0 \leq k \leq N-1 \). For \( m \geq 3 \), we give here a list of the terms which must be considered in giving a possible non-zero image of \( f \); these are used without further comment throughout the paper. (We assume \( m \geq 3 \); the cases \( m = 1 \) and \( m = 2 \) are different and are considered separately later.) For each \( i = 0, 1, \ldots, m-1 \):

- **m even, \( m \neq 2 \)**
  - \( t \) even
    - \( f(e_i \otimes e_{i+sm}) \), \( -p \leq \alpha \leq p \)
  - \( t \) odd, \( t \neq m-1 \)
    - \( f(e_i \otimes e_{i+sm-1}) \), \( -p \leq \beta \leq p \)
    - \( f(e_i \otimes e_{i+sm+1}) \), \( -p \leq \gamma \leq p \)
  - \( t = m-1 \)
    - \( f(e_i \otimes e_{i+sm}) \), \( -p \leq \beta \leq p+1 \)
    - \( f(e_i \otimes e_{i+sm-1}) \), \( -p-1 \leq \gamma \leq p \)
    - \( f(e_i \otimes e_{i+sm+1}) \), \( -p-1 \leq \gamma \leq p \)

- **m odd, \( m \neq 1 \)**
  - \( t \) even, \( t \neq m-1 \)
    - \( f(e_i \otimes e_{i+sm}) \), \( 0 \leq \alpha \leq p \)
    - \( f(e_i \otimes e_{i+sm-1}) \), \( 0 \leq \beta \leq p-1 \)
    - \( f(e_i \otimes e_{i+sm+1}) \), \( 0 \leq \gamma \leq p-1 \)
  - \( t \) odd
    - \( f(e_i \otimes e_{i+sm}) \), \( 0 \leq \alpha \leq p \)
    - \( f(e_i \otimes e_{i+sm-1}) \), \( 0 \leq \beta \leq p \)
    - \( f(e_i \otimes e_{i+sm+1}) \), \( 0 \leq \gamma \leq p \)
  - \( t = m-1 \)
    - \( f(e_i \otimes e_{i+sm}) \), \( -1 \leq \beta \leq p-1 \)
    - \( f(e_i \otimes e_{i+sm-1}) \), \( 0 \leq \gamma \leq p \)

2.2. Kernel of \( \partial^{pm+t+1} : \text{Hom}(P^{pm+t}, \Lambda_N) \to \text{Hom}(P^{pm+t+1}, \Lambda_N) \). An element \( f \) in \( \text{Hom}(P^{pm+t}, \Lambda_N) \) is determined by \( f(e_i \otimes e_{i+n-2r}) \) for appropriate \( i \in \{0, 1, \ldots, m-1\} \). Each term \( f(e_i \otimes e_{i+n-2r}) \) can be written as a linear combination of the basis elements in \( e_i \Lambda_N e_{i+n-2r} \), and we need to find the conditions on the coefficients in this linear combination when \( f \) is in \( \ker \partial^{pm+t+1} \). To do this, we separate the study into several cases: when \( m \) is even, we consider...
the three cases (i) \( t = m - 1 \), (ii) \( t \) even, and (iii) \( t \) odd with \( t \neq m - 1 \), and, when \( m \) is odd, we look at the four cases (i) \( t = m - 1 \), (ii) \( t = m - 2 \), (iii) \( t \) even with \( t \neq m - 1 \), and (iv) \( t \) odd with \( t \neq m - 2 \). Note that in some cases a factor \( N \) appears so that the results depend on the characteristic of \( K \). Note also that, when \( m \) is odd, the case \( K = 2 \) needs to be considered separately.

For \( f \in \operatorname{Hom}(P^m+t, \Lambda_N) \), we write
\[
\begin{align*}
  f(e_i \otimes e_{i+m}) &= \sum_{k=0}^{N-1} \sigma_{i,k}^0 (a_k \bar{a})^k + \sum_{k=1}^\infty \tau_{i,k}^0 (a_{i-1}a_{i+1})^k, \\
  f(e_i \otimes e_{i+m+1}) &= \sum_{k=0}^{N-1} \lambda_{i,k}^\beta (\bar{a}a)^k a_{i-1}, \\
  f(e_i \otimes e_{i+\gamma m+1}) &= \sum_{k=0}^{N-1} \mu_{i,k}^\gamma (\bar{a}a)^k a_i,
\end{align*}
\]
with coefficients \( \sigma_{i,k}^0 \), \( \tau_{i,k}^0 \), \( \lambda_{i,k}^\beta \) and \( \mu_{i,k}^\gamma \) in \( K \), for \( i = 0, 1, \ldots, m - 1 \).

We shall now do two of the cases above, in order to illustrate the general method. In both these cases, for consistency with our convention on the writing of elements \( e_i \otimes e_{i+k} \) with \( i \in \{0, 1, \ldots, m - 1\} \) and \( k \in \mathbb{Z} \), and to simplify, we shall use the following notation: if \( n \in \mathbb{Z} \), \( \langle n \rangle \) denotes the representative of \( n \) modulo \( m \) in \( \{0, 1, \ldots, m - 1\} \). In particular, \( (m) = (0) \), and this must be borne in mind when computing kernels and images. We write \( \partial f \) (rather than \( \partial f^{m+t+1} \)) for the image of \( f \) under the map \( \partial(\cdot)^{m+t+1} : \operatorname{Hom}(P^m+t, \Lambda_N) \to \operatorname{Hom}(P^{m+t}, \Lambda_N) \).

**Case** \( m \) **even and** \( t \) **even.** In this case, \( f \) is entirely determined by the \( f(e_i \otimes e_{i+\alpha m}) \) for \( -p \leq \alpha \leq p \) and \( i \in \{0, 1, \ldots, m - 1\} \) and \( \partial f \) is determined by
\[
\begin{align*}
  \partial f(e_i \otimes e_{i+am}) &= \begin{cases} \\
  \sum_{k=0}^{N-1} \sigma_{i,k}^0 \bar{a}(a_k \bar{a})^k & \text{if } \beta > 0 \\
  \sum_{k=0}^{N-1} \tau_{i,k}^0 \bar{a}a_k & \text{if } \beta < 0 \\
  \sum_{k=0}^{N-1} \lambda_{i,k}^\beta (a_{k-1}a_k)^k a_k & \text{if } \gamma > 0 \\
  \sum_{k=0}^{N-1} \mu_{i,k}^\gamma (a_k \bar{a})^k a_k & \text{if } \gamma < 0.
  \end{cases}
\end{align*}
\]

For any \( \alpha' \) with \( -p \leq \alpha' \leq p \), we can see that \( \sigma_{\alpha'}^0 \) determines the other \( \sigma_{\alpha'}^p \) (setting \( \beta \) or \( \gamma \) to be \( \alpha' \)). Combining the cases \( \beta \leq 0 \) and \( \gamma \geq 0 \) tells us that the \( \sigma_{\alpha'}^p \) for \( k = 1, \ldots, N - 1 \) are determined by the \( \sigma_{i,k}^0 \). Finally the \( \tau_{i,k}^0 \) are arbitrary, and the \( \lambda_{i,k}^\beta \) and \( \mu_{i,k}^\gamma \) for \( k = 1, \ldots, N - 1 \) are arbitrary. Therefore the dimension of \( \ker \partial^{m+t+1} \) in this case is \( (2p+1) + (2p+1)m + (2p+1)m(N-1) \).

**Case** \( m \) **even and** \( t \) **odd,** \( t \neq m - 1 \). In this case, \( f \) is entirely determined by
\[
\begin{align*}
  f(e_i \otimes e_{i+am}) &= \begin{cases} \\
  \sum_{k=0}^{N-1} \sigma_{i,k}^0 \bar{a}(a_k \bar{a})^k + \sum_{k=0}^{N-2} \lambda_{i,k}^\alpha (a_k \bar{a})^{k+1} + (1) (2p+1) \sum_{k=0}^{N-2} \mu_{i,k}^\gamma (a_k \bar{a})^{k+1} & \text{if } \alpha = 0 \\
  \sum_{k=0}^{N-1} \lambda_{i,k}^\alpha (a_k \bar{a})^{k+1} + (1) (2p+1) \sum_{k=0}^{N-2} \mu_{i,k}^\gamma (a_k \bar{a})^{k+1} & \text{if } \alpha > 0 \\
  \sum_{k=0}^{N-2} \lambda_{i,k}^\alpha (a_k \bar{a})^{k+1} + (1) (2p+1) \sum_{k=0}^{N-2} \mu_{i,k}^\gamma (a_k \bar{a})^{k+1} & \text{if } \alpha < 0.
  \end{cases}
\end{align*}
\]

So for \( \alpha = 0 \), \( \lambda_{i,0}^0 \), for \( i \in \{0, \ldots, m - 1\} \) and \( \mu_{i,0}^0 \) determine the other \( \mu_{i,0}^p \), unless the characteristic of \( K \) divides \( N \). For \( \alpha > 0 \), \( \lambda_{i,0}^\alpha \) and \( \lambda_{i,N-1}^\alpha \) determine the other \( \Lambda_{i,N-1}^\alpha \), and for \( \alpha < 0 \),
\( \lambda_{i,0}^\alpha \) and \( \mu_{0,N-1}^\alpha \) determine the other \( \mu_{i,N-1}^\alpha \). Moreover, for \( \alpha > 0 \) the \( \lambda_{i,k}^\alpha \) for \( k = 0, \ldots, N-2 \) are all 0, and for \( \alpha < 0 \) the \( \mu_{i,k}^\alpha \) for \( k = 0, \ldots, N-2 \) are 0. The remaining \( \lambda_{i,k}^\alpha, \mu_{i,k}^\alpha \) are arbitrary, and there are \( 2m(N-1) + pm(N-1) + pm(N-1) = 2(p+1)m(N-1) \) of these coefficients. This gives dimension: \( (m+1) + p(m+1) + p(m+1) + 2(p+1)m(N-1) + m-1 \) if \( \text{char } K \nmid N \) and \( (m+1) + p(m+1) + p(m+1) + 2(p+1)m(N-1) + m-1 \) if \( \text{char } K \mid N \).

Note that when computing the dimensions in the case \( m \) odd and \( \text{char } K \neq 2 \), we often need to introduce additional cases depending on the value of \( p \) modulo 4 and the value of \( m \) modulo 4. We summarise the results in the following two propositions, which separate the cases \( \text{char } K \mid N \) and \( \text{char } K \nmid N \).

**Proposition 2.1.** If \( \text{char } K \mid N \), the dimension of \( \text{Ker } \partial_{p^{m+1}+1} \) is as follows.

If \( m \) even
- \((2p+1)(mN+1)\) if \( t \) even,
- \((2p+1)(mN+1) + m(N-1)\) if \( t \) odd, \( t \neq m-1 \)
- \((2p+1)(mN+1) + m(N-1) + 2\) if \( t = m-1 \).

If \( m \) odd
- \((2p+1)(mN+1)\) if \( t \neq m-1, p + t \) even,
- \((2p+1)(mN+1) + m(N-1)\) if \( t \neq m-1, p + t \) odd,
- \((2p+1)(mN+1) + 2\) if \( t = m-1, p \) even,
- \((2p+1)(mN+1) + m(N-1) + 2\) if \( t = m-1, p \) odd.

If \( m \) odd
- \((2p+1)mN + \frac{p-1}{2} + 1\) if \( t \equiv 0 \pmod{4}, t \neq m-1, p \) even,
- \((2p+1)mN + \frac{p-1}{2}\) if \( t \equiv 2 \pmod{4}, t \neq m-1, p \) even,
- \((2p+1)mN + m(N-1) + \frac{p+1}{2}\) if \( m + t \equiv 1 \pmod{4}, t \neq m-1, p \) odd,
- \((2p+1)mN + m(N-1) + \frac{p+1}{2}\) if \( m + t \equiv 3 \pmod{4}, t \neq m-1, p \) odd,
- \((2p+1)mN + m(N-1) + \frac{p+1}{2}\) if \( t \equiv 1 \pmod{4}, p \) even,
- \((2p+1)mN + m(N-1) + \frac{p+1}{2}\) if \( t \equiv 3 \pmod{4}, p \) even,
- \((2p+1)mN + m(N-1) + \frac{p-1}{2}\) if \( t \equiv m \pmod{4}, p \) odd,
- \((2p+1)mN + \frac{p+1}{2}\) if \( t \equiv m + 2 \pmod{4}, p \) odd,
- \((2p+1)mN + \frac{p+1}{2} + 3\) if \( t = m-1 \equiv 0 \pmod{4}, p \) even,
- \((2p+1)mN + \frac{p+1}{2} + 2\) if \( t = m-1 \equiv 2 \pmod{4}, p \) even,
- \((2p+1)mN + m(N-1) + \frac{p+1}{2}\) if \( t = m-1, p \) odd.
Proposition 2.2. If char $K \mid N$, the dimension of $\text{Ker} \partial^{p^{m+t+1}}$ is given by the following.

If $m$ even
\[
\begin{align*}
(2p+1)(mN+1) & \quad \text{if } t \text{ even}, \\
(2p+2)mN+2p & \quad \text{if } t \text{ odd, } t \neq m-1 \\
(2p+2)(mN+1) & \quad \text{if } t = m-1.
\end{align*}
\]

If $m$ odd and char $K = 2$
\[
\begin{align*}
(2p+1)(mN+1) & \quad \text{if } t \neq m-1, \ p+t \text{ even}, \\
(2p+2)mN+2p & \quad \text{if } t \neq m-1, \ p+t \text{ odd,} \\
(2p+1)(mN+1)+2 & \quad \text{if } t = m-1, \ p \text{ even,} \\
(2p+2)(mN+1) & \quad \text{if } t = m-1, \ p \text{ odd.}
\end{align*}
\]

If $m$ odd and char $K \neq 2$
\[
\begin{align*}
(2p+1)mN+\frac{p}{2}+1 & \quad \text{if } t \equiv 0 \pmod{4}, \ p \text{ even,} \\
(2p+1)mN+\frac{p}{2} & \quad \text{if } t \equiv 2 \pmod{4}, \ p \text{ even,} \\
(2p+2)mN+\frac{p-1}{2} & \quad \text{if } t \text{ even, } p \equiv 1 \pmod{4} \\
(2p+2)mN+\frac{p+1}{2} & \quad \text{if } m+t \equiv 1 \pmod{4}, \ p \equiv 3 \pmod{4} \\
(2p+2)mN+\frac{p-1}{2} & \quad \text{if } m+t \equiv 3 \pmod{4}, \ p \equiv 3 \pmod{4} \\
(2p+2)mN+\frac{p}{2} & \quad \text{if } t \text{ odd, } p \equiv 0 \pmod{4} \\
(2p+2)mN+\frac{p}{2} & \quad \text{if } t \equiv 1 \pmod{4}, \ p \equiv 2 \pmod{4} \\
(2p+2)mN+\frac{p}{2}+1 & \quad \text{if } t \equiv 3 \pmod{4}, \ p \equiv 2 \pmod{4} \\
(2p+1)mN+\frac{p-1}{2} & \quad \text{if } t \equiv m \pmod{4}, \ p \text{ odd} \\
(2p+1)mN+\frac{p+1}{2} & \quad \text{if } t \equiv m+2 \pmod{4}, \ p \text{ odd} \\
(2p+2)mN+\frac{p-1}{2} & \quad \text{if } t = m-1, \ p \equiv 3 \pmod{4} \\
(2p+2)mN+\frac{p}{2} & \quad \text{if } t = m-1, \ p \equiv 1 \pmod{4} \\
(2p+1)mN+\frac{p}{2}+3 & \quad \text{if } t = m-1 \equiv 0 \pmod{4}, \ p \text{ even} \\
(2p+1)mN+\frac{p}{2}+2 & \quad \text{if } t = m-1 \equiv 2 \pmod{4}, \ p \text{ even.}
\end{align*}
\]

2.3. Image of $\partial^{p^{m+t}}$: $\text{Hom}(P^{p^{m+t}-1}, \Lambda_N) \rightarrow \text{Hom}(P^{p^{m+t}}, \Lambda_N)$. We explain here how to compute $\text{Im} \partial^{p^{m+t}}$ explicitly (rather than simply compute its dimension with the rank and nullity theorem) on the same cases as for the kernels, in order to be able to give a basis of cocycle representatives for each Hochschild cohomology group $HH^n(\Lambda_N)$. For an element $g \in \text{Hom}(P^{p^{m+t}-1}, \Lambda_N)$, we consider $\partial^{p^{m+t}}g$, and find the dimension of the subspace of $\text{Ker}(\partial^{p^{m+t+1}})$ spanned by all the $\partial^{p^{m+t}}g$. We shall again use the notation $\langle n \rangle$ for the representative modulo $m$ of the integer $n$, and write $\partial g$ for $\partial^{p^{m+t}}g$.

