A Model for Two-Phase Fluid Flow in Porous Media With Desorption

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Aim:

Numerical simulation of natural gas recovery of type CBM (Coal Bed Methane) with the following features:

- Modelling of immiscible two-phase fluid flow in porous media (water + gas)
- Gas is recovered by desorption from coalbed matrices
- Model for 2-D configurations
- Capillary pressure is neglected
- Numerical approximation by finite elements

Modelling for immiscible two-phase flows in porous media is (more or less) classical in reservoir engineering.
New: Modelling of desorption

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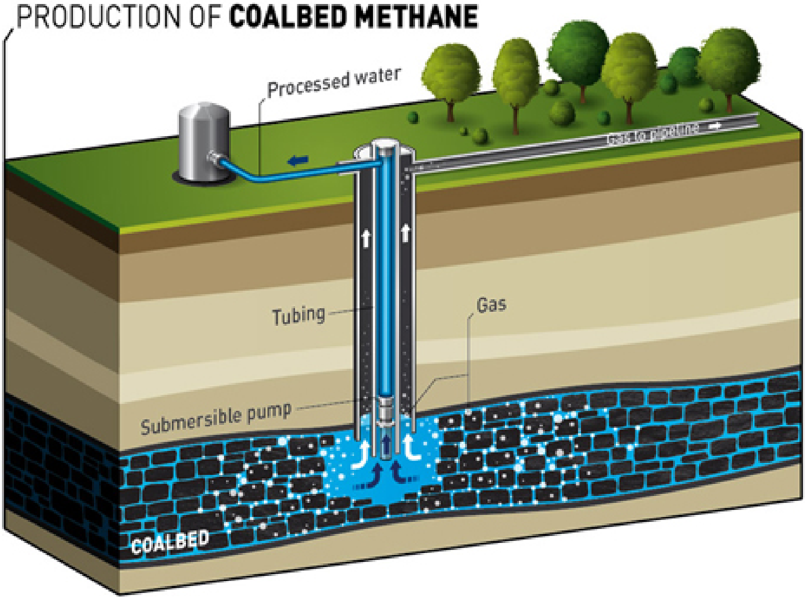
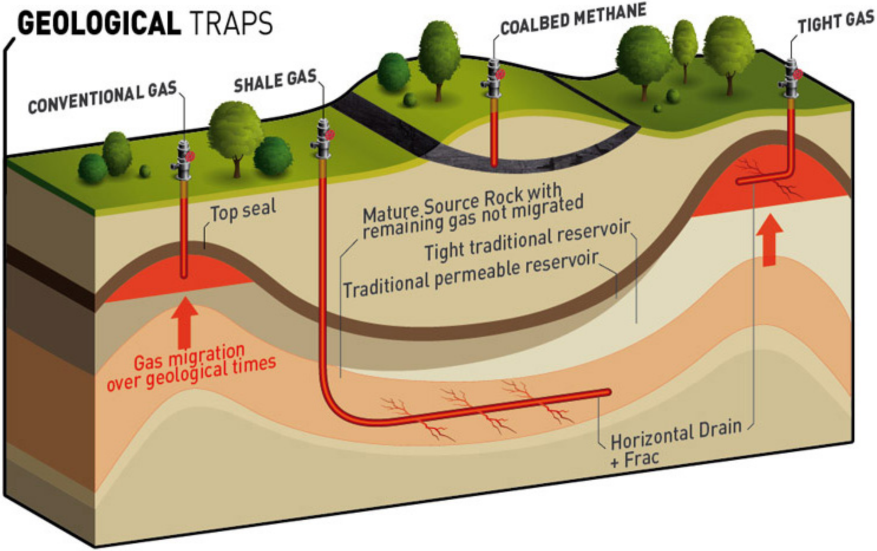
Principle of the CBM process




GEOLOGICAL TRAPS
CONVENTIONAL GAS
SHALE GAS
COALBED METHANE
TIGHT GAS
Top seal
Mature Source Rock with remaining gas not migrated
Tight traditional reservoir
Traditional permeable reservoir
Gas migration over geological times
Horizontal Drain + Frac

