Dynamic Multilevel Methods and Non-Homogeneous Turbulence

F. Bouchon, T. Dubois and F. Jauberteau

Laboratoire de Mathématiques Appliquées, Université Blaise Pascal and CNRS (UMR 6620),
63177 Aubière cedex, FRANCE
email: *bouchon, thdubois, jauber@ucfma.univ-bpclermont.fr*

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Abstract. Multilevel methods have been used in the numerical simulation of turbulent flows. The separation of scales can lead to different strategies, such as large eddy simulation or adaptative schemes for example. The large eddy simulation propose to resolve the large scale equation, by modeling the subgrid stress tensor. Multilevel methods propose a different approach: by analyzing the time and space behavior of the different scales, we propose to compute them differently.

In this paper, we describe the strategy in the case of non-homogeneous turbulence (the channel flow problem), after giving some results for the one-dimensional Burgers equation.

1 Introduction

Dynamic multilevel (DML) method have been applied to the numerical simulation of periodic flows (homogeneous turbulence).The strategy described in [Dubois] consists in using a spatial filtering $y = P(u)$ and a separation of scales $u = y + z$ such that both the large (y) and the small (z) scales satisfy the boundary conditions. Moreover, the projector P satisfies nice properties (such as commutativity with the spatial derivatives), which simplify the theoretical study and the numerical implementation. This strategy was not only used in the case of periodic flows, but either for the simulation of non-homogeneous turbulence (the channel flow problem) for which a multilevel methodology was used in the two periodic directions.

Here, we propose to develop algorithms using a separation of scales in the non-periodic direction. We study the case of the Chebyshev-Galerkin basis described in [Shen (95-2)], obtained as linear combinations of the Chebyshev polynomials.

We apply this basis to the numerical approximation of the channel flow problem. We first describe some numerical results, and comparisons with direct numerical simulations (DNS) for the two periodic directions (section 3). In section 4 we consider the one-dimensional Burgers equation with Dirichlet boundary conditions as a test model for the treatment of the direction normal to the wall. We present numerical results for this problem. Then we consider the non-homogeneous direction in the channel flow problem.

2 The 3D channel flow problem: equations and numerical approximation

We consider 3D flows driven by a constant pressure gradient in an infinite channel. The velocity field satisfy the incompressible Navier-Stokes equations supplemented with periodic conditions in the streamwise x_1 and spanwise x_3 directions and homogeneous Dirichlet boundary conditions in the direction x_2 normal to the walls. For this problem, the main difficulty is to correctly reproduce the mean velocity profile which strongly depends on the boundary layers appearing near the walls.

Let us note $\mathbf{u} = (u_1, u_2, u_3)$ the velocity field and $\Lambda = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ the nonlinear term. Following [Kim (87)], to resolve this problem we solve the second order equation associated with $g = \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1}$ (the component of the vorticity normal to the wall) and the fourth order equation associated with u_2

$$\frac{\partial g}{\partial t} - \nu \Delta g + h_g(\mathbf{u},\mathbf{u}) \;=\; 0 \tag{1}$$

and

$$\frac{\partial \Delta u_2}{\partial t} - \nu \Delta^2 u_2 + h_{u_2}(\mathbf{u},\mathbf{u}) \;=\; 0. \tag{2}$$

In (1) and (2), the nonlinear terms are defined as

$$h_g(\mathbf{u},\mathbf{u}) \;=\; \frac{\partial \Lambda_1}{\partial x_3} - \frac{\partial \Lambda_3}{\partial x_1}$$

and

$$h_{u_2}(\mathbf{u},\mathbf{u}) \;=\; -\frac{\partial}{\partial x_1}\left(\frac{\partial \Lambda_1}{\partial x_2} - \frac{\partial \Lambda_2}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial \Lambda_2}{\partial x_3} - \frac{\partial \Lambda_3}{\partial x_2}\right).$$

For the equation (2), the boundary conditions on $\dfrac{\partial u_2}{\partial x_2}$ are obtained by imposing the continuity equation at the walls. Then, knowing u_2 and g, we deduce the two other components of the velocity using the continuity equation. Finally, the pressure is obtained from the equation associated with $f = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_3}{\partial x_3}$, namely

$$\frac{\partial f}{\partial t} - \nu \Delta f + h_f(\mathbf{u},\mathbf{u}) + \left(\frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_3^2}\right) p \;=\; 0.$$

The spatial discretization is obtained with a spectral method. However, instead of using a tau method as in [Kim (87)], we apply a Galerkin method. Indeed, the tau method is unstable for fourth order problems. Moreover, the tau method does not allow to separate the small and large scales, which is at the basis of the multilevel methods. The Galerkin method proposed to solve the equations (1) and (2) uses the Galerkin basis developed in [Shen (95-1)] and [Shen (95-2)]. These basis are obtained as linear combination of the Chebyshev or Legendre polynomials. For the time integration, a Crank-Nicholson scheme is used for the linear terms and a third order explicit Runge-Kutta scheme for the nonlinear terms (the second order Adams-Bashforth scheme being less stable when the viscosity is decreased).

