Loop Quantum Gravity
1. Classical framework : Ashtekar-Barbero connection

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Why Quantum Gravity ?

Gravitation vs. Quantum Physics : the two infinities

- ▷ Gravitation : large scales of the Universe via General Relativity
  - Gravity is geometry and space-time is a dynamical entity
- ▷ Quantum physics : microscopic interactions via QFT
  - Particles and gauge fields live in a flat fixed Minkowski space-time
- ▷ Very successful theories but they do not see each other !

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However, gravity and the quantum world meet in some situations

- ▷ At the origin of the Universe
  - initial singularity where gravity fails to be predictive
  - it corresponds to the Planck scale $\ell_p \sim \sqrt{r_s \lambda_c} \sim \sqrt{\hbar G / c^3}$
- ▷ Near black holes
  - at the core singularity where the curvature diverges
  - at the horizon where there is a thermal radiation (gravitons ?)
- ▷ In general at all unavoidable space-time singularities
  - Penrose-Hawking singularity theorem

How quantum gravity ?

Standard (old-fashion) techniques for quantizing gravity fail

- ▷ Path integral quantization around the flat Minkowski metric
  - gravity is perturbatively non-renormalizable
- ▷ Canonical or Hamiltonian quantization
  - technically too complicated : too much quantum ambiguities
- ▷ Deep reasons behind these frustrating no-go theorems
  - we do not understand the meaning of quantizing space-time
  - quantizations break general covariance : what is the role of time ?
  - how to deal with the enormous symmetry group (diffeomorphisms) ?

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Some paths towards quantum gravity BUT no experiments

- ▷ Search for non-perturbative renormalization
- ▷ Gravity is not a fundamental theory but it is effective (law energy)
  - it has to be modified at Planck scale : new structure of space-time
- ▷ Quantization rules have to be adapted to gravity
  - the Fock space quantization is not suitable for general relativity

Loop quantum gravity in a nutshell

Gravity is a fundamental theory

- ▷ General Relativity could be quantized as it is
- ▷ If one respects the main features of the classical theory :
  - background independence, general covariance etc...
- ▷ The quantization should resolve the space-time singularities
  - as quantum mechanics resolves the classical instability of atoms
  - one does not modify the electrostatic potential $V(r)$
  - one shows the existence of a fundamental level and then stability

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Main characteristics of Loop Quantum Gravity

▷ Starting point : Einstein-Hilbert action in Ashtekar-Barbero variables

▷ Canonical or Hamiltonian quantization of pure gravity

- locally space-time looks like $\mathcal{M} = \Sigma \times [0,1]$ and Σ is space
- X is an $SU(2)$ connection and P the corresponding electric field

▷ Non-perturbative and background independent quantization

- no-background metric needed (no-trivial vacuum)

Success and failure of LQG

A beautiful and mathematically well-defined kinematic

- ▷ Kinematical states are one-dimensional excitations
  - they form a Hilbert space with an unique diff-invariant measure
- ▷ Geometric operators (area and volume) are kinematical observables
  - with a discrete spectrum : space is discrete at the Planck scale !
- ▷ The discreteness of quantum geometry is fundamental to
  - resolve the big-bang singularity : loop quantum cosmology (bounce)
  - understand black holes thermodynamics : entropy and radiation

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Failure and open issues

- ▷ Quantum dynamics is certainly the most important open issue
  - Spin-Foams : most promising attempts to define the dynamics
- ▷ Semi-classical limit still poorly understood
  - what is the quantum analogue of Minkowski, de Sitter etc... ?
- ▷ What about matter fields and other interactions ? $\ell_p \sim 10^{-20} \ell_{proton}$?
  - emergence of particles at classical limit : phase transition (tensors) ?

Overview of the course

1. Classical framework: Ashtekar-Barbero connection

- *Why does ADM canonical quantization fail ?*
- *From complex Ashtekar connection to Ashtekar-Barbero connection*
- *The holonomy-flux algebra : the polymer hypothesis*
- *Classical gravity in three space-time dimensions*

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2. Loop Quantum Gravity

- *A view in 3 dimensions where the program works*
- *Kinematics : discreteness of space*
- *Dynamics from Spin-Foam models*

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3. Black Hole thermodynamics from LQG

- *Heuristic Rovelli's model*
- *Black Hole partition function : counting microstates*
- *Back to complex variables : area law and thermal radiation*

ADM variables (1961)

Lagrangian formulation : M is the 4D space-time

▷ Einstein-Hilbert action without matter : functional of the metric g

$$S_{EH}[g] = \int d^4x \sqrt{|g|}(R - 2\Lambda)$$

▷ Variational principle leads to Einstein equations in vacuum

$$\frac{\delta S_{EH}}{\delta g_{\mu\nu}} = 0 \Longrightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

