Hochschild and cyclic homology of a family of Auslander algebras

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Abstract

In this paper, we compute the Hochschild and cyclic homologies of the Auslander algebras of the Taft algebras. We also describe the first Chern character for the Taft algebras and for their Auslander algebras.

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1 Introduction

The object of this paper is to compute the Hochschild homology, the cyclic homology and the Chern characters of the Auslander algebras $\Gamma_n$ of the Taft algebras $\Lambda_n$ and of their Auslander algebras $\Gamma_n$, in order to study a possible influence of the Hopf algebra structure of $\Lambda_n$ on them.

Note that Auslander algebras are useful when considering Artin algebras of finite representation type, since there is a bijection between the Morita equivalence classes of such algebras and the Morita equivalence classes of Auslander algebras (cf. [ARS]).

The Hopf algebra structure on an algebra $\Lambda$ conveys an additional structure on the Grothendieck groups $K_0(\Lambda)$ and $\overline{K_0}(\Lambda)$ of isomorphism classes of projective (resp. all) indecomposable modules, since the tensor product over the base ring $k$ of two $\Lambda-$modules is again a $\Lambda-$module, via the comultiplication of $\Lambda$. Furthermore, there is a one-to-one correspondence between the indecomposable modules over any algebra and the indecomposable projective modules over its Auslander algebra; in the case of a Hopf algebra, therefore, the Grothendieck group of projective modules of $\Gamma_\Lambda$ is endowed with a multiplicative structure. However, this correspondence does not preserve the underlying vector spaces, and this multiplicative structure does not appear to be natural.

In this paper, we study the example of the Taft algebras; they are Hopf algebras which are neither commutative, nor cocommutative. They are interesting for various reasons; for instance, $\Lambda_p$ is an example of a non-semisimple Hopf algebra whose dimension is the square of a prime (cf. [M]). They are of finite representation type; furthermore, when $n$ is odd, $\Lambda_n$ is isomorphic to the half-quantum group $u_q^+(\mathfrak{sl}_2)$ ($q$ primitive $n^{th}$-root of unity), and is the only half-quantum group $u_q^+(\mathfrak{g})$ at a root of unity which is not of wild representation type (cf. [C1]). Then, for each $n$, $\Lambda_n$ is not braided, but its Grothendieck group is a commutative ring nonetheless (cf. [C2, G]).

These examples show that the non-commutative, non-cocommutative Hopf algebra structure of $\Lambda_n$ does not yield a natural multiplicative structure on its cyclic homology. There is a product, however, obtained by transporting that of $K_0(\Lambda)$ via the Chern characters, which are onto.
The paper is organised as follows: first, we recall the quiver of the Auslander algebras \( \Gamma_{\Lambda_n} \) and give a minimal projective resolution of \( \Gamma_{\Lambda_n} \) as a \( \Gamma_{\Lambda_n} \)-bimodule. Then, we compute the Hochschild and cyclic homologies of these algebras, and finally we compute the Chern characters of the algebras \( \Lambda_n \) and \( \Gamma_{\Lambda_n} \).

Throughout this text, \( k \) is an algebraically closed field.

2 The quivers of the Auslander algebras \( \Gamma_{\Lambda_n} \)

In this paragraph, we shall describe the objects of our study: the Auslander algebras of the Taft algebras.

The Taft algebra \( \Lambda_n \) is described by quiver and relations as follows: the quiver is an oriented cycle with \( n \) vertices and \( n \) arrows, and the relations are all the paths of length greater than or equal to \( n \).

Its Auslander algebra has been described in [GR] (see [ARS] p232 for a general definition). Its quiver is

\[
\begin{array}{c}
(0,n-1) \\
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\end{array}
\]

where both vertical outer edges are identified (the quiver is on a cylinder). Let \( Q_n \) denote this quiver, let \( \{e_{i,u}/(i,u) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\} \) be the set of vertices of \( Q_n \), and let \( \{a_{i,u}, b_{i,u}, (i,u) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\} \) be the set of edges of \( Q_n \), as in the figure above.

The mesh relations on this quiver are: \( a_{i,i-2}b_{i,i-1} = 0 \) for all \( i \in \mathbb{Z}/n\mathbb{Z} \) (the composition of two edges of any ‘triangle’ under the top diagonal is zero), and \( a_{i,u-1}b_{i,u} + b_{i-1,u}a_{i,u} = 0 \) for all \( i \) and \( u \) in \( \mathbb{Z}/n\mathbb{Z} \) (the squares are anticommutative).