Case $m$ even and $t$, $\odd$. The map $\partial g$ is determined by $\partial g(e_i \otimes e_{i+m-1})$ for $-p \leq \alpha \leq p$ and $g$ is determined by $\partial g(e_i \otimes e_{i+\alpha m+1})$ for $-p \leq \alpha \leq p$ with $i \in \{0, \ldots, m-1\}$. Now \[
\partial g(e_i \otimes e_{i+\alpha m}) = \begin{cases} 
N \left[ (-1)^p \bar{\mu}^{0}_{(i-1,0)} + \mu^{0}_{i,0} + \lambda^{0}_{i,0} + (-1)^p \bar{\lambda}^{0}_{(i+1),1} \right] (aa)^N & \text{if } \alpha = 0 \\
\left[ (-1)^{p+\alpha} \bar{\mu}^{0}_{(i-1,0)} + \mu^{0}_{i,0} + \lambda^{0}_{i,N-1} + (-1)^{(p+\alpha)} \bar{\lambda}^{0}_{(i+1),N-1} \right] (aa)^N & \text{if } \alpha > 0 \\
\left[ (-1)^{p+\alpha} \bar{\mu}^{0}_{(i-1),N-1} + \mu^{0}_{i,N-1} + \lambda^{0}_{i,0} + (-1)^{(p+\alpha)} \bar{\lambda}^{0}_{(i+1),0} \right] (aa)^N & \text{if } \alpha < 0 \\
\end{cases}
\]

We can check that in each case, the sum or the alternate sum over $i \in \{0, \ldots, m-1\}$ of the coefficients of the $(aa)^N$ is zero (recall that $\langle m \rangle = \langle 0 \rangle$ and that $m$ is even), so for each $\alpha$ we add $m-1$ to the dimension, except when char $K \mid N$ and $\alpha = 0$, in which case, $\partial g(e_i \otimes e_i) = 0$ and this makes no contribution to $\dim \text{Im} \partial^{p^{m+t}}$. Moreover, for $\alpha \neq 0$, we can take arbitrary coefficients
for the \((aa)^{k+1}\) with \(0 \leq k \leq N - 2\), and the coefficients of the \((aa)^{k+1}\) are completely determined by those of the \((aa)^{k+1}\).

So \(\dim_K \Im \partial^{pm+t} = \begin{cases} (2p+1)(m-1) + 2pm(N-1) & \text{if } \operatorname{char} \mathbb{K} \nmid N, \\ 2p(mN-1) & \text{if } \operatorname{char} \mathbb{K} \mid N. \end{cases}\)

**Case \(m\) even and \(t\) odd, \(t \neq m - 1\).**

The element \(\partial g\) is determined by \(\{\partial g(e_i \otimes e_{i+\beta m-1})\} \) for \(-p \leq \beta \leq p\) \(\text{and } g\) is determined by \(g(e_i \otimes e_{i+\gamma m+1})\) for \(-p \leq \alpha' \leq p\). Now

\[
\begin{aligned}
\partial g(e_i \otimes e_{i+\beta m-1}) &= \begin{cases} (-1)^{(p+\beta+1)} \sum \sigma_{i,0} + \sigma_{i,0} (\bar{aa})^{-1} \bar{a} & \text{if } \beta > 0 \\
\sum \sigma_{i,0} + \sigma_{i,0} (\bar{aa})^{-1} \bar{a} - \sum_{k=1}^{N-1} \left[ (-1)^{(p+\beta+1)} \sum \sigma_{i,k} + \sigma_{i,k} (\bar{aa})^{-1} \bar{a} \right] & \text{if } \beta \leq 0 \\
n & \text{if } \beta < 0. \end{cases} \\
\end{aligned}
\]

For \(0 < \alpha' \leq p\) \(\text{resp. } -p \leq \alpha' < 0\), the coefficient of \((\bar{aa})^{N-1} \bar{a}\) (resp. \(\bar{a}\)) in \(\partial g(e_i \otimes e_{i+\alpha'm-1})\) is equal, up to sign, to the coefficient of \(a\) (resp. \(a(\bar{aa})^{N-1}\)) in \(\partial g(e_i \otimes e_{i+\alpha'm+1})\). The coefficient of \(\bar{a}\) in \(\partial g(e_i \otimes e_{i+1})\) (that is, when \(\alpha' = 0\)) is equal, up to sign, to that of \(a\) in \(\partial g(e_i \otimes e_{i+1})\). Moreover, either the sum or the alternate sum over \(i \in \{0, \ldots, m-1\}\) of these coefficients is zero (depending on the parity of \(p + \alpha' + 1\)). They contribute \(m-1\) to the dimension of \(\Im \partial^{pm+t}\) for each \(\alpha'\). The remaining coefficients of the \((\bar{aa})^{k} \bar{a}\) and \(a(\bar{aa})^{k}\), for \(k = 1, \ldots, N-1\), are arbitrary except for the case \(\gamma = 0\), where the coefficient of \(a(\bar{aa})^{k}\) is equal, up to sign, to that of \((\bar{aa})^{k} \bar{a}\) in \(\partial g(e_i \otimes e_{i+1})\) (for \(\beta = 0\)).

So \(\dim_K \Im \partial^{pm+t} = (2p+1)(m-1) + (2p+1)m(N-1)\).

### 2.4. Dimension of \(\text{HH}^0(\Lambda_N)\)

We will now give the dimension of each of the Hochschild cohomology groups of \(\Lambda_N\). Note that the dimensions can be verified using Propositions 2.1 and 2.2, together with the rank and nullity theorem in the form \(\dim \Im (\partial^{pm+t}) = \dim \text{Hom}(P^{pm+t-1}, \Lambda_N) - \dim \text{Ker}(\partial^{pm+t})\).

**Proposition 2.3.** If \(m\) is even, or if \(m\) is odd and \(\operatorname{char} \mathbb{K} = 2\), then \(\dim_K \text{HH}^{0}(\Lambda_N) = Nm + 1\) and, for \(pm + t \neq 0\),

\[
\dim_K \text{HH}^{pm+t}(\Lambda_N) = \begin{cases} 4p + 2 + m(N-1) & \text{if } t \neq m - 1, \text{ char } \mathbb{K} \nmid N, \\
4p + 4 + m(N-1) & \text{if } t = m - 1, \text{ char } \mathbb{K} \nmid N, \\
4p + 1 + mN & \text{if } t \neq m - 1, \text{ char } \mathbb{K} \mid N, \\
4p + 3 + mN & \text{if } t = m - 1, \text{ char } \mathbb{K} \mid N. \end{cases}
\]
Proposition 2.4. If $m$ is odd, $\text{char } \mathbb{K} \neq 2$ and $\text{char } \mathbb{K} \nmid N$, then $\dim K \text{HH}^0(\Lambda_N) = Nm + 1$ and, for $pm + t \neq 0$,

$$\dim K \text{HH}^{pm+t}(\Lambda_N) = m(N-1) + p + \begin{cases} 
1 & \text{if } t = 0, \ p \text{ even}, \\
3 & \text{if } t = 0, \ p \text{ odd}, \frac{m-1}{2} \text{ even}, \\
1 & \text{if } t = 0, \ p \text{ odd}, \frac{m-1}{2} \text{ odd}, \\
1 & \text{if } t \text{ even, } t \neq 0, \ m - 1, \ p \text{ even}, \\
-1 & \text{if } m + t \equiv 3 \pmod{4} \text{ even, } t \neq 0, \ m - 1, \ p \text{ odd}, \\
1 & \text{if } m + t \equiv 1 \pmod{4} \text{ even, } t \neq 0, \ m - 1, \ p \text{ odd}, \\
2 & \text{if } t \equiv 1 \pmod{4}, \ p \text{ even}, \\
0 & \text{if } t \equiv 3 \pmod{4}, \ p \text{ even}, \\
0 & \text{if } t \text{ odd, } p \text{ odd,} \\
3 & \text{if } t = m - 1, \ p \text{ even}, \\
1 & \text{if } t = m - 1, \ p \text{ odd}.
\end{cases}$$

Proposition 2.5. If $m$ is odd, $\text{char } \mathbb{K} \neq 2$ and $\text{char } \mathbb{K} \mid N$, then $\dim K \text{HH}^0(\Lambda_N) = Nm + 1$ and, for $pm + t \neq 0$,

$$\dim K \text{HH}^{pm+t}(\Lambda_N) = mN + p + \begin{cases} 
1 & \text{if } t = 0, \ p \equiv 0, 1 \pmod{4} \\
0 & \text{if } t = 0, \ p \equiv 2 \pmod{4} \\
3 & \text{if } t = 0, \ p \equiv 3 \pmod{4}, \ m \equiv 1 \pmod{4} \\
0 & \text{if } t = 0, \ p \equiv 3 \pmod{4}, \ m \equiv 3 \pmod{4} \\
1 & \text{if } t \text{ even, } t \neq 0, \ m - 1, \ p + t \equiv 0 \pmod{4} \\
0 & \text{if } t \text{ even, } t \neq 0, \ m - 1, \ p + t \equiv 2 \pmod{4} \\
0 & \text{if } m + t \equiv 1 \pmod{4}, \ t \neq 0, \ m - 1, \ p \equiv 1 \pmod{4}, \\
-1 & \text{if } m + t \equiv 3 \pmod{4}, \ t \neq 0, \ m - 1, \ p \equiv 1 \pmod{4} \\
1 & \text{if } m + t \equiv 1 \pmod{4}, \ t \neq 0, \ m - 1, \ p \equiv 3 \pmod{4} \\
-2 & \text{if } m + t \equiv 3 \pmod{4}, \ t \neq 0, \ m - 1, \ p \equiv 3 \pmod{4} \\
1 & \text{if } t \equiv 1 \pmod{4}, \ p \equiv 0 \pmod{4} \\
0 & \text{if } t \equiv 3 \pmod{4}, \ p \equiv 0 \pmod{4} \\
2 & \text{if } t \equiv 1 \pmod{4}, \ p \equiv 2 \pmod{4} \\
-1 & \text{if } t \equiv 3 \pmod{4}, \ p \equiv 2 \pmod{4} \\
-1 & \text{if } t \text{ odd, } p + m - t \equiv 1 \pmod{4} \\
0 & \text{if } t \text{ odd, } p + m - t \equiv 3 \pmod{4} \\
3 & \text{if } t = m - 1, \ p + m \equiv 1 \pmod{4} \\
2 & \text{if } t = m - 1, \ p + m \equiv 3 \pmod{4} \\
0 & \text{if } t = m - 1, \ p \equiv 1 \pmod{4} \\
1 & \text{if } t = m - 1, \ p \equiv 3 \pmod{4}.
\end{cases}$$

3. The Centre of the Algebra $\Lambda_N$

We start by describing $Z(\Lambda_N)$, the centre of $\Lambda_N$, since it is well known that $\text{HH}^0(\Lambda_N) = Z(\Lambda_N)$. Note that the next result requires $m \geq 2$; the case $m = 1$ is dealt with separately in Theorem 7.1.
Theorem 3.1. Suppose that \( N \geq 1, m \geq 2 \). Then \( \dim \text{HH}^0(\Lambda_N) = Nm + 1 \) and the set
\[
\{1,(a_i\bar{a})^N,[(a_i\bar{a})^s + (\bar{a}a_i)^s] \text{ for } i = 0,1,\ldots,m - 1 \text{ and } s = 1,\ldots,N - 1 \}
\]
is a \( \mathbb{K} \)-basis of \( \text{HH}^0(\Lambda_N) \).

Moreover, \( \text{HH}^0(\Lambda_N) \) is generated as an algebra by the set
\[
\{1, (a_i\bar{a})^N, [a_i\bar{a} + \bar{a}a_i] \text{ for } i = 0,1,\ldots,m - 1 \}
\]
if \( N = 1 \);
\[
\{1, (a_i\bar{a})^N, [a_i\bar{a} + \bar{a}a_i] \text{ for } i = 0,1,\ldots,m - 1 \}
\]
if \( N > 1 \).

The proof is straightforward since, for \( N > 1 \), we have \([(a_i\bar{a})^s + (\bar{a}a_i)^s] = [(a_i\bar{a} + \bar{a}a_i)]^s \).

4. The Hochschild cohomology ring \( \text{HH}^*(\Lambda_N) \) for \( m \geq 3 \) and \( m \) even.

In this section, we assume that \( N \geq 1, m \geq 3 \) and \( m \) even (so that, necessarily, \( m \geq 4 \)) and give all the details of \( \text{HH}^*(\Lambda_N) \). In the subsequent sections we will consider the case \( N \geq 1, m \geq 3 \) and \( m \) odd, and leave many of the details to the reader since the computations are similar.

4.1. Basis of \( \text{HH}^*(\Lambda_N) \). For each \( n \geq 1 \), we describe the elements of a basis of the Hochschild cohomology group \( \text{HH}^*(\Lambda_N) \) in terms of cocycles in \( \text{Hom}(P^n, \Lambda_N) \), that is, we give a set of elements in \( \ker \partial^{n+1} \), each of which is written as a map in \( \text{Hom}(P^n, \Lambda_N) \), such that the corresponding set of cosets in \( \ker \partial^{n+1} / \text{im} \partial^n \) with these representatives forms a basis of \( \ker \partial^{n+1} / \text{im} \partial^n = \text{HH}^n(\Lambda_N) \).

Following standard usage, our notation does not distinguish between an element of \( \text{HH}^n(\Lambda_N) \) and a cocycle in \( \text{Hom}(P^n, \Lambda_N) \) which represents that element of \( \text{HH}^n(\Lambda_N) \). However the precise meaning is always clear from the context. For each cocycle in \( \text{Hom}(P^n, \Lambda_N) \), we simply write the images of the generators \( e_i \otimes e_{i+n-2r} \) in \( P^n \) which are non-zero. We keep to these conventions throughout the whole paper.

Proposition 4.1. Suppose that \( N \geq 1, m \geq 3 \) and \( m \) is even. For each \( n \geq 1 \), the following elements define a basis of \( \text{HH}^n(\Lambda_N) \).