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Hamiltonian formulation : $M = \Sigma \times T$ with $\partial\Sigma = \emptyset$

▷ ADM parametrization of the metric :

$$ds^2 = N^2 dt^2 - h_{ab}(N^a dt + dx^a)(N^b dt + dx_b)$$

▷ h_{ab} is induced space metric, N is the lapse and N^a the shift

▷ The ten components of $g_{\mu\nu}$ parametrize by h_{ab}, N and N

The Legendre transformation is non invertible

▷ the canonical variables are h_{ab} and $\pi^{ab} = h^{-1/2}(K^{ab} - Kh^{ab})$

$$S_{ADM}[h,\pi;N,N_a] = \int dt \int d^3x(\dot{h}_{ab}\pi^{ab} + N_a H^a + NH)$$

▷ where K^{ab} is the intrinsic curvature and $X = X_a^a$ for any tensor

▷ the lapse and the shift are Lagrange multipliers which enforce

$$H^a = -2\nabla_b^{(3)}(h^{-1/2}\pi^{ab}) \simeq 0 \ , \ H = -h^{-1/2}[R^{(3)} + \frac{\pi^2 - 2\pi_{ab}\pi^{ab}}{2h}] \simeq 0$$

▷ where the index $^{(3)}$ refers to the 3-metric h_{ab}

Canonical analysis in ADM variables

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Symplectic structure and constraints analysis

▷ Poisson bracket : $\{\pi^{ab}, h_{cd}\} \propto \delta_{cd}^{ab}$ the symmetric tensor

▷ H^a is the vectorial constraint and H is the scalar constraint

▷ Dirac analysis : no more secondary constraints

▷ Then $6 \times 2 - 4 \times 2 = 4$ dof in phase space as expected

Too complicated constraints

Constraints and symmetries

- H and H generate space-time diffeomorphisms (on-shell)
- For instance, the action of $H[v] = \int d^3x \, v^a H_a$ on X

$$\delta_v X = \{H[v], X\} = \mathcal{L}_v X \text{ with } \mathcal{L} \text{ the Lie derivative}$$

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Formal physical phase space

$$\{(h_{ab}, \pi^{cd}) | H^a \simeq 0 \simeq H\}/\text{Diff}$$

- ▷ No explicit parametrization of the phase space
- ▷ Enormous symmetry group difficult to deal with
- ▷ Highly non linear expression of the constraints

All this leads to the impossibility of the quantization à la ADM

- ▷ Simplification : Wheeler-de Witt equation for the Universe

First order gravity in metric variables

The metric g and the connection Γ are independent variables

The Lagrangian point of view

- ▷ Hilbert Palatini action $S_{HP}[g,\Gamma]$ with Γ symmetric

$$S_{HP}[g,\Gamma] = \int d^4x\, \sqrt{|g|}\,(R[\Gamma] - 2\Lambda)$$

- ▷ Γ is torsion free then it is Levi-Civita : equivalence to Einstein-Hilbert

$$\frac{\delta S_{HP}}{\delta \Gamma} = 0 \Longrightarrow \Gamma(g)$$

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The Hamiltonian point of view in ADM parametrization

▷ Presence of secondary second class constraints $\psi \simeq 0$

$$\psi \simeq 0 \Longrightarrow \Gamma^{(3)} = \Gamma^{(3)}(g)$$

▷ Second class constraints must be resolved prior to quantization

▷ Redundant variables in considering g and Γ independent

▷ Back to ADM phase space : we gain nothing !

First order gravity in tetrad variables

The tetrad and the spin-connection

- ▷ The tetrad e_μ^I (4×4 matrix) such that $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$
- ▷ e is defined up to Lorentz transformations : $SL(2,\mathbb{C})$ gauge symmetry
- ▷ The $so(3,1)$ spin-connection ω_μ^{IJ} related to Γ by $\omega_\mu^{IJ} = \Gamma_\mu(e^I, e^J)$

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Hilbert-Palatini action in terms of tetrad

$$S_{HP}[e,\omega] = \int \langle \star(e \wedge e) \wedge F(\omega) \rangle = \int d^4x \, \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} e_\mu^I e_\nu^J F_{\nu\rho}^{KL}(\omega)$$

- ▷ The curvature 2-form $F(\omega) = d\omega + \omega \wedge \omega$
- ▷ The Hodge dual $\star : so(3,1) \to so(3,1)$
- ▷ The Killing form $\langle ; \rangle : so(3,1) \times so(3,1) \to \mathbb{C}$ s.t. $\langle a; b \rangle \propto tr(ab)$

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Canonical analysis in tetrad formalism

- First class constraints : H, H_a and the Gauss constraint G^{IJ}
- Second class constraints : $T_{ab} = 0 \Longrightarrow \omega_a^{ij}(e)$
- This formalism reduces to the ADM formalism : gain nothing again !

