HOMOLOGICAL INVARIANTS OF DISCRETE QUANTUM GROUPS

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Abstract. These are the notes for a mini-course given at the conference "Topological quantum groups and harmonic analysis", May 15-19, 2017 at SNU (Seoul National University), Korea. We introduce and discuss the classical homological invariants associated to discrete quantum groups (compact Hopf algebras), with a special emphasis on cohomological dimension, and present some recent computations.

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Introduction

These are the notes for a series of lectures given at the conference "Topological quantum groups and harmonic analysis", May 15-19, 2017 at SNU (Seoul National University), Korea. We introduce and discuss the classical homological invariants associated to discrete quantum groups (compact Hopf algebras): cohomological dimension, cohomology, L^2-Betti numbers, bialgebra cohomology, with a special emphasis on cohomological dimension, and we present some recent computations involving the "free" quantum groups O_n^+, U_n^+ and S_n^+.

These notes are organized as follows. Section 1 gives an overview of the Hopf algebraic approach to the theory of compact/discrete quantum groups. In Section 2, we discuss exact sequences of (compact) Hopf algebras. Section 3 presents the basic homological algebra material: projective modules, projective dimension, Ext-spaces, and we define the cohomology of Hopf algebras. In Section 4 we define the cohomological dimension of a Hopf algebra, discuss some basic facts and present computations for our main examples. Section 5 gives a brief definition of L^2-Betti numbers and surveys some recent computations. Section 6 presents bialgebra cohomology, and the final Section 7 lists a number of open questions.

Warning. Before starting, I would like to say a few words about what can be expected from the homological invariants we will discuss. As they are invariants, the most optimistic hope is that they will help to classify the objects we study (just like the theorem of invariance of domain can be proved by using singular homology). However, this is not so often the case. Just as in the classical theory of discrete groups, the homological invariants rather serve, on one hand, as a measure of some kind of complexity for discrete quantum groups, providing a rough classification into subclasses with well-identified properties, and on the other hand, as a tool to attack various non-trivial problems, as in usual group theory [25].

1. Hopf algebras and discrete/compact quantum groups

In this section we give an overview of the Hopf algebraic approach to the theory of compact/discrete quantum groups, through what we call compact Hopf algebras. The main reference is the book [46], and [68, 39] form convenient references as well.

We warn the reader that the presentation is designed to highlight the facts that we believe to be the most important and useful for the rest of the notes, but does not follow the logical order that one would need for a complete course on the subject.

1.1. Hopf $*$-algebras: basic definitions and examples.

Definition 1.1. A **Hopf algebra** is an algebra A together with algebra maps

(1) $\Delta : A \longrightarrow A \otimes A$ (comultiplication)
(2) $\varepsilon : A \longrightarrow \mathbb{C}$ (counit)
(3) $S : A \longrightarrow A^{\mathrm{op}}$ (antipode)

satisfying the following axioms:

(a) $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$ (Coassociativity)
(b) $(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta$ (counit axiom)
(c) $m \circ (\mathrm{id}_A \otimes S) \circ \Delta = u \circ \varepsilon = m \circ (S \otimes \mathrm{id}_A) \circ \Delta$ (antipode axiom),

where $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ are the respective multiplication and unit of A.

The most popular Hopf algebra textbook is [64]. The interested reader will find an historical account of the theory of Hopf algebras in [4].

Definition 1.2. A **Hopf $*$-algebra** is a $*$-algebra A, which is a Hopf algebra as well, and such that the comultiplication $\Delta : A \to A \otimes A$ is $*$-algebra map.

Remark 1.3. If A is Hopf $*$-algebra, it is automatic that $\varepsilon : A \to \mathbb{C}$ is a $*$-algebra map and that S is bijective, with $S^{-1}(a) = S(a^*)^*$ for any $a \in A$.

Example 1.4. Let Γ be a discrete group, and let $\mathbb{C}\Gamma$ be its group $*$-algebra with $\mathbb{C}$-basis $\{e_g, g \in \Gamma\}$, and $e_g^* = e_{g^{-1}}$. This is a Hopf $*$-algebra with, for any $g \in \Gamma$,

$$\Delta(e_g) = e_g \otimes e_g, \quad \varepsilon(e_g) = 1, \quad S(e_g) = e_{g^{-1}}$$

Example 1.5. Let G be a compact group. Recall that a representative function on G is a continuous function $f \in C(G)$ such that the set

$$Gf = \{x.f,\ x \in G\},\ \text{where } x.f(y) = f(x^{-1}y)$$

generates a finite-dimensional subspace of $C(G)$. This is equivalent to say that f is a coefficient of a finite-dimensional representation of G.