3 Multilevel methods applied in periodic directions. Numerical results

The computational domain is $\Omega = (0, L_1) \times (-h, +h) \times (0, L_3)$ with $h = 1$, $L_1 = 4\pi$ (streamwise direction) and $L_3 = 2\pi$ (spanwise direction). The boundary conditions are periodic in the streamwise and spanwise directions. As for the direction normal to the walls, we retained a (no-slip) homogeneous Dirichlet boundary condition. The flow is sustained by a mean pressure gradient K_p. Here we have chosen $K_p = -1$. The channel lengths L_1 and L_3 must be chosen in order to insure that the turbulence fluctuations are uncorrelated at a separation of one half-period in the two homogeneous directions. The viscosity is chosen equal to $\nu = 1/180 = 5.5 \times 10^{-3}$. The Reynolds number based on the friction velocity $u_\star = 1$, $\mathrm{Re}_\star = hu_\star/\nu$, is equal to 180. The number of grid points retained in each direction for this simulation is $128 \times 129 \times 128$, i.e. $N = 128$ (periodic directions) and $M = 128$ (non periodic direction). Following the algorithms previously developed in the case of periodic flows, a multilevel method has been applied in the two periodic directions. One cut-off level have been chosen equal to $N_1 = 64 < N$. A quasi-static (QS) approximation (see [Dubois]) is used to compute the small scales and the nonlinear interaction terms. The multilevel method uses approximately 2.5 less CPU time than the DNS code.

In order to compare with the results of direct numerical simulation, several quantities have been computed: one-dimensional energy spectra, mean-velocity profile, root-mean-square of the fluctuations of the velocity, of the the pressure and of the vorticity, Reynolds shear stress, skewness and flatness factors. Let us denote $\langle - \rangle$ the spatial average in the two periodic directions x_1 and x_3. On Fig: 1 (left) we present the mean-velocity profile $\langle u_1 \rangle (x_2)$ normalized with the bulk mean velocity $U_m = \frac{1}{2h} \int_{-h}^{+h} \langle u_1 \rangle dx_2$, computed with DNS and multilevel method. In the same manner, on Fig:1 (right) we present the root-mean-square velocity fluctuations (turbulence intensity) $u_{i,\mathrm{rms}} = \langle u_i'^2 \rangle^{1/2}$, $i = 1, 2, 3$ normalized with the friction velocity $u_\star$, computed with the two methods. We have noted $u_i' = u_i - \langle u_i \rangle$ the velocity fluctuations. On Fig: 2 we compare the skewness $F_3(u_i') = \langle u_i'^3 \rangle / \langle u_i'^2 \rangle^{3/2}$ and the flatness $F_4(u_i') = \langle u_i'^4 \rangle / \langle u_i'^2 \rangle^2$ factors for the fluctuations quantities u_i', computed with DNS and multilevel method. Firstly, the DNS results are in agreement with the results presented in [Kim (87)]. Then we can see that the DNS and multilevel profiles are very similar throughout the width of the channel.

4 Multilevel methods applied in non-homogeneous direction. Numerical results

In order to apply a multilevel method in the non-homogeneous direction, an algorithm has been developed for the one-dimensional Burgers equation, with homogeneous Dirichlet boundary conditions. Using the Chebyshev-Galerkin basis previously introduced, we separate the velocity field into large and small scales. Both, the large and small scales satisfy the boundary conditions. A cut-off level is defined and, as previously, a QS approximation is used to compute the small scales and the nonlinear interaction terms. However, in opposition with the case of Fourier polynomials, the projection operator does not commute with the partial differential operators. This lack of commutativity between projection and differentiation causes every spatial derivative operator in the Navier-Stokes equations to generate terms that cannot be expressed solely in terms of the projected field. Therefore, a closure problem is introduced not only for the nonlinear terms but for the linear terms as well. This closure is achieved with a QS approximation. On Fig: 3, we present the $L^2([0,T], L^2_\omega[-1,+1])$ error between the exact solution and the solution computed with a classical Chebyshev-Galerkin method. We recall that $|u|_{L^2_\omega[-1,+1]} = \int_{-1}^{+1} |u(x_2)|^2 w(x_2) dx_2$ with $w(x_2) = (1 - x_2^2)^{-1/2}$ the weight function in the Chebyshev case. On Fig: 4, we report the error, for the same norm, obtained with the multilevel method for $M = 100$. We used four different cut-off levels $M_1 < M$. The parameter p is the number of iterations during which the QS approximation is performed.