We consider a flow in porous medium of an immiscible mixture of water and gas.
Let S_w and S_g stand for the respective saturations of water and gas:

$$S_w + S_g = 1.$$

Mass conservation for each phase:

$$\frac{\partial}{\partial t}(\phi \varrho_w S_w) + \nabla \cdot (\varrho_w v_w) = 0,$$
$$\frac{\partial}{\partial t}(\phi \varrho_g S_g) + \nabla \cdot (\varrho_g v_g) = f_D,$$

where:

ϱ_w, ϱ_g	densities (water and gas)
ϕ	Porosity ($0 < \phi_0 \le \phi(x) \le 1$)
f_D	Rate of desorbed gas

In the following $S = S_w$ ($S_g = 1 - S$).

The model

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Darcy equation:

$$\mathbf{v}_w = -\frac{k_w}{\mu_w}\mathbf{K}\nabla p_w, \quad \mathbf{v}_g = -\frac{k_g}{\mu_g}\mathbf{K}\nabla p_g$$

where

$\mathbf{v}_w$, $\mathbf{v}_g$	Velocity of water and gas
p_w, p_g	Pressures
k_w, k_g	Relative permeabilities
μ_w, μ_g	Viscosities
$\mathbf{K}$	Absolute permeability tensor (assumed diagonal)

In general, we assume:

$$k_w = k_w(S), \ k_g = k_g(S).$$

We define the *capillary pressure* : $p_c(S) = p_g - p_w$.
The function $p_c(S)$ is assumed positive and non increasing.
We next define the mobilities:

$$m_w(S) = \frac{k_w(S)}{\mu_w}, \ m_g(S) = \frac{k_g(S)}{\mu_g}, \ m(S) = m_w(S) + m_g(S)$$

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The total velocity is defined by

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_g$$
$$= -m(S)\mathbf{K}\left(\nabla p_g - \frac{m_w(S)}{m(S)}\nabla p_c(S)\right)$$

We now want to define a global pressure: Let $\tilde{p}(S)$ be defined by

$$\tilde{p}'(S) = \frac{m_w(S)}{m(S)}\, p_c'(S)$$

The function $p = p_g - \tilde{p}$ satisfies then:

$$\nabla p = \nabla p_g - \nabla \tilde{p}$$
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Thus

$$\mathbf{v} = -m(S)\mathbf{K}\nabla p,$$

and

$$\begin{aligned}
\mathbf{v}_w &= -m_w(S)\mathbf{K}\nabla p - \alpha(S)\mathbf{K}\nabla S,\\
\mathbf{v}_g &= -m_g(S)\mathbf{K}\nabla p + \alpha(S)\mathbf{K}\nabla S,
\end{aligned}$$

where

$$\alpha(S) = -\frac{m_w(S)m_g(S)}{m(S)}p_c'(S) \geq 0.$$

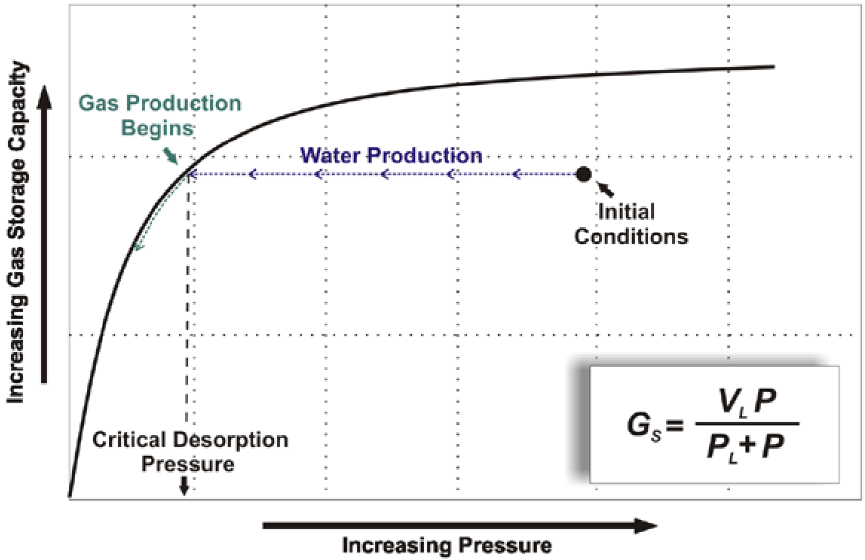
Modelling of desorption

Let V denote the adsorbed gas volume. We have, at equilibrium, the Langmuir isotherm:

$$V = \frac{V_L p}{p_L + p},$$

where:

- p_L: Langmuir adsorption constant
- V_L: Available gas volume


Increasing Gas Storage Capacity
Gas Production
Begins
Water Production
Initial
Conditions
Critical Desorption
Pressure
$G_S = \frac{V_L P}{P_L + P}$
Increasing Pressure

In a thermodynamical nonequilibrium situation, we have

$$\frac{\partial V}{\partial t} = -\frac{1}{\tau}\Big(V - \frac{V_L p}{p_L + p}\Big)$$

where $\tau > 0$ if a diffusion time.

Note that if we want to exclude adsorption, we must replace by

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We have the system of equations:

$$\frac{\partial}{\partial t}(\phi\varrho_w S) - \nabla\cdot(\varrho_w m_w \mathbf{K}\nabla p) - \nabla\cdot(\varrho_w \alpha \mathbf{K}\nabla S) = 0$$

$$\frac{\partial}{\partial t}(\phi\varrho_g (1-S)) - \nabla\cdot(\varrho_g m_g \mathbf{K}\nabla p) + \nabla\cdot(\varrho_g \alpha \mathbf{K}\nabla S) = -\varrho_m\varrho_b\frac{\partial V}{\partial t}$$

$$\frac{\partial V}{\partial t} + \frac{1}{\tau}\left(V - \frac{V_L p}{p_L + p}\right) = 0$$

It remains to define an equation of state for each phase.

We assume that the water is slightly compressible, *i.e.*

$$c_w = \frac{1}{\varrho_w}\frac{d\varrho_w}{dp} = \text{Const.} > 0$$

In the same way, we define the gas and rock compressibility coefficients by

$$c_g = c_g(p) = \frac{1}{\varrho_g}\frac{d\varrho_g}{dp}, \qquad c_f = c_f(p) = \frac{1}{\phi}\frac{d\phi}{dp}$$

We thus obtain

$$\frac{\partial\varrho_w}{\partial t} = c_w\varrho_w\frac{\partial p}{\partial t}, \quad \nabla\varrho_w = c_w\varrho_w\nabla p$$

The complete model

We have the system of equations:

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\begin{aligned}
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Neglecting nonlinear quadratic terms and dividing by ϱ_w we get:

$$\phi \frac{\partial S}{\partial t} + (c_w + c_f)\phi S \frac{\partial p}{\partial t} - \nabla \cdot (m_w \mathbf{K} \nabla p) - \nabla \cdot (\alpha \mathbf{K} \nabla S) = 0$$

For the gas, we consider a real gas model:

$$p = \varrho_g RTZ(p), \qquad \text{où} \quad 0 < Z(p) \leq 1.$$

Therefore

$$c_g(p) = \frac{1}{RT} \frac{Z(p) - Z'(p)p}{pZ^2(p)}$$

The equation of gas becomes:

$$\begin{aligned}\frac{\partial}{\partial t}(\phi(1-S)\varrho_g) &= \phi(1-S)\frac{\partial \varrho_g}{\partial t} + \phi\varrho_g(1-S)\frac{\partial \phi}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t} \\ &= \varrho_g(1-S)(c_g + c_f)\phi\frac{\partial p}{\partial t} - \phi\varrho_g\frac{\partial S}{\partial t}\end{aligned}$$