(1) For \( n \) even, \( n \geq 2 \):

(a) For \(-p \leq \alpha \leq p\):
\[
\chi_{n,\alpha} : e_i \otimes e_{i + \alpha m} \mapsto (-1)^{(\frac{n-\alpha}{2})} e_i \quad \text{for } i = 0,1,\ldots,m - 1;
\]
(b) For \(-p \leq \alpha < p\):
\[
\tau_{n,\alpha} : e_0 \otimes e_{\alpha m} \mapsto (a_\alpha a_\bar{a})^N;
\]
(c) For each \( j = 0,1,\ldots,m - 2 \) and each \( s = 1,\ldots,N - 1 \):
\[
F_{n,j,s} : e_j \otimes e_j \mapsto (a_j \bar{a}_j)^s;
\]
\[
e_{j+1} \otimes e_{j+1} \mapsto (-1)^{\frac{n}{2}} (\bar{a}_j a_j)^s;
\]
(d) For each \( s = 1,\ldots,N - 1 \):
\[
F_{n,m-1,s} : e_{m-1} \otimes e_{m-1} \mapsto (a_{m-1} a_{m-1})^s;
\]
\[
e_0 \otimes e_0 \mapsto (-1)^{\frac{n}{2}} (a_0 a_0)^s;
\]
(e) Additionally in the case \( n \mid N \), for each \( j = 1,\ldots,m - 1 \): \( \theta_{n,j} : e_j \otimes e_j \mapsto (a_j \bar{a}_j)^N \).

(2) For \( n \) odd, \( n \geq 1 \):

(a) For \(-p \leq \gamma < 0 \) and, if \( t = m - 1 \), \( \gamma = -p - 1 \) also:
\[
\varphi_{n,\gamma} : e_i \otimes e_{i + \gamma m + 1} \mapsto (-1)^{(\frac{n-1-\gamma}{2})} a_i (a_\bar{a} a_i^{N-1}) \quad \text{for all } i = 0,1,\ldots,m - 1;
\]
(b) For \( 0 \leq \gamma < p \):
\[
\psi_{n,\gamma} : e_i \otimes e_{i + \gamma m + 1} \mapsto (-1)^{(\frac{n-1-\gamma}{2})} a_i \quad \text{for all } i = 0,1,\ldots,m - 1;
\]
(c) For \( 0 < \beta \leq p \) and, if \( t = m - 1 \), \( \beta = p + 1 \) also:
\[
\varphi_{n,\beta} : e_i \otimes e_{i + \beta m - 1} \mapsto (-1)^{(\frac{n-1-\beta}{2})} a_{i-1} a_{i-1}^{N-1} \quad \text{for all } i = 0,1,\ldots,m - 1;
\]
(d) For \( -p \leq \beta < 0 \):
\[
\psi_{n,\beta} : e_i \otimes e_{i + \beta m - 1} \mapsto (-1)^{(\frac{n-1-\beta}{2})} a_{i-1} \quad \text{for all } i = 0,1,\ldots,m - 1;
\]
(e) For each \( j = 0,1,\ldots,m - 1 \) and each \( s = 1,\ldots,N - 1 \):
\[
\varepsilon_{n,j,s} : e_j \otimes e_j \mapsto (a_j \bar{a}_j)^{s a_j};
\]
(f) Additionally in the case \( n \mid N \), for each \( j = 1,\ldots,m - 1 \):
\[
\omega_{n,j} : e_j \otimes e_{j+1} \mapsto a_j.
\]
4.2. Computing products in cohomology. In order to compute the products in the Hochschild cohomology ring, we require a lifting of each element in the basis of $HH^n(\Lambda_N)$, that is, for each cocycle $f \in \text{Hom}(P^n, \Lambda_N)$, we give a map $\mathcal{L}^* f : P^{*+n} \to P^*$ such that

1. the composition $P^n \xrightarrow{\mathcal{L}^* f} P^0 \xrightarrow{\partial^q} \Lambda_N$ is equal to $f$, and
2. the square $\xymatrix{P^{q+n} \ar[r]^{\mathcal{L}^* f} \ar[d]_{\partial^q} & P^q \ar[d]_{\partial^q} \\ P^{q-1+n} \ar[r]_{\mathcal{L}^* f} & P^{q-1}}$ commutes for every $q \geq 1$.

It should be noted that such liftings always exist but are not unique. It is easy to check that the maps given in this paper are indeed liftings of the appropriate cocycle.

For homogeneous elements $\eta \in HH^n(\Lambda_N)$ and $\theta \in HH^k(\Lambda_N)$ represented by cocycles $\eta : P^n \to \Lambda_N$ and $\theta : P^k \to \Lambda_N$ respectively, the cup product (or Yoneda product) $\eta \theta \in HH^{n+k}(\Lambda_N)$ is the coset represented by the composition $P^{n+k} \xrightarrow{\mathcal{L}^* \eta \theta} P^n \xrightarrow{\eta} \Lambda_N$. The fact that the cup product is well-defined means that the composition $\eta \theta$ does not depend on the choice of representative cocycles for $\eta$ and $\theta$ or on the choice of the liftings of these cocycles.

4.3. Liftings for $N = 1$. We continue to assume that $m$ is even in the details that follow.

**Proposition 4.2.** For $n \geq 1$ and the algebra $\Lambda_1$, $m \geq 3$ and $m$ even, the following maps are liftings of the cocycle basis of $HH^n(\Lambda_1)$ given in Proposition 4.1.

1. For $n$ even, $n \geq 2$, and for $-p \leq \alpha \leq p$:
   - $\mathcal{L}^q_{\chi_{n,\alpha}} : e_i \otimes e_{i+am+q-2\ell} \mapsto (-1)\left(\frac{\ell}{m}\right)_{\alpha}e_i \otimes e_{i+q-2\ell}$ for all $i = 0, 1, \ldots, m-1$ and all $0 \leq \ell \leq q$.
   - $\mathcal{L}^q_{\pi_{n,\alpha}} : e_0 \otimes e_{am+q-2\ell} \mapsto a_0a_0e_0 \otimes e_{q-2\ell}$ for all $0 \leq \ell \leq q$.

2. For $n$ odd:
   - For $-p \leq \gamma \leq p$ if $t \neq m-1$:
     - $\mathcal{L}^q_{\varphi_{n,\gamma}} : e_i \otimes e_{i+am+q+2\ell} \mapsto (-1)^{\left(\frac{\ell}{m}\right)_{\gamma}}e_i \otimes e_{i+q+2\ell} a_iq-2\ell$ for all $i = 0, 1, \ldots, m-1$ and all $0 \leq \ell \leq q$.
   - For $-p \leq \beta \leq p$ if $t \neq m-1$:
     - $\mathcal{L}^q_{\psi_{n,\beta}} : e_i \otimes e_{i+am+q-2\ell} \mapsto (-1)^{\left(\frac{\ell}{m}\right)_{\beta}}e_i \otimes e_{i+q-2\ell} a_iq-2\ell-1$ for all $i = 0, 1, \ldots, m-1$ and all $0 \leq \ell \leq q$.

4.4. Liftings for $N > 1$.

**Proposition 4.3.** For $n \geq 1$ and the algebra $\Lambda_N$ with $N > 1$, $m \geq 3$ and $m$ even, the following maps give a lifting of the cocycles in Proposition 4.1 above. Once again, to simplify cases and for consistency with our conventions, we use the notation $\langle n \rangle \in \{0, 1, \ldots, m-1\}$ for the representative of the integer $n$ modulo $m$.

1. For $n$ even, $n \geq 2$: 
   - For $-p \leq \alpha \leq p$:
     - $\mathcal{L}^q_{\chi_{N,\alpha}} : e_i \otimes e_{i+am+q-2\ell} \mapsto (-1)^{\left(\frac{\ell}{m}\right)_{\alpha}}e_i \otimes e_{i+q-2\ell}$ for all $i = 0, 1, \ldots, m-1$ and all $0 \leq \ell \leq q$.
For \(-p \leq \alpha \leq p:\)

\[
\mathcal{L}^q_{\lambda,n,\alpha}(e_i \otimes e_{i+q-2\ell+n}) = \begin{cases}
(-1)^{\frac{n-q-2\ell}{2}} e_i \otimes e_{i+q-2\ell} & \text{if } (q-2\ell)\alpha \geq 0 \\
0 & \text{if } (q-2\ell)\alpha < 0 \text{ and } |q-2\ell| > 2 \\
(-1)^{\frac{n-m-1}{2}} (a^2)^{N-1} e_i \otimes e_{i+2} (a^2)^{N-1} & \text{if } \alpha < 0 \text{ and } q-2\ell = 2 \\
(-1)^{\frac{n-m-1}{2}} (a^2)^{N-1} e_i \otimes e_{i-1} (a^2)^{N-1} & \text{if } \alpha > 0 \text{ and } q-2\ell = -2 \\
(-1)^{\frac{n-m-1}{2}} \sum_{k=0}^{N-2} (a^2)^k e_i \otimes e_{i+k} (a^2)^{N-k-1} & \text{if } \alpha < 0 \text{ and } q-2\ell = 1 \\
(-1)^{\frac{n-m-1}{2}} \sum_{k=0}^{N-2} (a^2)^k e_i \otimes e_{i-1} (a^2)^{N-k-1} & \text{if } \alpha > 0 \text{ and } q-2\ell = -1 \\
\end{cases}
\]

for all \(i = 0, 1, \ldots, m-1\) and all \(0 \leq \ell \leq q\).

For \(-p \leq \alpha \leq p:\)

\[
\mathcal{L}^q_{\pi,n,\alpha}(e_0 \otimes e_{q-2\ell+n}) = \begin{cases}
(a^2)^{N} e_0 \otimes e_{q-2\ell} & \text{if } \alpha(q-2\ell) \geq 0 \\
(a^2)^{N} e_0 \otimes e_1 (a^2)^{N-1} & \text{if } \alpha(q-2\ell) = 1 \text{ and } \alpha < 0 \\
(a^2)^{N} e_0 \otimes e_{-1}(a^2)^{N-1} & \text{if } \alpha(q-2\ell) = -1 \text{ and } \alpha > 0 \\
0 & \text{otherwise}
\end{cases}
\]

for all \(0 \leq \ell \leq q\).

For each \(j = 0, 1, \ldots, m-2\) and each \(s = 1, \ldots, N-1:\)

\[
\mathcal{L}^q_{F_{n,j,s}} : e_j \otimes e_{j+q-2\ell} \mapsto (a_j \bar{a}_j)^s e_j \otimes e_{j+q-2\ell}
\]

for all \(0 \leq \ell \leq q\).

For each \(s = 1, \ldots, N-1:\)

\[
\mathcal{L}^q_{F_{n,m-1,s}} : e_0 \otimes e_{q-2\ell} \mapsto (-1)^{q} (\bar{a}_m \bar{a}_{m-1})^s e_0 \otimes e_{q-2\ell}
\]

for all \(0 \leq \ell \leq q\).

In the case \(\text{char } k \mid N\), for each \(j = 1, \ldots, m-1:\)

\[
\mathcal{L}^q_{\theta_{n,j}} : e_j \otimes e_{j+q-2\ell} \mapsto (a_j \bar{a}_j)^N e_j \otimes e_{j+q-2\ell}
\]

for all \(0 \leq \ell \leq q\).

(2) For \(n\) odd:

For \(-p \leq \gamma < 0\) and, if \(t = m - 1, \gamma = -p - 1\) also:

\[
\mathcal{L}^q_{\varphi,n,\gamma}(e_i \otimes e_{i+\gamma m+1+q-2\ell}) = \begin{cases}
0 & \text{if } q-2\ell > 1 \\
(-1)^{\frac{n-q-2\ell}{2}} (a^2)^{N-1} e_i \otimes e_{i+1} (a^2)^{N-1} & \text{if } q-2\ell = 1 \\
(-1)^{q} (a^2)^{N-1} e_i \otimes e_{i+q-2\alpha} (a^2)^{N-1} & \text{if } q-2\ell \leq 0
\end{cases}
\]

for all \(i = 0, 1, \ldots, m-1\) and all \(0 \leq \ell \leq q\).

For \(0 \leq \gamma \leq p:\)

\[
\mathcal{L}^q_{\varphi,n,\gamma}(e_i \otimes e_{i+\gamma m+1+q-2\ell}) = \begin{cases}
(-1)^{q} (a^2)^{N-1} e_i \otimes e_{i+q-2\alpha} & \text{if } q-2\ell > 0 \\
(-1)^{\frac{n-q-2\ell}{2}} \sum_{k=0}^{N-1} (a^2)^k e_i \otimes e_{i-1} (a^2)^{N-k-1} a + \\
(-1)^{q} \sum_{k=0}^{N-2} (a^2)^k e_i \otimes e_{i-1} (a^2)^{N-k-1} a & \text{if } q-2\ell = -1 \\
(-1)^{\frac{n-q-2\ell}{2}} (a^2)^{N-1} e_i \otimes e_{i-2} (a^2)^{N-1} a & \text{if } q-2\ell = -2 \\
0 & \text{if } q-2\ell < -2
\end{cases}
\]

for all \(i = 0, 1, \ldots, m-1\) and all \(0 \leq \ell \leq q\).
If $\gamma = 0$ then $L^q\phi_{n,0}(e_i \otimes e_{i+1+q-2\ell}) = 
\begin{cases}
(-1)^{q-\frac{n+1}{2}}e_i \otimes e_{i+q-2\ell}a \\
(-1)^{q-\frac{n+1}{2}}[e_i \otimes e_{i+q-2\ell}a - (-1)^q(N-1)e_i \otimes e_{i+q-2\ell-2}a(\bar{a})^{N-1}] \\
(-1)^{q-\frac{n+1}{2}}N\epsilon_i \otimes e_{i+q-2\ell-2}a(\bar{a})a^{N-1}
\end{cases}
$
if $q - 2\ell = q$
if $0 \leq q - 2\ell < q$
if $q - 2\ell < -1$

$$
\begin{align*}
(-1)^{q-\frac{n+1}{2}}i_{N-1,k-1} & \left\{ \sum_{k=1}^{N-1} \sum_{v=0}^{N-1} [(\bar{a}a)^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& -(-1)^{q+\frac{n+1}{2}}(\bar{a}a)^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& (-1)^{\frac{n+1}{2}}(\bar{a}a)^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& + \sum_{k=0}^{N-1} (aa)^k e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-k-1} \right\}
\end{align*}
$$
for all $i = 0, 1, \ldots, m - 1$ and all $0 \leq \ell \leq q$.

- For $0 < \beta \leq p$ and, if $t = m - 1$, $\beta = p + 1$ also:

$$
L^q\psi_{n,\beta}(e_i \otimes e_{i+\beta m-1+q-2\ell}) = 
\begin{cases}
0 \\
(-1)^{\frac{1}{2}(\beta-1)N} e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-1} \\
(-1)^{\frac{1}{2}(\beta-1)N} e_i \otimes e_{i+q-2\ell}a(\bar{a})a^{N-1}
\end{cases}
$$
if $q - 2\ell < -1$
if $q - 2\ell = -1$
if $q - 2\ell > -1$

for all $i = 0, 1, \ldots, m - 1$ and all $0 \leq \ell \leq q$.