A multilevel method has been applied in the non-homogeneous direction of the three-dimensional channel flow. The QS approximation is used to compute the small scales. Numerical results has been obtained. On Fig: 5 (left) we present the mean-velocity profile $\langle u_1 \rangle(x_2)$ (left) and the Reynolds shear stress $\langle u_1' u_2' \rangle(x_2)$ (right) computed with DNS and multilevel method. As we can see, the results are very similar.

5 Conclusion

In this paper, we have studied the application of the multilevel methodology in the case of the non-homogeneous turbulence.

Firstly, we study the use of a multilevel method in the two periodic directions. As for the homogeneous case [Dubois], the numerical results obtained show that the QS approximation is well suited for the small scales approximation.

Then we study a multilevel method applied in the non homogeneous direction in the case of the one-dimensional Burgers equation. The numerical results confirmed that QS approximation of the small scales and of the interaction terms appearing in the large scale equation correctly reproduce the action of the small scales on the large ones during the time evolution. Such a method is actually studied for the non-homogeneous direction in the channel flow problem. QS approximation of the small scales gives good results. However, some numerical instabilities appear with QS approximation of the linear and nonlinear interaction terms in the large scale equation. More elaborate closure on these terms are actually tested. This work is in progress.

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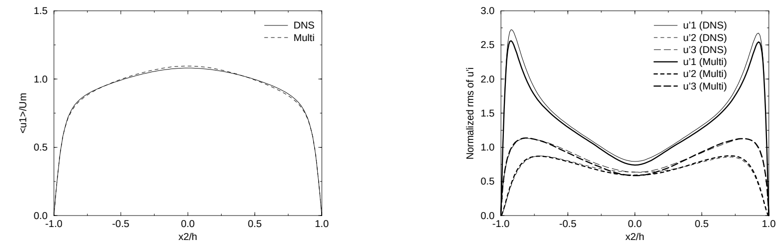

Fig. 1. The mean-velocity profile non-dimensionalized by the bulk mean velocity (left) and the root-mean-square velocity fluctuations (right) in global coordinates x_2/h, computed with DNS and multilevel method

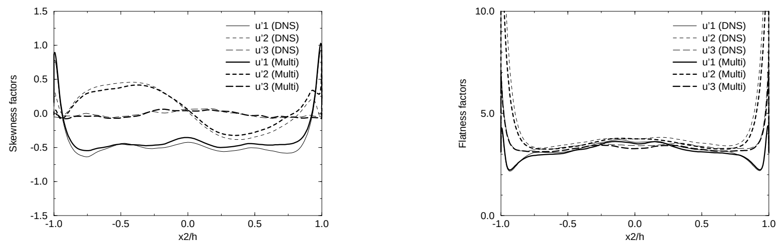

Fig. 2. The skewness $F_3(u_i')$(left) and flatness $F_4(u_i')$ (right) factors, $i = 1, 2, 3$, in global coordinates x_2/h, computed with DNS and multilevel method

M	20	40	60	80	100
Error	$1,2411.10^{-5}$	$9.4279.10^{-7}$	$1,9930.10^{-7}$	$6,5301.10^{-8}$	$2,7387.10^{-8}$
CPU	297s	385s	575s	675s	860s

Fig. 3.

$p\backslash M_1$	20	40	60	80
50	$2,4847.10^{-7}$	$2,8164.10^{-8}$	$2,7399.10^{-8}$	$2,7388.10^{-8}$
100	$3,7112.10^{-7}$	$2,9232.10^{-8}$	$2,7415.10^{-8}$	$2,7388.10^{-8}$
200	$5,1161.10^{-7}$	$3,1613.10^{-8}$	$2,7451.10^{-8}$	$2,7389.10^{-8}$
400	$6,7305.10^{-7}$	$3,5757.10^{-8}$	$2,7536.10^{-8}$	$2,7392.10^{-8}$

Fig. 4.

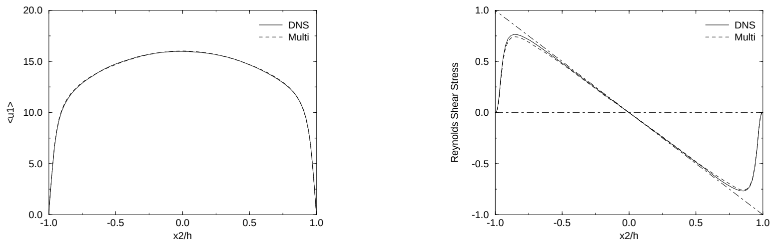

Fig. 5. The mean-velocity profile (left) and the Reynolds shear stress tensor (right) in global coordinates x_2/h, computed with DNS and multilevel method