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$$c_t\phi\frac{\partial p}{\partial t} - \nabla\cdot\left(m\mathbf{K}\nabla p\right) = \frac{\varrho_m\varrho_b}{\tau\varrho_g}\left(V - \frac{V_L p}{p_L + p}\right)$$

$$\frac{\partial V}{\partial t} + \frac{1}{\tau}\left(V - \frac{V_L p}{p_L + p}\right) = 0$$

where

$$c_t = c_w S + c_g(1-S) + c_f \qquad \text{Total compressibility}$$

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Adding these two equations we obtain:

$$\phi\frac{\partial S}{\partial t} + (c_w\phi + c_f)S\,\frac{\partial p}{\partial t} - \nabla\cdot(m_w\mathbf{K}\nabla p) - \nabla\cdot(\alpha\mathbf{K}\nabla S) = 0$$
$$c_t\phi\frac{\partial p}{\partial t} - \nabla\cdot\big(m\mathbf{K}\nabla p\big) = \frac{\varrho_m\varrho_b}{\tau\varrho_g}\Big(V - \frac{V_L p}{p_L + p}\Big)$$
$$\frac{\partial V}{\partial t} + \frac{1}{\tau}\Big(V - \frac{V_L p}{p_L + p}\Big) = 0$$

where

$$c_t = c_w S + c_g(1-S) + c_f \qquad \text{Total compressibility}$$

Remarks

1. The main advantage of this formulation is that

$$m \geq m_0 > 0 \quad \text{although } m_g \geq 0, \quad m_w \geq 0.$$

   i.e. the equation is not degenerate.

2. We have

$$\nabla \cdot (m_w \mathbf{K} \nabla p) = m_w \nabla \cdot (\mathbf{K} \nabla p) + \mathbf{K} \nabla p \cdot \nabla m_w = m_w \nabla \cdot (\mathbf{K} \nabla p) + m_w'(S) \mathbf{K} \nabla p \cdot \nabla S$$

   which is a diffusion-convection problem. This implies the necessity of using an upwind scheme, if the capillary pressure is null (or small enough).

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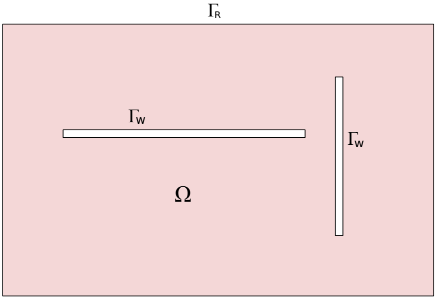

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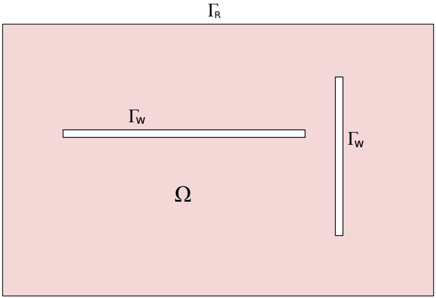

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In realistic situations, the domain (reservoir) contains wells with a small diameter (with respect to the reservoir's diameter). This is generally at the origin of serious numerical difficulties.

Consider, for instance, the case of a vertical well. We assume that the flow is radial in the vicinity of the well. We also assume that the flow is incompressible in this neighborhood and has constant properties.

The flow is then modelled in this neighborhood, for the water phase by

$$-\nabla \cdot (\varrho_w m_w \mathbf{K} \nabla p) = q_w \delta$$

where δ is the Dirac distribution at the center of the well and q_w is the well's production rate for the water.

We obtain the analytical solution

$$p(r) = p(r_w) - \frac{q_w}{2\pi \varrho_w m_w \kappa H} \ln\left(\frac{r}{r_w}\right), \qquad r = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

where r_w is the well's radius, $\kappa = K_{11}$ and H is the reservoir's height.

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Well modelling

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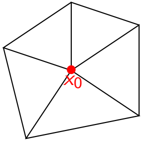
Let φ_0 denote the $\mathbb{P}_1$ basis function at node x_0 (well node), we have

$$\varrho_w m_w H \sum_{e \subset \Omega_0} \int_e \mathbf{K}\nabla p \cdot \varphi_0 \, dx = q_w$$

where Ω_0 is the support de φ_0.



We assume that the analytical solution is a good approximation of the pressure at neighboring nodes.