- For $-p \leq \beta < 0$:

$$
\begin{align*}
\text{If } \beta < 0, \text{ then } L^q\psi_{n,\beta}(e_i \otimes e_{i+\beta m-1+q-2\ell}) = 
& \left\{ (-1)^{\frac{1}{2}+\frac{n+1}{2}}e_i \otimes e_{i+q-2\ell}a \\
& (-1)^{\frac{1}{2}+\frac{n+1}{2}}[e_i \otimes e_{i+q-2\ell}a - (-1)^q(N-1)e_i \otimes e_{i+q-2\ell}a(\bar{a})a^{N-1}] \\
& (-1)^{\frac{1}{2}+\frac{n+1}{2}}N\epsilon_i \otimes e_{i+q-2\ell}a(\bar{a})a^{N-1}
\end{align*}
$$
if $q - 2\ell < 1$
if $q - 2\ell = 1$
if $q - 2\ell > 1$

for all $i = 0, 1, \ldots, m - 1$ and all $0 \leq \ell \leq q$.

- If $\beta = 0$ then $L^q\psi_{n,0}(e_i \otimes e_{i+1+q-2\ell}) = 
\begin{align*}
& \left\{ (-1)^{\frac{1}{2}+\frac{n+1}{2}}e_i \otimes e_{i+q-2\ell}a \\
& (-1)^{\frac{1}{2}+\frac{n+1}{2}}[e_i \otimes e_{i+q-2\ell}a - (-1)^q(N-1)e_i \otimes e_{i+q-2\ell}a(\bar{a})a^{N-1}] \\
& (-1)^{\frac{1}{2}+\frac{n+1}{2}}N\epsilon_i \otimes e_{i+q-2\ell}a(\bar{a})a^{N-1}
\end{align*}
$$
if $q - 2\ell = -q$
if $-q < q - 2\ell \leq 0$
if $q - 2\ell > 1$

$$
\begin{align*}
(-1)^{\frac{1}{2}+\frac{n+1}{2}} & \left\{ \sum_{k=1}^{N-1} \sum_{v=0}^{N-1} [(\bar{a})a^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& (-1)^{\frac{1}{2}+\frac{n+1}{2}}(\bar{a})a^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& +(-1)^{\frac{1}{2}+\frac{n+1}{2}}(\bar{a})a^v e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-v-1} + \\
& + \sum_{k=0}^{N-1} (aa)^k e_i \otimes e_{i+1}\bar{a}(a\bar{a})^{N-k-1} \right\}
\end{align*}
$$
for all $i = 0, 1, \ldots, m - 1$ and all $0 \leq \ell \leq q$. 

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For each \( j = 0, 1, \ldots, m - 1 \) and each \( s = 1, \ldots, N - 1 \):
\[
\mathcal{L}^q \mathcal{E}_{n,j,s} = \begin{cases} 
\sum_{k=0}^{N-s} \sum_{\nu=k}^{N-s} [(aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1}] & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-s} \sum_{\nu=k}^{N-s} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} a e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-s} \sum_{\nu=k}^{N-s} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-s} \sum_{\nu=k}^{N-s} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} a e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-s} \sum_{\nu=k}^{N-s} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\end{cases}
\]

In the case \( \text{char } \mathbb{K}/ \mathcal{N} \), for each \( j = 1, \ldots, m - 1 \):
\[
\mathcal{L}^q \omega_{n,j} = \begin{cases} 
\sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} [(aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1}] & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} a e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} a e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} \left( (-1)^{\frac{\nu+1}{2}} (aa)^{s+\nu} e_j \otimes e_{j+1} a(aa)^{N-v-1} \right) & \text{if } q \text{ is odd} \\
\end{cases}
\]

4.5. The Hochschild cohomology ring \( HH^*(\Lambda_1) \) for \( N = 1, m \geq 3, m \text{ even} \).

**Theorem 4.4.** For \( N = 1, m \geq 3 \) and \( m \) even, \( HH^*(\Lambda_1) \) is a finitely generated algebra with generators:

\[
\begin{align*}
1, a, \bar{a}, & \text{ for } i = 0, 1, \ldots, m - 1 \text{ in degree } 0, \\
\varphi_{1,0}, & \text{ in degree } 1, \\
\chi_{2,0}, & \text{ in degree } 2, \\
\varphi_{m-1,1}, & \text{ in degree } m - 1, \\
\chi_{m,1}, & \text{ in degree } m.
\end{align*}
\]

It is known from [21, Proposition 4.4] that those generators in \( HH^*(\Lambda_1) \) whose image is in \( \tau \) are nilpotent, and it may be seen that the remaining generators of \( HH^*(\Lambda_1) \), namely \( \chi_{2,0}, \chi_{m,1} \) and \( \chi_{m,1} \), are not nilpotent. Moreover \( \chi_{2,0}^m = \chi_{m,1} \chi_{m,1} \). Thus we have the following corollary.

**Corollary 4.5.** For \( N = 1, m \geq 3 \) and \( m \) even,

\[
HH^*(\Lambda_1)/\mathcal{N} \cong \mathbb{K}[\chi_{2,0}, \chi_{m,1}, \chi_{m,1}]/(\chi_{2,0}^m - \chi_{m,1} \chi_{m,1})
\]

and hence \( HH^*(\Lambda_1)/\mathcal{N} \) is a commutative finitely generated algebra of Krull dimension 2.
4.6. The Hochschild cohomology ring $\text{HH}^*(\Lambda_N)$ for $N > 1$, $m \geq 3$, $m$ even. In this section we describe the main arguments behind the presentation of $\text{HH}^*(\Lambda_N)$ by generators and relations. The first lemma indicates the method required to show that $\text{HH}^*(\Lambda_N)$ is a finitely generated algebra whilst the second lemma considers the relations between the generators.

**Lemma 4.6.** Suppose $N > 1$, $m \geq 3$ and $m$ even. Then

(1) $\chi_{2,0}\chi_{2,0} = \chi_{4,0}$;

(2) $\pi_{2,0} = (a_0a_0)^{N}\chi_{2,0}$.

**Proof** (1) By definition, $\chi_{2,0}\chi_{2,0} = \chi_{2,0} \cdot \ell^2\chi_{2,0}$. Now, from Proposition 4.3, $\ell^2\chi_{2,0}(e_i \otimes e_{i+2-2\ell}) = (-1)^i e_i \otimes e_{i+2-2\ell}$ for $i = 0, \ldots, m - 1$ and $0 \leq \ell \leq 2$, and $\chi_{2,0}(e_i \otimes e_i) = (-1)^i e_i$ for $i = 0, \ldots, m - 1$. Thus

$$
\chi_{2,0} \cdot \ell^2\chi_{2,0}(e_i \otimes e_{i+2-2\ell}) = \begin{cases} (-1)^i(-1)^i e_i = e_i & \text{if } \ell = 1, \\ 0 & \text{otherwise,} \end{cases}
$$

for all $i = 0, \ldots, m - 1$. Hence $\chi_{2,0}\chi_{2,0} = \chi_{4,0}$.

(2) We know that $\chi_{2,0}(e_i \otimes e_i) = (-1)^i e_i$ for $i = 0, \ldots, m - 1$, $\pi_{2,0} : e_0 \otimes e_0 \mapsto (a_0a_0)^{N}$, and $(a_0a_0)^{N} \in \text{HH}^0(\Lambda_N)$. Then

$$
((a_0a_0)^{N}\chi_{2,0})(e_i \otimes e_i) = \chi_{2,0}((a_0a_0)^{N} e_i \otimes e_i) = \begin{cases} (-1)^0e_0(a_0a_0)^{N} & \text{if } i = 0, \\ 0 & \text{if } i = 1, \ldots, m - 1. \end{cases}
$$

Thus $\pi_{2,0} = (a_0a_0)^{N}\chi_{2,0}$. \hfill \Box

We remark that, for even $n$, $2 \leq n < m$ and writing $n = pm + t$ as usual, we necessarily have $p = 0$, and thus, from Proposition 4.1, we have $\{\chi_{n,0} \mid -p \leq \alpha \leq p\} = \{\chi_{n,0}\}$. Inductively, Lemma 4.6 (1) gives, for $n = 2k$ even, $2 \leq n \leq m - 2$, that $\chi_{n,0} = (\chi_{2,0})^{k}$.

**Lemma 4.7.** Suppose $N > 1$, $m \geq 3$ and $m$ even. Then

(1) $\chi_{m,1}\chi_{m,-1} = 0$;

(2) $\varphi_{1,0}^2 = 0$;

(3) $\varphi_{1,0}\psi_{1,0} = Nm(a_0a_0)^{N}\chi_{2,0}$;

(4) for char $K \mid N$, $S = \sum_{k=1}^{N} k$ and $e_i = (a_i\bar{a}_i)^{N}$ for $i = 0, 1, \ldots, m - 1$, we have

$$
\varphi_{1,0}\omega_{1,i} = \begin{cases} S\chi_{2,0}(e_i + e_{i+1}) & \text{if char } K = 2 \text{ and } i \neq m - 1, \\ S\chi_{2,0}(e_{m-1} + e_0) & \text{if char } K = 2 \text{ and } i = m - 1, \\ 0 & \text{if char } K \neq 2. \end{cases}
$$

**Proof** (1) We have $\chi_{m,1}\chi_{m,-1} = \chi_{m,1} \cdot \ell^m\chi_{m,-1}$. Since $m$ is even, Proposition 4.3 gives

$$
\ell^m\chi_{m,-1}(e_i \otimes e_{i+m-2\ell-m}) = \begin{cases} e_i \otimes e_{i+m-2\ell} & \text{if } m - 2\ell \leq 0, \\ 0 & \text{if } m - 2\ell > 0 \text{ and } |m - 2\ell| > 2, \\ (a\bar{a})^{N-1}e_i \otimes e_{i+2}\bar{(a\bar{a})}^{N-1} & \text{if } m - 2\ell = 2. \end{cases}
$$

for all $i = 0, 1, \ldots, m - 1$ and all $0 \leq \ell \leq m$. Also $\chi_{m,1}(e_i \otimes e_{i+m}) = e_i$ for $i = 0, 1, \ldots, m - 1$. Hence $\chi_{m,1} \cdot \ell^m\chi_{m,-1}(e_i \otimes e_{i+m-2\ell-m}) = 0$ and $\chi_{m,1}\chi_{m,-1} = 0$.

(2) If char $K \neq 2$, then $\varphi_{1,0}^2 = 0$ by graded commutativity of $\text{HH}^*(\Lambda_N)$ so we may assume char $K = 2$. Thus we have $\varphi_{1,0}(e_i \otimes e_{i+1}) = a_i$ for $i = 0, 1, \ldots, m - 1$, and, from Proposition 4.3,
\[ L^1 \varphi_{1,0}(e_i \otimes e_{i+1+1-2\ell}) = \begin{cases} 
 e_i \otimes e_{i+1}a_{i+1} & \text{if } \ell = 0, \\
 - \sum_{k=1}^{N-1} \sum_{v=0}^{k-1} [(aa)^v e_i \otimes e_{i+1} \bar{a}(aa)^{N-v-1} - (aa)^v ae_{(i+1) \otimes e_{(i+1)-1}}(aa)^{N-v-1}] & \text{if } \ell = 1, \\
 + \sum_{k=0}^{N-1} (aa)^k e_i \otimes e_{i-1} \bar{a}(aa)^{N-k-1} & \end{cases} \]

for all \( i = 0, 1, \ldots, m - 1 \). Hence

\[ \varphi_{1,0} \cdot L^1 \varphi_{1,0}(e_i \otimes e_{i+1+1-2\ell}) = \begin{cases} 
 a_i a_{i+1} - \sum_{k=1}^{N-1} \sum_{v=0}^{k-1} [(aa)^v e_i \otimes e_{i-1} \bar{a}(aa)^{N-v-1} - (aa)^v ae_{(i-1) \otimes e_{(i-1)-1}}(aa)^{N-v-1}] & \text{if } \ell = 0, \\
 - \sum_{k=1}^{N-1} \sum_{v=0}^{k-1} [(aa)^v e_i \otimes e_{i+1} \bar{a}(aa)^{N-v-1} - (aa)^v ae_{(i+1) \otimes e_{(i+1)-1}}(aa)^{N-v-1}] & \text{if } \ell = 1, \\
 + \sum_{k=0}^{N-1} (aa)^k e_i \otimes e_{i-1} \bar{a}(aa)^{N-k-1} & \end{cases} \]

for \( i = 0, 1, \ldots, m - 1 \). Thus \( \varphi_{1,0} \cdot L^1 \varphi_{1,0}(e_i \otimes e_{i+1+1-2\ell}) = 0 \) if \( \ell = 1 \). On the other hand, if \( \ell = 0 \), then

\[ \varphi_{1,0} \cdot L^1 \varphi_{1,0}(e_i \otimes e_i) = \sum_{k=0}^{N-1} \sum_{v=0}^{k-1} [(aa)^v a_i a_{i-1} \bar{a}(aa)^{N-v-1} + (aa)^v a_i \bar{a}(aa)^{N-v-1}] + \sum_{k=0}^{N-1} (aa)^k a_i \bar{a}(aa)^{N-k-1} = \sum_{k=0}^{N-1} (a_i \bar{a})^k N = N(a_i \bar{a})^N. \]

Now define \( g \in \text{Hom}(P^1, \Lambda_N) \) by \( e_i \otimes e_{i+1} \mapsto (m-1-i)a_i \) for \( i = 0, 1, \ldots, m - 2 \) (with our usual convention that all other summands are mapped to zero). Then \( \partial^2 g \in \text{Hom}(P^2, \Lambda_N) \) and, for \( i = 1, \ldots, m - 2 \), we have

\[ \partial^2 g(e_i \otimes e_i) = \sum_{k=0}^{N-1} \sum_{v=0}^{k-1} [(aa)^v a_i a_{i-1} \bar{a}(aa)^{N-v-1} + (aa)^v a_i \bar{a}(aa)^{N-v-1}] + \sum_{k=0}^{N-1} (aa)^k a_i \bar{a}(aa)^{N-k-1} = -N(a_i \bar{a})^N. \]

In a similar way, we also have \( \partial^2 g(e_{m-1} \otimes e_{m-1}) = -N(a_{m-1} \bar{a}_{m-1})^N \). For \( i = 0 \) we have

\[ \partial^2 g(e_0 \otimes e_0) = \sum_{k=0}^{N-1} \sum_{v=0}^{k-1} [(aa)^v a_0 a_{m-1} \bar{a}(aa)^{N-v-1} + (aa)^v a_0 \bar{a}(aa)^{N-v-1}] + \sum_{k=0}^{N-1} (aa)^k (m-1)a_0 \bar{a}(aa)^{N-k-1} = N(m-1)(a_0 \bar{a}_0)^N. \]
Moreover, $\partial^2 g$ is zero on all other summands. Hence

$$\partial^2 g(e_i \otimes e_i) = \begin{cases} N(m-1)(a_0a_0)^N & \text{if } i = 0, \\ -N(a_i \bar{a}_i)^N & \text{if } i = 1, \ldots, m-1. \end{cases}$$

So $\varphi_{1,0} \cdot \mathcal{L}^1 \psi_{1,0} + \partial^2 g$ is the map given by $e_0 \otimes e_0 \mapsto Nm(a_0a_0)^N$ and thus $\varphi_{1,0} \cdot \mathcal{L}^1 \psi_{1,0} + \partial^2 g = Nm\pi_{2,0}$. Using Lemma 4.6(2), we have that the cocycles $\varphi_{1,0} \psi_{1,0}$ and $Nm(a_0a_0)^N \chi_{2,0}$ represent the same elements in Hochschild cohomology so that $\varphi_{1,0} \psi_{1,0} = Nm(a_0a_0)^N \chi_{2,0}$ as required.