The algebra \( \Gamma_{\Lambda_n} \) is the quotient of the path algebra \( kQ_n \) by the ideal generated by these relations. We shall assume \( n \geq 2 \).

3 Hochschild and cyclic homologies of \( \Gamma_{\Lambda_n} \)

We are going to use the following theorem due to Happel to compute a minimal projective resolution of \( \Gamma_{\Lambda_n} \) as a \( \Gamma_{\Lambda_n} \)-bimodule (the situation in [H] is more general):

Theorem 3.1 ([H] 1.5) If

\[
\cdots \rightarrow R_p \rightarrow R_{p-1} \rightarrow \cdots \rightarrow R_1 \rightarrow R_0 \rightarrow \Gamma_{\Lambda_n} \rightarrow 0
\]
is a minimal projective resolution of $\Gamma_{\Lambda_n}$ as a $\Gamma_{\Lambda_n}$-bimodule, then

$$R_p = \bigoplus_{(i, u)} (\Gamma_{\Lambda_n} e_{j,v} \otimes e_{i,u} \Gamma_{\Lambda_n})^{\dim_k \text{Ext}^p_{\Lambda_n}(S_{i,u};S_{j,v})}.$$ 

In our case, we have:

**Corollary 3.2** The complex

$$\cdots \to 0 \to \bigoplus \{ (i, u) \} \Gamma_{\Lambda_n} e_{i-1,u-1} \otimes e_{i,u} \Gamma_{\Lambda_n} \to \bigoplus (\Gamma_{\Lambda_n} e_{i-1,u} \otimes e_{i,u} \Gamma_{\Lambda_n}) \oplus (\Gamma_{\Lambda_n} e_{i,u-1} \otimes e_{i,u} \Gamma_{\Lambda_n}) \to \Gamma_{\Lambda_n} \to 0$$

is a minimal projective resolution of $\Gamma_{\Lambda_n}$ as a $\Gamma_{\Lambda_n}$-bimodule.

**Proof:** We need to compute the Ext groups between the simple $\Gamma_{\Lambda_n}$-modules. First, let us compute the projective resolutions of the simple modules:

**Lemma 3.3** Let $P_{i,u}$ denote the indecomposable projective $\Gamma_{\Lambda_n}$-module at the vertex $e_{i,u}$, and let $S_{i,u} = \text{top}(P_{i,u})$ be the corresponding simple module. The minimal projective resolutions of the simple modules are:

$$0 \to P_{i,1-i} \to P_{i,1} \to S_{i,1} \to 0$$

$$0 \to P_{i,1-i-j-1} \oplus P_{i,1-j} \to P_{i,1-i-j} \to S_{i,1-j} \to 0$$

for $2 \leq j \leq n - 1$.

**Proof of Lemma:** We consider only the $S_{n-1,u}$, because the other cases may be obtained by translating the quiver along the cylinder on which it lies. It is then straightforward to compute their minimal projective resolutions. \qed

We can now compute the Ext groups:

**Lemma 3.4** Let $S$ be a simple $\Gamma_{\Lambda_n}$-module. Then:

$$\text{Ext}_{\Gamma_{\Lambda_n}}^0(S_{i,u};S) = \begin{cases} k & \text{if } S = S_{i,u}, \\ 0 & \text{if } S \neq S_{i,u}, \end{cases}$$

$$\text{Ext}_{\Gamma_{\Lambda_n}}^1(S_{i,i};S) = \begin{cases} k & \text{if } S = S_{i-1,i}, \\ 0 & \text{if } S \neq S_{i-1,i}, \end{cases}$$

$$\text{Ext}_{\Gamma_{\Lambda_n}}^1(S_{i,i-2};S) = \begin{cases} k & \text{if } S = S_{i,i-2}, \\ 0 & \text{if } S \neq S_{i,i-2}, \end{cases}$$

$$\text{Ext}_{\Gamma_{\Lambda_n}}^1(S_{i,i-1};S) = \begin{cases} k & \text{if } S = S_{i-1,i-1} \text{ or } S = S_{i-1,i-j}, \\ 0 & \text{if } S \neq S_{i-1,i-1} \text{ and } S \neq S_{i-1,i-j}, \end{cases} \quad \text{if } 1 \leq j \leq n - 1$$

$$\text{Ext}_{\Gamma_{\Lambda_n}}^2(S_{i,i-j};S) = 0$$

$$\text{Ext}_{\Gamma_{\Lambda_n}}^n(S_{i,u};S) = 0 \text{ if } p \geq 3.$$
Applying Happel’s Theorem (3.1) we get the minimal projective resolution for $\Gamma_{\Lambda_n}$.