Using the expansion

$$p = \sum_i p_i \varphi_i \qquad \text{in } \Omega_0$$

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$$\varrho_w m_w H \sum_{e \subset \Omega_0} \sum_i \Big(\int_e \mathbf{K}\varphi_i \cdot \nabla \varphi_0 \, dx \Big) p_i = q_w.$$

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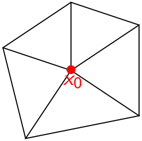

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Well model for the water phase

$$q_w = \frac{\sum_{i\neq 0} T_i}{1 + \frac{1}{2\pi\kappa}\sum_{i\neq 0} T_i \ln(r_i/r_w)} \varrho_w m_w H(p_w - p_0)$$

For the gas phase, the situation is more delicate: One cannot assume that ϱ_g is constant in the vicinity of a well.

We use the Kirchhoff transformation by defining

$$\tilde{p} = \int_{p_0}^{p} \varrho_g(s)\, ds$$

Then, we have

$$-m_g \kappa \Delta p = q_g \delta$$

We deduce then

$$\tilde{p}(r) = \tilde{p}(r_w) - \frac{q_g}{2\pi m_g \kappa H} \ln\left(\frac{r}{r_w}\right)$$

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We use a $\mathbb{P}_1$ finite element method with a *Streamline Upwind* stabilization term:
Let $\mathscr{T}(\Omega)$ denote a triangulation of Ω and let us define the finite dimensional space:

$$\mathcal{S} = \{\psi \in \mathcal{C}^0(\overline{\Omega});\ \psi_{|K} \in \mathbb{P}_1\ \forall\ K \in \mathscr{T}(\Omega)\},$$
$$\mathcal{P} = \{q \in \mathcal{C}^0(\overline{\Omega});\ q_{|K} \in \mathbb{P}_1\ \forall\ K \in \mathscr{T}(\Omega)\},$$
$$\mathcal{V} = \{W;\ W_{|K} = \text{Const.}\ \forall\ K \in \mathscr{T}(\Omega)\}.$$

Consider the diffusion-convection equation

$$-\varepsilon \Delta u + \mathbf{a} \cdot \nabla u = f \qquad \text{in } \Omega$$

It is well known that if the local Péclet number

$$Pe = \frac{|\mathbf{a}|h}{2\varepsilon} > 1$$

then a standard (centered) discretization leads to a nonmonotone matrix and then to instabilities.

To remedy to this, a Petrov-Galerkin formulation has been proposed in the 80's by T.J.R. Hughes *et al.* and analyzed by C. Johnson. It consists in the following variational formulation:

$$\int_\Omega \varepsilon \nabla u_h \cdot \nabla v \, dx + \int_\Omega (\mathbf{a} \cdot \nabla u_h) v \, dx + \sum_K \frac{h_K}{2|\mathbf{a}|} \int_K (\mathbf{a} \cdot \nabla u_h)(\mathbf{a} \cdot \nabla v) \, dx = \int_\Omega f v \, dx \quad \forall\, v \in \mathscr{V}_h$$

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We define the variational formulation (We keep the same notation for the unknowns and their approximations):
We seek $S(\cdot,t)\in\mathcal{S}$, $p(\cdot,t)\in\mathcal{P}$ and $V(t)\in\mathcal{V}$ such that forl all $\varphi\in\mathcal{S}$ et $\psi\in\mathcal{P}$:

$$
\begin{aligned}
&\int_\Omega \phi\frac{\partial S}{\partial t}\varphi\,dx+\int_\Omega (c_w\phi+c_f)S\frac{\partial p}{\partial t}\varphi\,dx+\int_\Omega m_w\mathbf{K}\nabla p\cdot\nabla\varphi\,dx\\
&\qquad+\sum_{K\in\mathscr{T}(\Omega)}\xi_K\int_K(\mathbf{K}\nabla p\cdot\nabla S)(\mathbf{K}\nabla p\cdot\nabla\varphi)\,dx=-\sum_{i=1}^{n_w}\frac{q_{wi}}{H}\varphi(x_{wi})\\
&\int_\Omega c_t\phi\frac{\partial p}{\partial t}\psi\,dx+\int_\Omega m\mathbf{K}\nabla p\cdot\nabla\psi\,dx\\
&\qquad=\frac{\varrho_m\varrho_b}{\tau}\int_\Omega\frac{1}{\varrho_g}\left(V-\frac{V_L\,p}{p_L+p}\right)\psi\,dx-\sum_{i=1}^{n_w}\frac{q_{gi}}{H}\varphi(x_{wi})\\
&\frac{\partial V}{\partial t}+\frac{1}{\tau\varrho_g}\left(\frac{V_L\,p}{p_L+p}-V\right)=0
\end{aligned}
$$

with

$$\xi_K=\frac{h_K}{2|\mathbf{K}\nabla p|}|m_w'(S)|$$

We choose the implicit Euler scheme:

$$
\begin{aligned}
&\frac{1}{\delta t}\int_\Omega \phi^{n+1}(S^{n+1}-S^n)\varphi\,dx + \frac{1}{\delta t}\int_\Omega (c_w\phi^{n+1}+c_f^{n+1})(p^{n+1}-p^n)\varphi\,dx \\
&\qquad + \int_\Omega m_w^{n+1}\mathbf{K}\nabla p^{n+1}\cdot\nabla\varphi\,dx \\
&\qquad + \sum_{K\in\mathscr{T}(\Omega)} \xi_K^n \int_K (\mathbf{K}\nabla p^n\cdot\nabla S^{n+1})(\mathbf{K}\nabla p^n\cdot\nabla\varphi)\,dx = -\sum_{i=1}^{n_w}\frac{q_{wi}^{n+1}}{H}\varphi(x_{wi})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int_\Omega c_t^{n+1}\phi^{n+1}(p^{n+1}-p^n)\psi\,dx + \int_\Omega m^{n+1}\mathbf{K}\nabla p^{n+1}\cdot\nabla\psi\,dx \\
&\qquad = \frac{\varrho_m\varrho_b}{\tau+\delta t}\int_\Omega \frac{1}{\varrho_g^{n+1}}\left(V^n-\frac{V_L\,p^{n+1}}{p_L+p^{n+1}}\right)\psi\,dx - \sum_{i=1}^{n_w}\frac{q_{gi}^{n+1}}{H}\varphi(x_{wi})
\end{aligned}
$$

$$
V^{n+1} = \frac{1}{\tau+\delta t}\left(\tau V^n + \delta t\,\frac{V_L\,p^{n+1}}{p_L+p^{n+1}}\right)
$$

for all $\varphi\in\mathcal{S}$ and $\psi\in\mathcal{P}_0$.

Note that the variable V is decoupled from S and p.

In order to optimize the computational time, an adaptive time stepping procedure is used. We use the following procedure:

For all n, we compute

$$\alpha^n = \frac{\delta t^n}{\varepsilon}\left(\frac{\|p^{n+1} - p^n\|}{\|p^n\|} + \frac{\|S^{n+1} - S^n\|}{\|S^n\|}\right)$$

On choisit

$$\delta t^{n+1} = \begin{cases} \min\left(\theta, \dfrac{\alpha^n}{\delta t_n}\right)\delta t^n & \text{if } \alpha^n > \delta t^n \\ \dfrac{\delta t^n}{\min\left(\theta, \dfrac{\alpha^n}{\delta t^n}\right)} & \text{if } \alpha^n \le \delta t^n \end{cases}$$

where ε is a given tolerance and θ is the maximal (given) value of $\delta t^{n+1}/\delta t^n$ or $\delta t^n/\delta t^{n+1}$.