(4) Finally, suppose $\text{char } \mathbb{k} \mid N, S = \sum_{k=1}^N k$ and let $e_i = (a_i \bar{a}_i)^N$ for $i = 0, 1, \ldots, m-1$. For ease of notation we consider the product $\varphi_{1,0} \omega_{1,i}$, where $1 \leq i \leq m-2$; the cases $i = 0$ and $i = m-1$ are similar. We know $\varphi_{1,0} (e_j \otimes e_{j+1}) = a_j$ for $j = 0, 1, \ldots, m-1$ and, from Proposition 4.3, we have

$$\mathcal{L}^1 \omega_{1,i} : \begin{cases} e_i \otimes e_i \mapsto \sum_{k=1}^{N-1} (\bar{a}a)^{v} e_{i+1} \otimes e_i (a\bar{a})^{N-v-1} - \sum_{k=1}^{N-1} (a\bar{a})^{v} e_{i} \otimes e_{i+1} (a\bar{a})^{N-v-1} \\ e_{i+1} \otimes e_{i+1} \mapsto -\sum_{k=1}^{N-1} (\bar{a}a)^{v} e_{i+1} \otimes e_i (a\bar{a})^{N-v-2} - \sum_{k=1}^{N-1} (a\bar{a})^{v} e_{i} \otimes e_{i+1} (a\bar{a})^{N-v-1} \\ e_{i-1} \otimes e_{i+1} \mapsto ae_i \otimes e_{i+1}. \end{cases}$$

Hence $\varphi_{1,0} \cdot \mathcal{L}^1 \omega_{1,i}(e_i \otimes e_i) = \sum_{k=1}^{N-1} (\bar{a}a)^{v} a\bar{a} (a\bar{a})^{N-v-1} - \sum_{k=1}^{N-1} (a\bar{a})^{v} a\bar{a} (a\bar{a})^{N-v-1} = -k(a_i \bar{a}_i)^N = -S(a_i \bar{a}_i)^N$. And

$$\varphi_{1,0} \cdot \mathcal{L}^1 \omega_{1,i}(e_{i+1} \otimes e_{i+1}) = (\bar{a}a)^{v} e_{i+1} \otimes e_i (a\bar{a})^{N-v-2} - (a\bar{a})^{v} e_{i} \otimes e_{i+1} (a\bar{a})^{N-v-1} = \sum_{k=1}^{N-1} k(a_i \bar{a}_i)^N = S(a_i \bar{a}_i)^N.$$

Finally, $\varphi_{1,0} \cdot \mathcal{L}^1 \omega_{1,i}(e_{i-1} \otimes e_{i+1}) = 0$. Hence $\varphi_{1,0} \cdot \mathcal{L}^1 \omega_{1,i} : \begin{cases} e_i \otimes e_i \mapsto -S(a_i \bar{a}_i)^N \\ e_{i+1} \otimes e_{i+1} \mapsto S(a_i \bar{a}_i)^N. \end{cases}$

Now, $\chi_{2,0}(e_i + e_{i+1}) = \begin{cases} e_i \otimes e_i \mapsto (-1)^i (a_i \bar{a}_i)^N \\ e_{i+1} \otimes e_{i+1} \mapsto (-1)^{i+1} (a_i \bar{a}_{i+1} + a_{i+1} \bar{a}_i)^N. \end{cases}$ If $\text{char } \mathbb{k} = 2$, then $\varphi_{1,0} \omega_{1,i} = \varphi_{1,0} \cdot \mathcal{L}^1 \omega_{1,i} = S\chi_{2,0}(e_i + e_{i+1})$. However, if $\text{char } \mathbb{k} \neq 2$ then, since $\text{char } \mathbb{k} \mid N$, we have $2S = (N-1)N = 0$ so that $S = 0$ and hence $\varphi_{1,0} \omega_{1,i} = 0$. \hfill \Box

By considering all possible products of the basis elements of $\text{HH}^*(\Lambda_N)$ from Proposition 4.1, we are now able to give the main result of this section. We do not explicitly list the relations which are a direct consequence of the graded commutativity of $\text{HH}^*(\Lambda_N)$. Note that we can check that we do indeed have all the relations by comparing $\text{dim } \text{HH}^0(\Lambda_N)$ from Proposition 2.3 with the dimension in degree $n$ of the algebra given in Theorem 4.8 below.

**Theorem 4.8.** Suppose $N > 1$, $m \geq 3$ and $m$ even.

1. $\text{HH}^*(\Lambda_N)$ is a finitely generated algebra with generators:

   \begin{itemize}
   \item $1, (a_i \bar{a}_i)^N$ for $i = 0, 1, \ldots, m-1$ in degree 0
   \item $\varphi_{1,0}, \psi_{1,0}, \omega_{1,j}$ for $j = 1, \ldots, m-1$ in degree 1
   \item $\chi_{2,0}$ in degree 2
   \item $\varphi_{m-1,-1}, \psi_{m-1,1}$ in degree $m-1$
   \item $\chi_{m,1}, \chi_{m,-1}$ in degree $m$.\end{itemize}

2. Let $e_i = (a_i \bar{a}_i)^N$ for $i = 0, 1, \ldots, m-1$ and $S = \sum_{k=1}^N k$. Then $\text{HH}^*(\Lambda_N)$ is a finitely generated algebra over $\text{HH}^0(\Lambda_N)$ with generators:
and hence

5.1. Proposition 5.1. In this section, $N \geq 1$, $m \geq 3$ and $m$ is odd. Then the action of $\text{HH}^0(\text{A}_N)$ on the generators of $\text{HH}^n(\text{A}_N)$ is described as follows, for all $i,j=0, \ldots, m-1$.

$$
\begin{align*}
\varphi_{i,j} & = \alpha \varphi_{i,j}, \\
\varphi_{i,j} & = \beta \varphi_{i,j},
\end{align*}
$$

and relations

$$
\begin{align*}
\varphi^2_{0,0} &= 0, & \varphi_{1,0} & = N \varphi_{0,0}, & \varphi_{1,0} & = 0, \\
\varphi^2_{1,0} &= 0, & \varphi_{1,0} & = 0, & \varphi_{1,0} & = 0, \\
\varphi_{m-1,1}^2 m & = 0, & \varphi_{0,1} & = 0, & \varphi_{0,1} & = 0, \\
\varphi_{m-1,1}^2 m & = 0, & \varphi_{0,1} & = 0, & \varphi_{0,1} & = 0, \\
\varphi_{1,0}^2 m & = 0, & \varphi_{1,0} & = 0, & \varphi_{1,0} & = 0.
\end{align*}
$$

(3) Let $\varepsilon_i = (a_i \tilde{a}_i)^N$ and $f_i = [a_i \tilde{a}_i + \tilde{a}_i a_i]$ for $i = 0, \ldots, m-1$. Then the action of $\text{HH}^0(\text{A}_N)$ on the generators of $\text{HH}^n(\text{A}_N)$ is described as follows, for all $i,j=0, \ldots, m-1$.

$$
\begin{align*}
\varepsilon_i & = 0 \quad \forall g \neq \chi_{m} \alpha, & \varepsilon_i & = 0 \quad \text{if } i \neq 0 \text{ and } \text{char } \mathbb{K} \nmid N, & \varepsilon_i & = \varepsilon_i, \\
\varepsilon_i & = \chi_{2,0} \varepsilon_i = 0 \quad \text{if } i \neq 0, & \varepsilon_i & = (1)^{\frac{m}{2}} \varepsilon_i, & \varepsilon_i & = \varepsilon_i, \\
f_i f_j & = 0 \quad \text{if } i \neq j, & f_i f_j & = f_j f_i, & f_i f_j & = f_j f_i, \\
f_i f_{m-1,1} & = 0, & f_i f_{m-1,1} & = 0, & f_i f_{m-1,1} & = 0, \\
\varepsilon_{m-1,1} f_j & = 0 \quad \text{for } i \neq j, & \varepsilon_{m-1,1} f_j & = f_j \varepsilon_{m-1,1}.
\end{align*}
$$

Corollary 4.9. For $N > 1$, $m > 2$ and $m$ even,

$$
\text{HH}^*(\text{A}_N)/N \cong \mathbb{K}[\chi_{2,0}, \chi_{m,1}, \chi_{m,-1}]/(\chi_{m,1} \chi_{m,-1})
$$

and hence $\text{HH}^*(\text{A}_N)/N$ is a commutative finitely generated algebra of Krull dimension 2.

5. For the Hochschild cohomology ring $\text{HH}^*(\text{A}_N)$ for $m \geq 3$, $m$ odd.

In this section, $N \geq 1$, $m \geq 3$ and $m$ is odd. Most of the details are similar to those in the previous section and are left to the reader. We consider separately the cases $\text{char } \mathbb{K} \neq 2$ and $\text{char } \mathbb{K} = 2$.

5.1. Basis of $\text{HH}^n(\text{A}_N)$ for $N \geq 1$, $m \geq 3$, $m$ odd and $\text{char } \mathbb{K} \neq 2$.

**Proposition 5.1.** Suppose that $N \geq 1$, $m \geq 3$, $m$ odd and $\text{char } \mathbb{K} \neq 2$. For $n \geq 1$, the following elements define a basis of $\text{HH}^n(\text{A}_N)$:

1. For $n$ even, $n \geq 2$:

   a. $\chi_{n,\delta} : \varepsilon_i \otimes \varepsilon_{i+\delta m} = \varepsilon_i$ for all $i = 0, \ldots, m-1$, with

   $$
   \delta = \begin{cases} 
   p - 2\alpha - 1, & \text{when } t \text{ is odd and } \alpha + \frac{m-t}{2} \text{ is odd,} \\
   p - 2\alpha, & \text{when } t \text{ is even and } \alpha + \frac{t}{2} \text{ is even};
   \end{cases}
   $$

   b. $\pi_{n,\delta} : \varepsilon_0 \otimes \varepsilon_{\delta m} \mapsto (a_0 \tilde{a}_0)^N$, with

   $$
   \delta = \begin{cases} 
   p - 2\alpha - 1, & \text{when } t \text{ is odd and } \alpha + \frac{m-t}{2} \text{ is even,} \\
   p - 2\alpha, & \text{when } t \text{ is even and } \alpha + \frac{t}{2} \text{ is odd};
   \end{cases}
   $$
(c) For each \( j = 0, 1, \ldots, m - 2 \) and each \( s = 1, \ldots, N - 1 \):

\[
F_{n,j,s}: \begin{cases}
\epsilon_j \otimes \epsilon_j \mapsto (a_j \bar{a}_j)^s; \\
\epsilon_{j+1} \otimes \epsilon_{j+1} \mapsto (-1)^{\frac{j}{2}} (\bar{a}_j a_j)^s;
\end{cases}
\]

(d) For each \( s = 1, \ldots, N - 1 \):

\[
F_{n,m-1,s}: \begin{cases}
\epsilon_{m-1} \otimes \epsilon_{m-1} \mapsto (\bar{a}_{m-1} a_{m-1})^s; \\
\epsilon_0 \otimes \epsilon_0 \mapsto (-1)^{\frac{m}{2}} (\bar{a}_{m-1} a_{m-1})^s;
\end{cases}
\]

(e) For \( t = m - 1 \) and \( \sigma = - (p + 1) \): \( \varphi_{n,\sigma}: \epsilon_i \otimes \epsilon_{i+\sigma m+1} \mapsto (a\bar{a})^{N-1} a_i \) for all \( i = 0, 1, \ldots, m - 1 \);

(f) For \( t = m - 1 \) and \( \tau = p + 1 \): \( \psi_{n,\tau}: \epsilon_i \otimes \epsilon_{i+\tau m+1} \mapsto (\bar{a}\bar{a})^{N-1} \bar{a}_{i-1} \) for all \( i = 0, 1, \ldots, m - 1 \);

(g) Additionally, if char \( \mathbb{K} \mid N \), and for each \( \begin{cases}
j = 0, 1, \ldots, m - 1 & \text{if } \frac{n}{2} \text{ is even} \\
j = 1, \ldots, m - 1 & \text{if } \frac{n}{2} \text{ is odd}
\end{cases} \)

\[\theta_{n,j}: \epsilon_j \otimes \epsilon_j \mapsto (a_j \bar{a}_j)^N.\]

(2) For \( n \) odd:

(a) For \( t = 0 \) and \( \delta = \pm p \): \( \pi_{n,\delta}: \epsilon_0 \otimes \epsilon_{\delta m} \mapsto (a_0 \bar{a}_0)^N \);

(b) \( \varphi_{n,\sigma}: \epsilon_i \otimes \epsilon_{i+\sigma m+1} \mapsto (a\bar{a})^{N-1} a_i \) for all \( i = 0, 1, \ldots, m - 1 \), with

\[
\begin{cases}
\sigma = p - 2\gamma & \text{if } t \text{ is odd, } 2\gamma > p, \gamma \leq p, \text{ and } \gamma + \frac{1}{2} \text{ is even,} \\
\sigma = p - 2\gamma - 1 & \text{if } t \text{ is even, } t \neq m - 1, \gamma \leq p - 1, 2\gamma > p - 1 \text{ and } \gamma + \frac{m - 1}{2} + \frac{1}{2} \text{ even,} \\
\sigma = p - 2\gamma - 1 & \text{if } t = m - 1, \gamma \leq p, 2\gamma > p - 1, \gamma \text{ even,}
\end{cases}
\]

(c) \( \varphi_{n,\sigma}: \epsilon_i \otimes \epsilon_{i+\sigma m+1} \mapsto a_i \) for all \( i = 0, 1, \ldots, m - 1 \), with

\[
\begin{cases}
\sigma = p - 2\gamma & \text{if } t \text{ is odd, } 2\gamma \leq p, \gamma \geq 0 \text{ and } \gamma + \frac{1}{2} \text{ is even,} \\
\sigma = p - 2\gamma - 1 & \text{if } t \text{ is even, } \gamma \geq 0, 2\gamma \leq p - 1 \text{ and } \gamma + \frac{m - 1}{2} + \frac{1}{2} \text{ even,}
\end{cases}
\]

(d) \( \psi_{n,\tau}: \epsilon_i \otimes \epsilon_{i+\tau m+1} \mapsto (\bar{a}\bar{a})^{N-1} \bar{a}_{i-1} \) for all \( i = 0, 1, \ldots, m - 1 \), with

\[
\begin{cases}
\tau = p - 2\beta & \text{if } t \text{ is odd, } 2\beta < p, \beta \geq 0, \text{ and } \beta + \frac{1}{2} \text{ is even,} \\
\tau = p - 2\beta - 1 & \text{if } t \text{ is even, } t \neq m - 1, \beta \geq 0, 2\beta < p - 1 \text{ and } \beta + \frac{m - 1}{2} + \frac{1}{2} \text{ even,} \\
\tau = p - 2\beta - 1 & \text{if } t = m - 1, \beta \geq -1, 2\beta < p - 1, \text{ and } \beta \text{ even,}
\end{cases}
\]

(e) \( \psi_{n,\tau}: \epsilon_i \otimes \epsilon_{i+\tau m+1} \mapsto \bar{a}_{i-1} \) for all \( i = 0, 1, \ldots, m - 1 \), with

\[
\begin{cases}
\tau = p - 2\beta & \text{if } t \text{ is odd, } 2\beta \geq p, \beta \leq p, 2\beta \geq p, \text{ and } \beta + \frac{1}{2} \text{ is even,} \\
\tau = p - 2\beta - 1 & \text{if } t \text{ is even, } \beta \leq p - 1, 2\beta \geq p - 1, \text{ and } \beta + \frac{m - 1}{2} + \frac{1}{2} \text{ even,}
\end{cases}
\]

(f) For each \( j = 0, 1, \ldots, m - 1 \) and each \( s = 1, \ldots, N - 1 \):

\[
E_{n,j,s}: \epsilon_j \otimes \epsilon_{j+1} \mapsto (a\bar{a})^s a_j;
\]

(g) Additionally, if char \( \mathbb{K} \mid N \), and for each \( \begin{cases}
j = 0, 1, \ldots, m - 1 & \text{if } \frac{n}{2} \text{ is odd} \\
j = 1, \ldots, m - 1 & \text{if } \frac{n}{2} \text{ is even}
\end{cases} \)

\[\omega_{n,j}: \epsilon_j \otimes \epsilon_{j+1} \mapsto a_j.\]

5.2. The Hochschild cohomology ring HH\(^*\)(\(\Lambda_N\)) for \( N > 1, m > 3, m \text{ odd and char } \mathbb{K} \neq 2 \).