Now applying the functor $\Gamma_{\Lambda_n} \otimes_{\Gamma_{\Lambda_n}} -$ to this resolution, we obtain a complex:

\[ \cdots 0 \rightarrow \cdots \rightarrow 0 \rightarrow kQ_0 \rightarrow 0. \]

Therefore:

**Proposition 3.5** The Hochschild homology of $\Gamma_{\Lambda_n}$ is:

\[
\begin{align*}
\text{HH}_0(\Gamma_{\Lambda_n}) &= kQ_0 \cong k^{n^2} \\
\text{HH}_p(\Gamma_{\Lambda_n}) &= 0 \quad \text{forall} \; p > 0,
\end{align*}
\]

and hence the cyclic homology of $\Gamma_{\Lambda_n}$ is:

\[
\begin{align*}
\text{HC}_{2p}(\Gamma_{\Lambda_n}) &= kQ_0 \cong k^{n^2} \\
\text{HC}_{2p+1}(\Gamma_{\Lambda_n}) &= 0 \quad \text{forall} \; p \geq 0.
\end{align*}
\]

**Remark 3.6** There doesn’t seem to be any connection between these results and those for $\Lambda_n$. Indeed, the Hochschild and cyclic homologies for the Taft algebras are given as follows (see [S] for the Hochschild homology and [T, T1] for the cyclic homology):

\[
\begin{align*}
\text{HH}_0(\Lambda_n) &= k^n \\
\text{HH}_p(\Lambda_n) &= k^{n-1} \quad \text{forall} \; p > 0.
\end{align*}
\]

and

\[
\begin{align*}
\text{HC}_{2p}(\Lambda_n) &= k^n, \\
\text{HC}_{2p+1}(\Lambda_n) &= k^{n-1} \quad \text{forall} \; p \geq 0.
\end{align*}
\]

4 **Chern characters of $\Lambda_n$ and $\Gamma_{\Lambda_n}$**

Let $K_0(\Lambda_n)$ (resp. $K_0(\Gamma_{\Lambda_n})$) be the Grothendieck group of projective $\Lambda_n$-modules (resp. $\Gamma_{\Lambda_n}$-modules). We are interested in the Chern characters $\text{ch}_{0,p} : K_0(\Lambda_n) \rightarrow \text{HC}_{2p}(\Lambda_n)$ (resp. $K_0(\Gamma_{\Lambda_n}) \rightarrow \text{HC}_{2p}(\Gamma_{\Lambda_n})$). We shall write $[P_i]$ (resp. $[P_{i,u}]$) for the isomorphism class of the projective module at the vertex $e_i$ (resp. $e_{i,u}$).

Set $\sigma^p = (y_p, z_p, \ldots, y_1, z_1, y_0) \in \mathbb{N}^{2p+1}$ with $y_p = (-1)^p(2p)!/p!$ and $z_p = (-1)^{p-1}(2p)!/2(p!)$. There is a system of generators of $\text{HC}_{2p}(\Lambda_n)$ (resp. $\text{HC}_{2p}(\Gamma_{\Lambda_n})$) given by the following set:

\[
\{ \sigma^p_i := \sigma^p(e_i, \ldots, e_i) \in (\text{Tot} \; CC(\Lambda_n))_{2p} \mid i = 0, \ldots, n - 1 \}
\]

(resp. by $\sigma^p_{i,u} := \sigma^p(e_{i,u}, \ldots, e_{i,u}) \in (\text{Tot} \; CC(\Gamma_{\Lambda_n}))_{2p} \mid i, u \in \{0, 1, \ldots, n - 1\}$).