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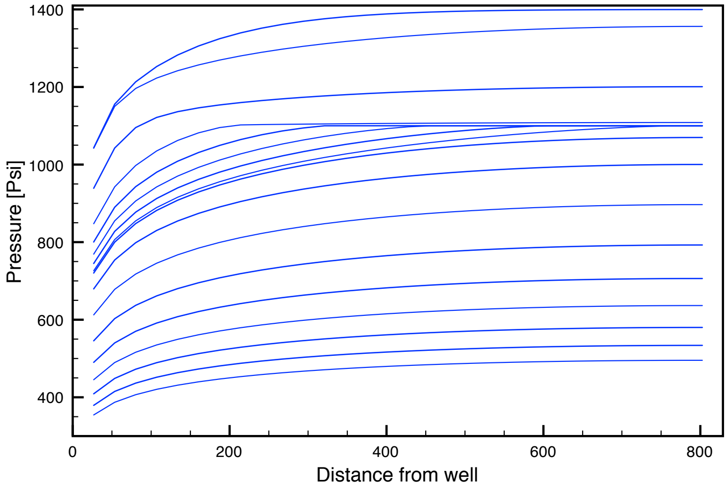
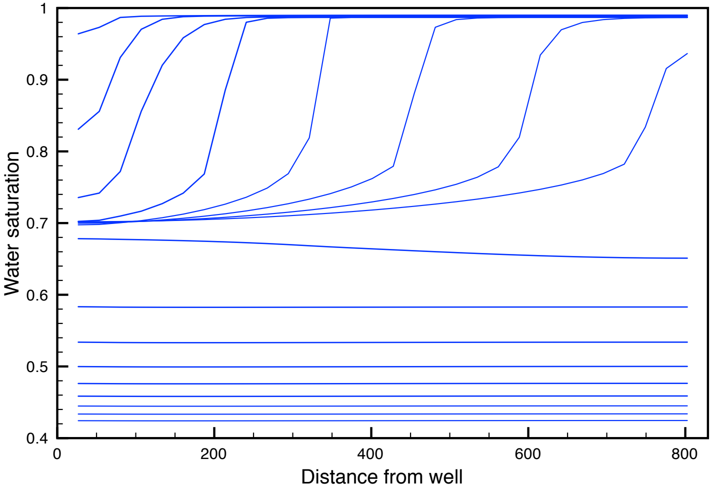
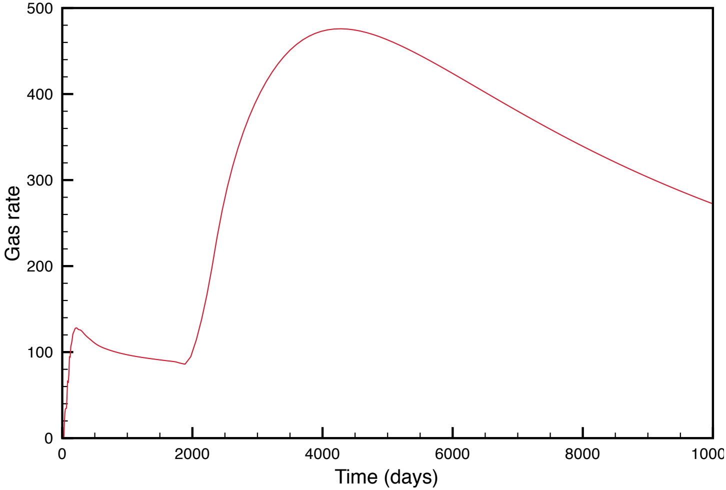
We look for a radial solution where the well is disk of radius $R_w = 0.15\,m$ located at the center of a reservoir of radius $R_e = 800\,m$, *i.e.* $R_w \ll R_e$.
We choose

$$p_c = 0, \quad S_0 = 1, \quad p_0 = 1400 \text{ psi}, \quad p_w = 100 \text{ psi}, \quad \tau = 1 \text{ jour}$$
$$T_{\max} = 10000 \text{ days (more than 27 years)}$$



Pressure [Psi]
1400
1200
1000
800
600
400
0
200
400
600
800
Distance from well


Water saturation
Distance from well
1
0.9
0.8
0.7
0.6
0.5
0.4
0
200
400
600
800


Gas rate
Time (days)
0
100
200
300
400
500
0
2000
4000
6000
8000
1000

Numerical simulations

- A vertical well
- A horizontal well
- A heterogeneous reservoir