**Theorem 5.2.** For \( N = 1, m > 3, m \text{ odd and char } \mathbb{K} \neq 2 \), HH\(^*\)(\(\Lambda_1\)) is a finitely generated algebra with generators:

1. \( a_i \bar{a}_i \) for \( i = 0, 1, \ldots, m - 1 \) in degree 0,
2. \( \varphi_{1,0} \), \( \psi_{1,0} \) in degree 1,
3. \( \pi_{2,0} \) in degree 2,
4. \( \chi_{1,0} \) in degree 4,
5. \( \varphi_{m-1,0}, \psi_{m-1,1} \) in degree \( m - 1 \),
6. \( \pi_{m,1}, \pi_{m,-1} \) in degree \( m \),
7. \( \chi_{2m,2}, \chi_{2m,-2} \) in degree 2m.
We note that if char(\(\mathbb{K}\)) \(\nmid m\), then the generators \(\pi_{2,0}, \pi_{m,1}\) and \(\pi_{m,-1}\) are redundant.

**Corollary 5.3.** For \(N = 1, m \geq 3, m\) odd and char \(\mathbb{K} \neq 2\),
\[
\mathrm{HH}^*(A_1)/N \cong \mathbb{K}[\chi_{4,0}, \chi_{2m,2}, \chi_{2m,-2}]/(\chi_{4,0}^m - \chi_{2m,2}\chi_{2m,-2})
\]
and hence \(\mathrm{HH}^*(A_1)/N\) is a commutative finitely generated algebra of Krull dimension 2.

**Theorem 5.4.** For \(N > 1, m \geq 3, m\) odd and char \(\mathbb{K} \neq 2\), \(\mathrm{HH}^*(A_N)\) is a finitely generated algebra with generators:

\[
\begin{align*}
1, (a_i \bar{a}_i)^N, [a_i \bar{a}_i + \bar{a}_i a_i] & \text{ for } i = 0, 1, \ldots, m - 1 \text{ in degree 0,} \\
\varphi_{1,0}, \psi_{1,0} & \text{ in degree 1,} \\
\omega_{1,j} & \text{ for } j = 1, \ldots, m - 1 \text{ if char } \mathbb{K} | N \text{ in degree 1,} \\
\pi_{2,0}, F_{2,1} & \text{ for } j = 1, \ldots, m - 1 \text{ in degree 2,} \\
\theta_{2,j} & \text{ for } j = 1, \ldots, m - 1 \text{ if char } \mathbb{K} | N \text{ in degree 2,} \\
\omega_{3,1} & \text{ for } j = 1, \ldots, m - 1 \text{ if char } \mathbb{K} | N \text{ in degree 3,} \\
\chi_{4,0} & \text{ in degree 4,} \\
\varphi_{m-1,-1}, \psi_{m-1,1} & \text{ in degree } m - 1, \\
\pi_{m,1}, \pi_{m,-1} & \text{ in degree } m, \\
\chi_{2m,2}, \chi_{2m,-2} & \text{ in degree } 2m, \\
\varphi_{m+1,-2}, \psi_{m+1,2} & \text{ if char } \mathbb{K} | N \text{ in degree } 2m + 1.
\end{align*}
\]

**Corollary 5.5.** For \(N > 1, m \geq 3, m\) odd and char \(\mathbb{K} \neq 2\),
\[
\mathrm{HH}^*(A_N)/N \cong \mathbb{K}[\chi_{4,0}, \chi_{2m,2}, \chi_{2m,-2}]/(\chi_{2m,2}\chi_{2m,-2})
\]
and hence \(\mathrm{HH}^*(A_1)/N\) is a commutative finitely generated algebra of Krull dimension 2.

### 5.3. Basis of \(\mathrm{HH}^n(A_N)\) for \(N \geq 1, m \geq 3, m\) odd and char \(\mathbb{K} = 2\).

**Proposition 5.6.** Suppose that \(N \geq 1, m \geq 3, m\) odd and char \(\mathbb{K} = 2\). For \(n \geq 1\), the following elements define a basis of \(\mathrm{HH}^n(A_N)\):

1. For all \(n \geq 1:\)
   
   a. \(\chi_{0,\delta}: e_i \otimes e_{i+\delta m} \mapsto e_i\) for all \(i = 0, 1, \ldots, m - 1\), with
   \[
   \delta = \begin{cases} 
   p - 2\alpha - 1, & 0 \leq \alpha \leq p - 1 \quad \text{when } t \text{ is odd,} \\
   p - 2\alpha, & 0 \leq \alpha \leq p \quad \text{when } t \text{ is even.}
   \end{cases}
   \]
   
   b. \(\pi_{0,\delta}: e_0 \otimes e_{\delta m} \mapsto (a_0 \bar{a}_0)^N,\) with \(\delta = \begin{cases} 
   p - 2\alpha - 1, & 0 \leq \alpha \leq p - 1 \quad \text{when } t \text{ is odd,} \\
   p - 2\alpha, & 0 \leq \alpha \leq p \quad \text{when } t \text{ is even.}
   \end{cases}
   \]
   
   c. \(\varphi_{0,\sigma}: e_i \otimes e_{i+\sigma m+1} \mapsto a_i\) for all \(i = 0, 1, \ldots, m - 1\) with
   \[
   \sigma = \begin{cases} 
   p - 2\gamma & \text{if } t \text{ is odd, } 2\gamma \leq p, \text{ and } \gamma \geq 0, \\
   p - 2\gamma - 1 & \text{if } t \text{ is even, } t \neq m - 1, \gamma \geq 0, \text{ and } 2\gamma \leq p, \\
   p - 2\gamma - 1 & \text{if } t = m - 1, \gamma \geq 0, \\
   p - 2\gamma - 1 & \text{if } t = m - 1, 2\gamma \leq p - 1, \gamma \geq 0;
   \end{cases}
   \]
   
   d. \(\varphi_{0,\sigma}: e_i \otimes e_{i+\sigma m+1} \mapsto (a a_i)^N-1a_i\) for all \(i = 0, 1, \ldots, m - 1\) with
   \[
   \sigma = \begin{cases} 
   p - 2\gamma & \text{if } t \text{ is odd, } 2\gamma > p, \text{ and } \gamma \leq p, \\
   p - 2\gamma - 1 & \text{if } t \text{ is even, } t \neq m - 1, \gamma \leq p - 1, \text{ and } 2\gamma > p - 1, \\
   p - 2\gamma - 1 & \text{if } t = m - 1, 2\gamma > p - 1, \gamma \leq p; \\
   \end{cases}
   \]
   
   e. \(\psi_{0,\tau}: e_i \otimes e_{i+\tau m-1} \mapsto a_{i-1}\) for all \(i = 0, 1, \ldots, m - 1\), with:
   \[
   \tau = \begin{cases} 
   p - 2\beta & \text{if } t \text{ is odd, } 2\beta \geq p, \text{ and } \beta \leq p, \\
   p - 2\beta - 1 & \text{if } t \text{ is even, } t \neq m - 1, \beta \leq p - 1, \text{ and } 2\beta \geq p - 1, \\
   \end{cases}
   \]

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\[(f) \, \psi_{n, \tau} : e_i \otimes e_i e_{i + \tau m - 1} \mapsto (\bar{a}a)^{N-1} \bar{a}_i^{-1} \text{ for all } i = 0, 1, \ldots, m - 1, \text{ with:} \]

\[
\tau = \begin{cases} 
  p - 2\beta & \text{if } t \text{ is odd, } 2\beta < p, \text{ and } \beta \geq 0, \\
  p - 2\beta - 1 & \text{if } t \text{ is even, } t \neq m - 1, \beta \geq 0, \text{ and } 2\beta < p - 1, \\
  p - 2\beta - 1 & \text{if } t = m - 1, 2\beta < p - 1, \beta \geq -1.
\end{cases}
\]

(2) For \( n \) even, \( n \geq 2; \)
(a) For each \( j = 0, 1, \ldots, m - 2 \) and each \( s = 1, \ldots, N - 1; \)
\[
F_{n,j,s} : e_j \otimes e_j \mapsto (a_j \bar{a}_j)^s, \quad e_{j+1} \otimes e_{j+1} \mapsto (\bar{a}_j a_j)^s;
\]
(b) For each \( s = 1, \ldots, N - 1; \)
\[
F_{n,m-1,s} : e_{m-1} \otimes e_{m-1} \mapsto (\bar{a}_{m-1} a_{m-1})^s;
\]
(c) Additionally, if \( \text{char } K \mid N \) for each \( j = 0, 1, \ldots, m - 1; \theta_{n,j} : e_j \otimes e_j \mapsto (a_j \bar{a}_j)^N. \)
(3) For \( n \) odd:
(a) For each \( j = 0, 1, \ldots, m - 1 \) and each \( s = 1, \ldots, N - 1; \)
\[
E_{n,j,s} : e_j \otimes e_{j+1} \mapsto (a\bar{a})^s a_j;
\]
(b) For each \( j = 0, 1, \ldots, m - 1; \omega_{n,j} : e_j \otimes e_{j+1} \mapsto a_j. \)

5.4. The Hochschild cohomology ring \( \text{HH}^*(\Lambda_1) \) for \( N \geq 1, m \geq 3, m \text{ odd and } \text{char } K = 2. \)

Theorem 5.7. For \( N = 1, \) \( m \geq 3, \text{ odd and char } K = 2, \) \( \text{HH}^*(\Lambda_1) \) is a finitely generated algebra with generators:

\[1, a_i \bar{a}_i \text{ for } i = 0, 1, \ldots, m - 1 \text{ in degree } 0, \]
\[\varphi_{1,0}, \psi_{1,0} \text{ in degree } 1, \]
\[\Lambda_{2,0}, \varphi_{m-1,-1}, \psi_{m-1,1} \text{ in degree } m - 1, \]
\[\chi_{m,1}, \chi_{m,-1} \text{ in degree } m. \]

Corollary 5.8. For \( N = 1, \) \( m \geq 3, \text{ odd and char } K = 2, \)
\[\text{HH}^*(\Lambda_1)/N \cong K[\chi_{2,0}, \chi_{m,1}, \chi_{m,-1}]/(\chi_{2,0}^m - \chi_{m,1} \chi_{m,-1})\]
and hence \( \text{HH}^*(\Lambda_1)/N \) is a commutative finitely generated algebra of Krull dimension 2.

Theorem 5.9. For \( N > 1, \) \( m \geq 3, \text{ odd and char } K = 2, \) \( \text{HH}^*(\Lambda_N) \) is a finitely generated algebra with generators:

\[1, (a_i \bar{a}_i)^N, [a_i \bar{a}_i + \bar{a}_i a_i] \text{ for } i = 0, 1, \ldots, m - 1 \text{ in degree } 0, \]
\[\varphi_{1,0}, \psi_{1,0} \text{ in degree } 1, \]
\[\omega_{1,j} \text{ for } j = 0, 1, \ldots, m - 1 \text{ if } \text{char } K \mid N \text{ in degree } 1, \]
\[\Lambda_{2,0}, \varphi_{m-1,-1}, \psi_{m-1,1} \text{ in degree } m - 1, \]
\[\chi_{m,1}, \chi_{m,-1} \text{ in degree } m. \]

Corollary 5.10. For \( N > 1, \) \( m \geq 3, \text{ odd and char } K = 2, \)
\[\text{HH}^*(\Lambda_N)/N \cong K[\chi_{2,0}, \chi_{m,1}, \chi_{m,-1}]/(\chi_{m,1} \chi_{m,-1})\]
and hence \( \text{HH}^*(\Lambda_N)/N \) is a commutative finitely generated algebra of Krull dimension 2.

6. The Hochschild cohomology ring \( \text{HH}^*(\Lambda_N) \) for \( m = 2 \)

Recall, from Theorem 3.1, that \( \dim \text{HH}^0(\Lambda_N) = 2N + 1. \) We start with the case \( N = 1. \)
6.1. The case $N = 1$. 

**Proposition 6.1.** For $m = 2$ and $N = 1$,

$$\dim \text{HH}^n(\Lambda_1) = \begin{cases} 3 & \text{if } n = 0, \\ 2(n + 1) & \text{if } n \neq 0. \end{cases}$$

**Proposition 6.2.** For $m = 2$ and $N = 1$, the following elements define a basis of $\text{HH}^n(\Lambda_1)$ for $n \geq 1$.

1. For $n = 2p$ even, $n \geq 2$:
   
   a. For $-p \leq \alpha \leq p$: $\chi_{n,\alpha} : e_i \otimes e_{i+2\alpha} \mapsto (-1)^{(p+\alpha)}e_i$ for $i = 0, 1$;
   
   b. For $-p \leq \alpha \leq p$: $\pi_{n,\alpha} : e_0 \otimes e_{2\alpha} \mapsto a_0a_0$.

2. For $n = 2p + 1$ odd:
   
   a. For $-p - 1 \leq \gamma \leq p$: $\varphi_{n,\gamma} : e_i \otimes e_{i+2\gamma+1} \mapsto (-1)^{(p+\gamma)}a_i$ for $i = 0, 1$;
   
   b. For $-p - 1 \leq \beta \leq p$: $\psi_{n,\beta} : e_i \otimes e_{i+2\beta+1} \mapsto (-1)^{(p+\beta+1)}a_{i+1}$ for $i = 0, 1$.

It is straightforward to give liftings of these elements and we omit the details. Further computations enable us to give generators for $\text{HH}^n(\Lambda_1)$.

**Theorem 6.3.** For $m = 2$ and $N = 1$, $\text{HH}^*(\Lambda_1)$ is a finitely generated algebra with generators:

- $1, a_0a_0, a_1a_1$ in degree 0,
- $\varphi_{1,0}, \varphi_{1,-1}, \psi_{1,0}, \psi_{1,-1}$ in degree 1,
- $\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}$ in degree 2.