The input of Complex Ashtekar variables

Self-dual or anti self-dual complex connection

- The (anti) self-dual action

$$S_{\pm}[e,{}^{(\pm)}\omega] = \int \langle \star(e\wedge e)\wedge F({}^{(\pm)}\omega)\rangle = \frac{1}{2}\int \langle \star(e\wedge e)\wedge F(\omega)\rangle \pm i \langle (e\wedge e)\wedge F(\omega)\rangle$$

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Quick Hamiltonian analysis

▷ Poisson bracket : $\{E_i^a(x), A_b^j(y)\} = \pm i\delta_b^a\delta_i^j\delta(x-y)$

$$E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^je_c^k \;\text{ and }\; A_a^i =^{(\pm)} \omega_a^i$$

▷ The three families of first class constraints (polynomial)

$$G_i = D_aE_i^a\ ,\quad H_a = E^b\cdot F_{ab}\ ,\quad H = E^a\times E^b\cdot F_{ab}$$

▷ No second class constraints. We gain something important !

The Ashtekar-Barbero connection

Problem with complex variables : non-compact gauge group

▷ Reality conditions : $A_a^i + \overline{A}_a^i = \Gamma_a^i(E)$ unsolved at quantum level

Making the connection real : the Barbero-Immirzi parameter

▷ Holst action with a free parameter γ

$$S_\gamma = \frac{1}{2}\int \langle \star(e \wedge e) \wedge F(\omega)\rangle + \frac{1}{\gamma}\langle (e\wedge e)\wedge F(\omega)\rangle$$

▷ Classically, γ is totally irrelevant by virtue of Bianchi identity

Hamiltonian analysis in the time gauge

▷ An $su(2)$-valued connection : $\{E_i^a(x), A_b^j(y)\} = \gamma \delta_b^a \delta_i^j \delta(x-y)$

$$E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k \;\text{ and }\; A_a^i = \Gamma_a^i + \gamma K_a^i$$

▷ The three families of first class constraints (polynomial)

$$G_i = D_a E_i^a \;,\;\; H_a = E^b \cdot F_{ab} \;,\;\; H = E^a \times E^b \cdot (F_{ab} + (\gamma^2+1)K_a \times K_b)$$

▷ The Hamiltonian constraint is no more polynomial

The polymer hypothesis

Classical phase space of Ashtekar gravity :

- ▷ Phase space : $\mathcal{P} = T^*(\mathcal{A})$ with $\mathcal{A} = \{SU(2)\, connections\}$
- ▷ Fundamental excitations are one-dimensional : polymer hypothesis
- ▷ Holonomy-flux algebra associated to edges e and surfaces S

$$A(e) = P\exp(\int_e A) \quad \text{and} \quad E_f(S) = \int_S \mathrm{Tr}(f \star E)\,.$$

- ▷ Cylindrical functions : $f \in \mathrm{Cyl}$ is a function of $A(e)$ with $e \subset \gamma$
- ▷ $E_f(S)$ acts as a vector field on f if $S \cap \gamma \neq 0$.

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Action of symmetries : $\mathcal{S} = \mathcal{G} \ltimes Diff(\Sigma)$ **with** $\mathcal{G} = C^\infty(\Sigma, SU(2))$

- ▷ Gauss constraint : $f(A(e)) \longmapsto f(g(s(e))^{-1}A(e)g(t(e)))$
- ▷ Diffeomorphisms : $f(A(e)) \longmapsto f(A(\varphi(e)))$
- ▷ Similar action for the variables $E_f(S)$
- ▷ Symmetries are automorphisms of classical algebra

Brief summary of the classical

The Ashtekar-Barbero connection

- ▷ Hypothesis : time-gauge $SL(2,\mathbb{C}) \to SU(2)$
- ▷ Obtained from Holst action with γ irrelevant
- ▷ Equivalently from canonical transformation
- ▷ A is an $su(2)$-valued connection
- ▷ At the kinematical level : gravity looks like $SU(2)$ Yang-Mills theory
- ▷ But the Hamiltonian constraint is no more polynomial...

The polymer hypothesis

- ▷ Excitations are one-dimensional
- ▷ Fundamental variables are holonomies of A
- ▷ Ready for the quantization...

3D gravity as a toy model