Consider the elements

\[
\begin{align*}
\epsilon_j : & \Lambda_n \rightarrow \Lambda_n \quad \text{and} \quad \epsilon_{i,u} : \Gamma_{\Lambda_n} \rightarrow \Gamma_{\Lambda_n} \\
\lambda & \mapsto \lambda e_j \\
\lambda & \mapsto \lambda e_{i,u}
\end{align*}
\]

in $\mathcal{M}_1(\Lambda_n)$ and $\mathcal{M}_1(\Gamma_{\Lambda_n})$; their ranges are the corresponding projective modules. Then by definition of the Chern characters (see [L] 8.3.4), we have:

\[
\begin{align*}
\text{ch}_{0,p}([P_j]) &= \text{ch}_{0,p}([\epsilon_j]) := \text{tr}(\epsilon([\epsilon_j])) = \sigma^p_j \quad \text{in} \; \text{HC}_{2p}(\Lambda_n) \\
\text{ch}_{0,p}([P_{i,u}]) &= \text{ch}_{0,p}([\epsilon_{i,u}]) = \sigma^p_{i,u} \quad \text{in} \; \text{HC}_{2p}(\Gamma_{\Lambda_n}).
\end{align*}
\]
using the isomorphisms $\mathcal{M}_m(\Lambda) \cong \mathcal{M}_m(k) \otimes \Lambda$. Here,
\[ c(e) = (y_0e_2^{2p+1}, \ldots, y_0e_1^{2p}) \in \mathcal{M}(\Gamma_n) \otimes \cdots \otimes \mathcal{M}(\Gamma_n). \]

**Remark 4.1** There is a decomposition formula for the tensor product of indecomposable modules on $\Lambda_n$ (see [C2, G]). From this formula, we get inductively:
\[ ch_{0,p}([L_1] \otimes \cdots \otimes [L_r]) = \frac{1}{n^r} \prod_{i=1}^r (\dim L_i) (\sigma_{\theta_1}^p, \ldots, \sigma_{\theta_n}^p), \text{ for } r \geq 2, \]
where the $L_i$ are arbitrary projective $\Lambda_n$-modules. Unfortunately, this product in the cyclic homology doesn’t seem natural.

**Remark 4.2** Let $\mathcal{K}_0(\Lambda_n)$ be the Grothendieck group of all $\Lambda_n$-modules (not just the projective ones). Then $\mathcal{K}_0(\Lambda_n) \cong K_0(\Gamma_n)$. Hence, if $N_{i,u}$ is the indecomposable $\Lambda_n$-module which starts at the vertex $i$ and ends at the vertex $u$, it corresponds to the projective $\Gamma_n$-module $P_{i,u}$, and we get a map:
\[ \mathcal{K}_0(\Lambda_n) \longrightarrow HC_{2p}(\Gamma_n) \]
\[ N_{i,u} \mapsto \sigma_{\theta_{i,u}}^p. \]

**Remark 4.3** Although $\Gamma_n$ is not a Hopf algebra, its Grothendieck group $K_0(\Gamma_n)$ does have a ring structure, which does not appear to be natural: for every $[P]$ in $K_0(\Gamma_n)$, there exists a $[B]$ in $K_0(\Lambda_n)$ such that $[P] = [\text{Hom}_{\Lambda_n}(M, B)]$, where $M$ is the sum of all isomorphism classes of indecomposable $\Lambda_n$-modules. If $[Q] = [\text{Hom}_{\Lambda_n}(M, C)]$ is another element in $K_0(\Gamma_n)$, we can set
\[ [P] \cdot [Q] = [\text{Hom}_{\Lambda_n}(M, B \otimes_k C)] \]
(the vector space $B \otimes_k C$ is a $\Lambda_n$-module since $\Lambda_n$ is a Hopf algebra). In fact, using the decomposition in [C2, G], the product can be written:
\[ [P_{i,u}] [P_{j,v}] = \begin{cases} \sum_{l=0}^{u-i} [P_{i+j+l, u+v+l}] & \text{if } u + v - (i + j) \leq n - 1 \\ \sum_{l=0}^{v-j} [P_{i+j+l, u+v+l}] + \sum_{m=e+1}^{u-i} [P_{i+j+m, u+v-m}] & \text{if } e := u + v - (i + j) - (n - 1) \geq 0 \end{cases} \]
where $u$ and $v$ represent elements in $\mathbb{Z}/n\mathbb{Z}$ such that $u - i$ and $v - j$ are in $\{0, 1, \ldots, n - 1\}$.

**References**


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