In order to give the structure of the Hochschild cohomology ring modulo nilpotence, it can now be verified that $1, \chi_{2,0}, \chi_{2,1}, \chi_{2,-1}$ are the non-nilpotent generators of $\text{HH}^*(\Lambda_1)$ and that $\chi_{2,0}^2 = \chi_{2,1}\chi_{2,-1}$. This gives the following result.

**Corollary 6.4.** For $m = 2$ and $N = 1$,

$$\text{HH}^*(\Lambda_1)/\mathcal{N} \cong \mathbb{K}[\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}]/(\chi_{2,0}^2 - \chi_{2,1}\chi_{2,-1})$$

with $\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}$ all in degree 2. Thus $\text{HH}^*(\Lambda_1)/\mathcal{N}$ is a commutative finitely generated algebra of Krull dimension 2.

6.2. The case $N > 1$. We start by giving a basis of cocycles for each cohomology group, together with a lifting for each basis element. We shall use once more the notation $(n) \in \{0, 1, \ldots, m - 1\}$ for the representative of $n$ modulo $m$.

**Proposition 6.5.** For $m = 2$ and $N > 1$,

$$\dim \text{HH}^n(\Lambda_N) = \begin{cases} 2N + 1 & \text{if } n = 0, \\ 2N + 2n & \text{if } n \geq 1 \text{ and } \text{char } \mathbb{K} \nmid N, \\ 2N + 2n + 1 & \text{if } n \geq 1 \text{ and } \text{char } \mathbb{K} \mid N. \end{cases}$$

**Proposition 6.6.** For $m = 2, N > 1$ and for each $n \geq 1$, we give a basis for the cohomology group $\text{HH}^n(\Lambda_N)$ together with a lifting for each basis element.

1. For $n$ even, $n \geq 2$:
   
   a. For $-p \leq \alpha \leq p$:
      
      $\diamond \chi_{n,\alpha} : e_i \otimes e_{i+2\alpha} \mapsto (-1)^{(2-\alpha)}e_i$ for $i = 0, 1$,
\[ L^q_{\pi_n,\alpha}(e_i \otimes e_{i+q-2\ell+2a}) = \]
\[
\begin{cases}
\begin{aligned}
&(-1)^{(q-\alpha)\ell}e_i \otimes e_{i+q-2\ell} \\
&0
\end{aligned}
\end{cases}
\]
\[
\begin{cases}
\begin{aligned}
&(-1)^{(q-\alpha)\ell}_1(a\bar{a})^{-1}e_i \otimes e_{i+2(a\bar{a})}^{-1} \\
&(-1)^{(q-\alpha)\ell}_2(a\bar{a})^{-1}e_i \otimes e_{i+2(a\bar{a})}^{-1} \\
&(-1)^{(q-\alpha)\ell}_3\left[ \sum_{k=0}^{N-1}(a\bar{a})^{k}e_i \otimes e_{i+1(a\bar{a})}^{N-k-1} \right] \\
&(-1)^{(q-\alpha)\ell}_4\left[ \sum_{k=0}^{N-2}(a\bar{a})^{k}e_{i+1} \otimes e_{i+2(a\bar{a})}^{N-k-2} \right]
\end{aligned}
\end{cases}
\]
\[
\begin{cases}
\begin{aligned}
&\text{if } (q-2\ell)\alpha \geq 0 \\
&\text{if } (q-2\ell)\alpha < 0 \text{ and } |q-2\ell| > 2 \\
&\text{if } \alpha < 0 \text{ and } q-2\ell = 2 \\
&\text{if } \alpha > 0 \text{ and } q-2\ell = -2 \\
&\text{if } \alpha < 0 \text{ and } q-2\ell = 1 \\
&\text{if } \alpha > 0 \text{ and } q-2\ell = -1 \\
\end{aligned}
\end{cases}
\]
for i = 0, 1 and 0 \leq \ell \leq q.

- For \( -p \leq \alpha \leq p, \)
  \[
  \pi_n,\alpha : e_0 \otimes e_{2\alpha} \mapsto (a\bar{a}a)^N
  \]

\[
L^q_{\pi_n,\alpha}(e_0 \otimes e_{q-2\ell+2a}) = \begin{cases}
\begin{aligned}
&(a\bar{a})^Ne_0 \otimes e_{q-2\ell} \\
&(a\bar{a})^Ne_0 \otimes e_{1(a\bar{a})}^{-1} \\
&(a\bar{a})^Ne_0 \otimes e_{-1(a\bar{a})}^{-1} \\
&0
\end{aligned}
\end{cases}
\]
\[
\begin{cases}
\begin{aligned}
&\text{if } \alpha(q-2\ell) \geq 0 \\
&\text{if } q-2\ell = 1 \text{ and } \alpha < 0 \\
&\text{if } q-2\ell = -1 \text{ and } \alpha > 0 \\
&\text{otherwise}
\end{aligned}
\end{cases}
\]
for all 0 \leq \ell \leq q.

- For each 1 \leq s \leq N - 1:
  \[
  F_{n,s} : e_0 \otimes e_0 \mapsto (a\bar{a}a)^s
  \]

\[
L^qF_{n,s} : e_0 \otimes e_{q-2\ell} \mapsto (a\bar{a}a)^s e_0 \otimes e_{q-2\ell}
\]
for all 0 \leq \ell \leq q.

- For each 1 \leq s \leq N - 1:
  \[
  F_{n,s} : e_0 \otimes e_0 \mapsto (a_1\bar{a}_1)^s
  \]

\[
L^qF_{n,s} : e_0 \otimes e_{q-2\ell} \mapsto (a_1\bar{a}_1)^s e_0 \otimes e_{q-2\ell}
\]
for all 0 \leq \ell \leq q.

- Additionally for char of $\mathbb{K} | N$:
  \[
  \theta_{n} : e_0 \otimes e_0 \mapsto (a_1\bar{a}_1)^N
  \]

\[
L^q\theta_{n} : e_0 \otimes e_{q-2\ell} \mapsto (a_1\bar{a}_1)^Ne_1 \otimes e_{q-2\ell}
\]
for all 0 \leq \ell \leq q.

(2) For $n$ odd:

- For $-p - 1 \leq \gamma < 0$:
  \[
  \varphi_{n,\gamma} : e_i \otimes e_{i+2\gamma+1} \mapsto (-1)^{(\frac{q-\alpha}{2})}a_i(a\bar{a})^{-1} \text{ for } i = 0, 1,
  \]

\[
L^q\varphi_{n,\gamma}(e_i \otimes e_{i+2\gamma+1+q-2\ell}) = \begin{cases}
\begin{aligned}
0 & \text{if } q-2\ell > 1 \\
(-1)^{(\frac{q-\alpha}{2})}e_i \otimes e_{i+1}(a\bar{a})^{-1} & \text{if } q-2\ell = 1 \\
(-1)^{(\frac{q-\alpha}{2})}e_i \otimes e_{i+q-2\ell}(a\bar{a})^{-1} & \text{if } q-2\ell \leq 0
\end{aligned}
\end{cases}
\]
for $i = 0, 1$ and for all $0 \leq \ell \leq q$.

- For $0 \leq \gamma \leq p$:
  \[
  \varphi_{n,\gamma} : e_i \otimes e_{i+2\gamma+1} \mapsto (-1)^{(\frac{q-\alpha}{2})}a_i \text{ for } i = 0, 1,
  \]

for $i = 0, 1$ and for all $0 \leq \ell \leq q$.  

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\* If $\gamma > 0$, then $L^\gamma \varphi_{n, \gamma}(e_i \otimes e_{i+2\gamma+1+q-2\ell}) =
\sum_{k=0}^{N-1} (\bar{a}a)^k e_i \otimes e_{i-1} (a\bar{a})^{N-k-1} a +
(\bar{a}a)^k \bar{a} e_{(i-1)} \otimes e_{(i-1)+1} (a\bar{a})^{N-k-1}
\] if $q - 2\ell > 0$
\] if $q - 2\ell = -1$
\] if $q - 2\ell = -2$
\] if $q - 2\ell < -2$
\) for $i = 0, 1$ and for all $0 \leq \ell \leq q$.

\* If $\gamma = 0$, then $L^\gamma \varphi_{n, \gamma}(e_i \otimes e_{i+1+q-2\ell}) =
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a (a\bar{a})^{N-1}
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a (a\bar{a})^{N-1}
\] if $q - 2\ell = q$
\] if $0 \leq q - 2\ell < q$
\] if $q - 2\ell < -1$
\) for $i = 0, 1$ and for all $0 \leq \ell \leq q$.

\* For $0 \leq \beta \leq p$:
\* If $\beta < 1$, then $L^\beta \psi_{n, \beta}(e_i \otimes e_{i+2\beta+1+q-2\ell}) =
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a (a\bar{a})^{N-1}
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a (a\bar{a})^{N-1}
\] if $q - 2\ell < -1$
\] if $q - 2\ell = -1$
\] if $q - 2\ell > 1$
\) for $i = 0, 1$ and for all $0 \leq \ell \leq q$.

\* For $-p - 1 \leq \beta < 0$:
\* If $\beta < -1$, then $L^\beta \psi_{n, \beta}(e_i \otimes e_{i+2\beta+1+q-2\ell}) =
(\bar{a}a)^{\ell} e_i \otimes e_{i+q-2\ell} a
\) if $q - 2\ell < -1$
\] if $q - 2\ell = -1$
\] if $q - 2\ell > 1$
\) for $i = 0, 1$ and for all $0 \leq \ell \leq q$. 

\newpage
• If $\beta = -1$, then $L^q\psi_{n-1}(e_i \otimes e_{i-1+q-2\ell}) =$
\[
(-1)^{\frac{n-1}{2}}e_i \otimes e_{i+q-2\ell}a \\
(-1)^{\frac{n+1}{2}}[e_i \otimes e_{i+q-2\ell}a - (-1)^q(N-1)e_i \otimes e_{i+q-2\ell+2a(\bar{a}a)^N]^1] \\
(-1)^{\frac{n+1}{2}}N e_i \otimes e_{i+q-2\ell+2a(\bar{a}a)^N} \\
(-1)^{\frac{n+1}{2}} \left\{ \sum_{k=1}^{N-1-k} \left( [\bar{a}a]^\ell e_i \otimes e_{i-1}(\bar{a}a)^{N-v-1} + \\
(-1)^{\frac{n+1}{2}}[a(\bar{a}a)^v e_j \otimes e_{j-1}(\bar{a}a)^{N-v-1} + \\
(-1)^{\frac{n+1}{2}}(a(\bar{a}a)^v e_j \otimes e_{j-1}+e_{j+1}(\bar{a}a)^{N-v-1}] \\
\right) \right\} \\
\sum_{k=0}^{N-1} (\bar{a}a)^k e_i \otimes e_{i+1}(\bar{a}a)^{N-k-1} \right\}
\]
for $i = 0, 1$ and for all $0 \leq \ell \leq q$.

• For each $j = 0, 1$ and each $1 \leq s \leq N - 1$:
  • $E_{n,j,s} : e_j \otimes e_{j-1} \mapsto (\bar{a}a)^se_j$,
  • $L^0 E_{n,j,s} :$
\[
\begin{align*}
e_j \otimes e_{j-1} &\mapsto (\bar{a}a)^se_j \otimes e_{j-1}
\end{align*}
\]
for $q$ is even
\[
\begin{align*}
e_j \otimes e_j &\mapsto \sum_{k=0}^{N-s-N-s} \sum_{v=0}^{N-s} [(\bar{a}a)^{v+s}e_j \otimes e_{j-1}(\bar{a}a)^{N-v-1} + \\
(-1)^{\frac{n+1}{2}}(\bar{a}a)^{v+s}e_j \otimes e_{j-1}(\bar{a}a)^{N-v-1}] \\
e_{j-1} \otimes e_{j-1} &\mapsto (-1)^{\frac{n+1}{2}} \left\{ \sum_{k=0}^{N-s-1-N-s} \sum_{v=0}^{N-s} \left[ (\bar{a}a)^{v+s}e_j \otimes e_{j+1}(\bar{a}a)^{N-v-1} \\
+ (\bar{a}a)^{v+s+1}e_{j+1} \otimes e_{j+1}(\bar{a}a)^{N-v-2} \right] \right\}
\end{align*}
\]
for all $0 \leq \ell \leq q$.

• Additionally for char $\mathbb{K} \mid N$:
  • $\omega_n : e_0 \otimes e_1 \mapsto a_0$,
  • $L^0 \omega_n :$
\[
\begin{align*}
e_0 \otimes e_0 &\mapsto \sum_{k=0}^{N-1-k-1} \sum_{v=0}^{N-1-k-1} [(-1)^{\frac{n+1}{2}}(\bar{a}a)^v e_0 \otimes e_{0}(\bar{a}a)^{N-v-1} - \\
(\bar{a}a)^v e_0 \otimes e_{j}(\bar{a}a)^{N-v-1}] \\
e_1 \otimes e_1 &\mapsto (-1)^{\frac{n+1}{2}} \sum_{k=0}^{N-1-k} \sum_{v=0}^{N-1-k-1} [((\bar{a}a)^{v+1}e_1 \otimes e_{0}(\bar{a}a)^{N-v-2} + \\
(\bar{a}a)^v e_0 \otimes e_{1}(\bar{a}a)^{N-v-1}] \\
e_{(-v)} \otimes e_{(-v)+2v} &\mapsto (-1)^{\frac{n+1}{2}} ([(-1)^{\frac{n+1}{2}}(\bar{a}a)^v e_{(-v)+2v} \otimes e_{(-v)+2v+1}(\bar{a}a)^{N-v-1}] \\
&\text{for } 1 \leq v \leq \frac{N+1}{2} \text{ and } q \text{ odd} \\
e_{(-v)} \otimes e_{(-v)+2v+1} &\mapsto (-1)^{\frac{n+1}{2}} [(-1)^{\frac{n+1}{2}}(\bar{a}a)^v e_{(-v)+2v+1} \otimes e_{(-v)+2v+2}] \\
&\text{for } 0 \leq v \leq \frac{N}{2} \text{ and } q \text{ even}
\end{align*}
\]
for all $0 \leq \ell \leq q$.  

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Theorem 6.7. For $m = 2$ and $N > 1$, $\text{HH}^*(\Lambda_N)$ is a finitely generated algebra, with generators:

$$1, (a_0a_0)^N, (a_1a_1)^N, (a_0a_0 + a_0a_0), (a_1a_1 + a_1a_1),$$

$$\varphi_{1,0}, \varphi_{1,1}, \psi_{1,0}, \psi_{1,1}, E_{1,0,0},$$

$$\omega_1$$ if $\text{char } K | N$ in degree 0,

$$\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}, F_{2,0,1}$$ in degree 1,

$$\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}, F_{2,0,1}$$ in degree 2.

It is easy to verify that the non-nilpotent generators of $\text{HH}^*(\Lambda_N)$ are $1, \chi_{2,0}, \chi_{2,1}$ and $\chi_{2,-1}$, and that $\chi_{2,1}\chi_{2,-1} = 0$. Thus we have the following theorem.

Corollary 6.8. For $m = 2$ and $N > 1$,

$$\text{HH}^*(\Lambda_N)/N \cong K[\chi_{2,0}, \chi_{2,1}, \chi_{2,-1}]/(\chi_{2,1}\chi_{2,-1}).$$

Hence $\text{HH}^*(\Lambda_N)/N$ is a commutative finitely generated algebra of Krull dimension 2.

7. The Hochschild cohomology ring $\text{HH}^*(\Lambda_N)$ for $m = 1$

We start by giving the centre of the algebra, since $\text{HH}^0(\Lambda_N) = Z(\Lambda_N)$.

Theorem 7.1. Suppose that $m = 1$. Then the dimension of $\text{HH}^0(\Lambda_N)$ is $N + 3$, and a $K$-basis of $\text{HH}^0(\Lambda_N)$ is given by

$$\{1, (aa)^N, [(aa)^s + (aa)^s] \text{ for } 1 \leq s \leq N - 1, a(aa)^{N-1}, a(aa)^{N-1}\}.$$

Thus $\text{HH}^0(\Lambda_N)$ is generated as an algebra by

$$\{1, a, a\} \text{ if } N = 1;$$

$$\{1, (aa)^N, [a\bar{a} + \bar{a}a], a(aa)^{N-1}, \bar{a}(aa)^{N-1}\} \text{ if } N > 1.$$

7.1. The case $N = 1$. The algebra $\Lambda_1$ is a commutative Koszul algebra. The quiver has a single vertex 1 and two loops $a$ and $\bar{a}$ such that $a^2 = 0$, $\bar{a}^2 = 0$, and $a\bar{a} = \bar{a}a$. The structure of the Hochschild cohomology ring for $\Lambda_1$ with $m = 1$ was determined in [1]; we state the result here for completeness. Note that we have $\text{HH}^0(\Lambda_1) = \Lambda_1$ in this case.

Theorem 7.2. [1] For $m = 1$ and $N = 1$, we have

$$\text{HH}^*(\Lambda_1) \cong \begin{cases} 
\Lambda_1\langle u_0, u_1, x_0, x_1 \rangle/\langle u_0^2, u_1^2, au_0, \bar{a}u_1, ax_0, ax_1 \rangle & \text{if char } K \neq 2, \\
\Lambda_1\langle y_0, y_1 \rangle & \text{if char } K = 2,
\end{cases}$$

where $x_0, x_1$ are in degree 2 and $u_0, u_1, y_0, y_1$ are in degree 1.

We remark that we could also have used the Künneth formula [18, Theorem 7.4 p297] together with $\text{HH}^*(A)$, where $A = K[x]/(x^2)$. The cohomology of $A$ has been studied in several places in the literature including in [4, 17].

We can now give the structure of the Hochschild cohomology ring of $\Lambda_1$ modulo nilpotence and its Krull dimension.

Corollary 7.3. For $m = 1$ and $N = 1$,

$$\text{HH}^*(\Lambda_1)/N \cong \begin{cases} 
K[x_0, x_1] & \text{if char } K \neq 2, \\
K[y_0, y_1] & \text{if char } K = 2,
\end{cases}$$

where $x_0, x_1$ are in degree 2 and $y_0, y_1$ are in degree 1. Hence $\text{HH}^*(\Lambda_1)/N$ is a commutative finitely generated algebra of Krull dimension 2.
7.2. The case \( N > 1 \) and \( \text{char} \mathbb{K} \neq 2 \). For the case \( N > 1 \) and \( m = 1 \) there are two subcases to consider, depending on whether the characteristic of the field is or is not equal to 2. Firstly we suppose that \( \text{char} \mathbb{K} \neq 2 \).

**Proposition 7.4.** For \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} \neq 2 \), the dimensions of the Hochschild cohomology groups are given by \( \dim \text{HH}^n(\Lambda_N) = N + 3 \) and, for \( n \geq 1 \),

\[
\dim \text{HH}^n(\Lambda_N) = \begin{cases} 
N + 4n + 3 & \text{if char } \mathbb{K} = 2, \\
N + n + 2 & \text{if char } \mathbb{K} \neq 2, \text{ char } \mathbb{K} \nmid N, \\
N + n + 2 & \text{if char } \mathbb{K} \neq 2, \text{ char } \mathbb{K} \mid N, \ n \equiv 1, 2 \pmod{4}, \\
N + n + 3 & \text{if char } \mathbb{K} \neq 2, \text{ char } \mathbb{K} \mid N, \ n \equiv 0, 3 \pmod{4}.
\end{cases}
\]

We now describe explicitly the elements of the Hochschild cohomology groups \( \text{HH}^n(\Lambda_N) \) for \( n \geq 1 \), in order to give a set of generators for \( \text{HH}^*(\Lambda_N) \) as a finitely generated algebra. Recall that \( \text{HH}^0(\Lambda_N) = Z(\Lambda_N) \) and was described in Theorem 7.1. For each cocycle \( f \in \text{Hom}(P^n, \Lambda_N) \), we continue our practice of writing only the image of each generator \( e_i \otimes e_{i+n-2r} \) in \( P^n \) where that image is non-zero. However, since we have a single vertex, we write 1 for \( e \) we continue our practice of writing only the image of each generator.

**Proposition 7.5.** Suppose that \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} \neq 2 \). The following elements define a basis of \( \text{HH}^n(\Lambda_N) \) for \( n \geq 1 \).

1. For all \( n \geq 1 \) and \( 0 \leq r \leq n \) with \( r \) even: \( \chi_{n,r} : 1 \otimes r 1 \mapsto 1 \).
2. For \( n \) even, \( n \geq 2 \):
   a. \( \psi_{n,0} : 1 \otimes 1 \mapsto (\bar{a}a)^{n-1} \);
   b. \( \varphi_{n,0} : 1 \otimes n 1 \mapsto a(\bar{a}a)^{n-1} \);
   c. For \( 1 \leq r \leq n \) and \( r \) odd: \( \pi_{n,r} : 1 \otimes r 1 \mapsto (\bar{a}a)^n \);
   d. For each \( s \) such that \( 1 \leq s \leq N - 1 \): \( F_{n,s} : 1 \otimes s 1 \mapsto (\bar{a}a)^s + (-1)^{s+1}(\bar{a}a)^s \);
   e. Additionally, for char \( \mathbb{K} \mid N, \) if \( n \equiv 0 \pmod{4} \): \( \theta_{n} : 1 \otimes s/2 1 \mapsto (\bar{a}a)^{n/2} \).
3. For \( n \) odd:
   a. For \( 0 \leq r \leq \frac{n-1}{2} \) and \( r \) even: \( \varphi_{n,r} : 1 \otimes r 1 \mapsto a \);
   b. For \( \frac{n+1}{2} \leq r \leq n - 1 \) and \( r \) even: \( \varphi_{n,r} : 1 \otimes r 1 \mapsto (\bar{a}a)^{n-1}a \);
   c. For \( 1 \leq r \leq \frac{n+1}{2} \) and \( r \) odd: \( \psi_{n,r} : 1 \otimes r 1 \mapsto (\bar{a}a)^{n-1}a \);
   d. For \( \frac{n+1}{2} \leq r \leq n \) and \( r \) odd: \( \psi_{n,r} : 1 \otimes r 1 \mapsto a \);
   e. \( \pi_{n,r} : 1 \otimes 0 1 \mapsto (\bar{a}a)^N \);
   f. \( \pi_{n,r} : 1 \otimes n 1 \mapsto (\bar{a}a)^N \);
   g. For each \( s \) such that \( 1 \leq s \leq N - 1 \): \( E_{n,s} : 1 \otimes (n-1)/2 1 \mapsto (\bar{a}a)^sa \).
   h. Additionally, for char \( \mathbb{K} \mid N, \) if \( n \equiv 3 \pmod{4} \) and \( r = \frac{n+1}{2} \): \( \psi_{n,r} : 1 \otimes 1 1 \mapsto a \).

We omit the liftings for this case, since the calculations are straightforward, and give generators of the Hochschild cohomology ring.

**Theorem 7.6.** For \( m = 1, N > 1 \), \( \text{char} \mathbb{K} \neq 2 \), \( \text{HH}^*(\Lambda_N) \) is a finitely generated algebra with generators:

- \( 1, a\bar{a} + \bar{a}a, a(\bar{a}a)^{N-1}, (\bar{a}a)^N \) in degree 0,
- \( \varphi_{1,0}, \psi_{1,1} \) in degree 1,
- \( \chi_{2,0}, \chi_{2,2}, \pi_{2,1}, \bar{F}_{2,1}, \varphi_{2,2}, \psi_{2,2} \) in degree 2,
- \( \varphi_{3,2}, \psi_{3,1} \) in degree 3,
- \( \psi_{3,2} \) if char \( \mathbb{K} \mid N \) in degree 3,
- \( \chi_{4,2}, \bar{F}_{4,1} \) in degree 4.
The non-nilpotent generators of \( \HH^*(\Lambda_N) \) are precisely those elements given in the above theorem whose image does not lie in \( \mathfrak{r} \), that is, the elements \( 1, \chi_{2,0}, \chi_{2,2} \) and \( \chi_{4,2} \). This gives the next result.

**Corollary 7.7.** For \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} \neq 2 \),

\[
\HH^*(\Lambda_N)/\mathcal{N} \cong \mathbb{K}[\chi_{2,0}, \chi_{2,2}, \chi_{4,2}]/(\chi_{2,0}\chi_{2,2})..
\]

Hence \( \HH^*(\Lambda_N)/\mathcal{N} \) is a commutative finitely generated algebra of Krull dimension 2.

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### 7.3. The case \( N > 1 \) and \( \text{char} \mathbb{K} = 2 \)

In this final case, we continue to keep the notation that \( 1 \otimes r = e_0 \otimes e_{n-2r} \) for \( r = 0, 1, \ldots, n \).

**Proposition 7.8.** For \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} = 2 \), then \( \dim \HH^0(\Lambda_N) = 4n + N + 3 \) for all \( n \geq 0 \).

**Proposition 7.9.** Suppose that \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} = 2 \). For \( n \geq 0 \), the following elements define a basis of \( \HH^0(\Lambda_N) \).

1. For all \( n \geq 0 \):
   1. For \( r = 0, \ldots, n \): \( \chi_{n,r} : 1 \otimes r \mapsto 1 \);
   2. For \( r = 0, \ldots, n \): \( \pi_{n,r} : 1 \otimes r \mapsto (a\bar{a})^N \).
2. For \( n \) even, \( n \geq 2 \):
   1. For \( r = 0, 1, \ldots, \frac{n-2}{2} \): \( \varphi_{n,r} : 1 \otimes r \mapsto a \);
   2. For \( r = \frac{n}{2}, \ldots, n-1, n \): \( \varphi_{n,r} : 1 \otimes r \mapsto (a\bar{a})^{N-1}a \);
   3. For \( r = 0, 1, \ldots, \frac{n}{2} \): \( \psi_{n,r} : 1 \otimes r \mapsto (a\bar{a})^{N-1}a \);
   4. For \( r = \frac{n+2}{2}, \ldots, n-1, n \): \( \psi_{n,r} : 1 \otimes r \mapsto a \);
   5. For \( j = 1, \ldots, N-1 \): \( F_{n,j} : 1 \otimes n/2 \mapsto (a\bar{a})^{j} + (a\bar{a})^{j} \).
3. For \( n \) odd:
   1. For \( r = 0, 1, \ldots, \frac{n-1}{2} \): \( \varphi_{n,r} : 1 \otimes r \mapsto a \);
   2. For \( r = \frac{n+1}{2}, \ldots, n-1, n \): \( \varphi_{n,r} : 1 \otimes r \mapsto (a\bar{a})^{N-1}a \);
   3. For \( r = 0, 1, \ldots, \frac{n-1}{2} \): \( \psi_{n,r} : 1 \otimes r \mapsto (a\bar{a})^{N-1}a \);
   4. For \( r = \frac{n+1}{2}, \ldots, n-1, n \): \( \psi_{n,r} : 1 \otimes r \mapsto a \);
   5. For \( j = 1, \ldots, N-1 \): \( E_{n,j} : 1 \otimes n+1 \mapsto (a\bar{a})^{j} \).

Again we omit the details of the liftings, and state directly the generators of \( \HH^*(\Lambda_N) \).

**Theorem 7.10.** For \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} = 2 \), the Hochschild cohomology ring \( \HH^*(\Lambda_N) \) is finitely generated as a \( \mathbb{K} \)-algebra with generators

\[
1, a\bar{a} + a\bar{a}(a\bar{a})^{N-1}a, (a\bar{a})^{N-1}a, (a\bar{a})^{N} \quad \text{in degree 0},
\]

\[
\chi_{1,0}, \chi_{1,1}, \varphi_{1,0}, \psi_{1,1} \quad \text{in degree 1},
\]

\[
\chi_{2,1}, \psi_{2,2} \quad \text{in degree 2}.
\]

**Corollary 7.11.** For \( m = 1, N > 1 \) and \( \text{char} \mathbb{K} = 2 \),

\[
\HH^*(\Lambda_N)/\mathcal{N} \cong \mathbb{K}[\chi_{1,0}, \chi_{1,1}, \chi_{2,1}]/(\chi_{1,0}\chi_{1,1})^2.
\]

Thus \( \HH^*(\Lambda_N)/\mathcal{N} \) is a commutative ring of Krull dimension 2.
8. Summary

In conclusion we have the following theorem.

**Theorem 8.1.** For all \( N \geq 1 \) and \( m \geq 1 \), \( \text{HH}^*(\Lambda_N) \) is a finitely generated \( \mathbb{K} \)-algebra. Moreover \( \text{HH}^*(\Lambda_N)/N \) is a finitely generated \( \mathbb{K} \)-algebra of Krull dimension 2.

In particular the theorem holds for the algebras \( \Lambda_1 \) which occur in the representation theory of the Drinfeld doubles of the generalised Taft algebras ([6]), and for the algebras of Farnsteiner and Skowroński ([10]). Moreover the conjecture of [21] has been proved for this class of algebras.

We conclude the paper by recalling the following simultaneous conditions which were considered in [7]:

(Fg1) There is a commutative Noetherian graded subalgebra \( H \) of \( \text{HH}^*(\Lambda_1) \) such that \( H^0 = \text{HH}^0(\Lambda_1) \).

(Fg2) \( \text{Ext}^*_{\Lambda_1}(\Lambda_1, \Lambda_1) \) is a finitely generated \( H \)-module.

In the case \( N = 1 \), it is immediate from the above results that the algebras \( \Lambda_1 \) satisfy the condition (Fg1) with \( H = \text{HH}^0(\Lambda_1) \). We know that \( \Lambda_1 \) is a Koszul algebra with \( \text{Ext}^*_{\Lambda_1}(\Lambda_1, \Lambda_1) \) being the Koszul dual of \( \Lambda_1 \). Hence \( \text{Ext}^*_{\Lambda_1}(\Lambda_1, \Lambda_1) = K\mathbb{Q}/T \) where \( \mathbb{Q} \) is the quiver of \( \Lambda_1 \) and \( T \) is the ideal of \( K\mathbb{Q} \) generated by \( a_i a_i + a_{i-1} a_{i-1} \) for all \( i = 0, 1, \ldots, m-1 \). It may be verified that the condition (Fg2) also holds with \( H = \text{HH}^0(\Lambda_1) \), the full details being left to the reader.

The conjecture of [21] concerning the finite generation of the Hochschild cohomology ring modulo nilpotence was motivated by the work on support varieties. The fact that \( \Lambda_1 \), which is self-injective, and \( H = \text{HH}^0(\Lambda_1) \) together satisfy (Fg1) and (Fg2) means that the support varieties for finitely generated \( \Lambda_1 \)-modules satisfy all the properties of [21] and the subsequent work of [7] on self-injective algebras.

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