The set of representative functions on G is denoted $\mathcal{O}(G)$. In fact $\mathcal{O}(G)$ is a $*$-subalgebra of $C(G)$, and is dense in $C(G)$ by the Peter-Weyl theorem. The group structure of G induces a Hopf $*$-algebra structure on $\mathcal{O}(G)$. The multiplication of $m : G \times G \to G$ induces a comultiplication

$$\begin{aligned} \Delta : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G) \\ f &\longmapsto f \circ m \longmapsto \Delta(f) \end{aligned}$$

The counit is defined by

$$\begin{aligned} \varepsilon : \mathcal{O}(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(1) \end{aligned}$$

and the antipode is induced by the inversion map in G

$$\begin{aligned} S : \mathcal{O}(G) &\longrightarrow \mathcal{O}(G) \\ f &\longmapsto S(f), S(f)(x) = f(x^{-1}) \end{aligned}$$

Example 1.6. Let again G be a compact group, with G a subgroup of the unitary group U_n. Let u_{ij}, $1 \leq i, j \leq n$, be the coordinate functions on G: for $g = (g_{ij}) \in G$, $u_{ij}(g) = g_{ij}$. The elements u_{ij} belong to $\mathcal{O}(G)$ (and in fact generate $\mathcal{O}(G)$ as a $*$-algebra), and we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u_{ij}) = (u^{-1})_{ij}$$

where u^{-1} stands for the inverse of the matrix $u = (u_{ij}) \in M_n(\mathcal{O}(G))$.

The above example exactly paves the way to construct examples, by the following useful result. The proof is left as an exercise.

Lemma 1.7. *Let A be $*$-algebra endowed with $*$-algebra maps $\Delta : A \longrightarrow A \otimes A$ and $\varepsilon : A \longrightarrow \mathbb{C}$, and an algebra map $S : A \longrightarrow A^{\mathrm{op}}$. Assume that there exists a matrix $u = (u_{ij}) \in M_n(A)$ such that the following conditions hold:*

(1) $u = (u_{ij})$ and $\overline{u} = (u_{ij}^*)$ are invertible matrices;
(2) A is generated, as a $*$-algebra, by the coefficients of the matrix u;
(3) for any i, j, we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u_{ij}) = (u^{-1})_{ij},\ S(u_{ij}^*) = (\overline{u}^{-1})_{ij}$$

Then A, endowed with the above structure maps, is a Hopf $*$-algebra.

One defines morphisms of Hopf $*$-algebras in the obvious way, we get a category, and the constructions of examples 1.4 and 1.5 define functors.

There are important groups naturally attached to a Hopf ($*$-)algebra.

Definition 1.8. Let A be a Hopf algebra.

(1) An element $a \in A$ is said to be **group-like** if $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$. The set of group-like elements in A is denoted $\mathrm{Gr}(A)$, the multiplication of A induces a group structure on $\mathrm{Gr}(A)$, with for $a \in \mathrm{Gr}(A)$, $a^{-1} = S(a)$.

(2) The set of algebra maps $A \longrightarrow \mathbb{C}$, denoted $G(A)$, is a group under the law

$$\phi \cdot \psi := (\phi \otimes \psi) \circ \Delta$$

The unit element is ε and the inverse of $\phi \in G(A)$ is $\phi \circ S$.

(3) If A is a Hopf $*$-algebra, let $G_{\mathbb{R}}(A)$ be the set of $*$-algebra maps $A \longrightarrow \mathbb{C}$: this a subgroup of the group $G(A)$. We endow $G_{\mathbb{R}}(A)$ with the weakest topology making the evaluations $G_{\mathbb{R}}(A) \longrightarrow \mathbb{C}$, $\phi \longmapsto \phi(a)$ $(a \in A)$ continuous, making $G_{\mathbb{R}}(A)$ into a topological group.

These constructions allow us to reconstruct the groups from the Hopf algebras in examples 1.4 and 1.5

Example 1.9. (1) If Γ is a discrete group, we have a group isomorphism $\Gamma \simeq \mathrm{Gr}(\mathbb{C}\Gamma)$, $x \longmapsto e_x$ (exercise). Therefore the Hopf algebra $\mathbb{C}\Gamma$ completely determines the group Γ.

(2) Let G be a compact group. Then $G_{\mathbb{R}}(\mathcal{O}(G))$ is a compact group, and

$$\begin{aligned} \iota : G &\longrightarrow G_{\mathbb{R}}(\mathcal{O}(G)) \\ x &\longmapsto \iota(x), \ \ \iota(x)(f) = f(x) \end{aligned}$$

is a compact group isomorphim: this is the *Tannaka duality theorem*, which asserts in particular that G can be reconstructed from the Hopf $*$-algebra $\mathcal{O}(G)$, and hence from its finite-dimensional representations, see [21].

1.2. **Compact Hopf algebras.** We now define compact Hopf algebras. We first introduce a piece of vocabulary, coming from Example 1.6.

Definition 1.10. Let A be a Hopf algebra and let $u = (u_{ij}) \in M_n(A)$ be a matrix. We say that u is a **multiplicative matrix** if for all $i, j \in \{1, \dots, n\}$, we have

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \ \ \varepsilon(u_{ij}) = \delta_{ij}$$

If A is a Hopf $*$-algebra, a multiplicative matrix $u \in M_n(A)$ is said to be unitarisable if the exists $F \in \mathrm{GL}_n(\mathbb{C})$ such that the matrix FuF^{-1} is unitary.

Theorem-Definition 1.11. *A Hopf $*$-algebra A is said to be* ***compact*** *if it satisfies the following equivalent conditions.*

(1) Any multiplicative matrix $u \in M_n(A)$ is unitarisable.

(2) There exists a faithful state $h : A \to \mathbb{C}$ such that $(\mathrm{id}_A \otimes h) \circ \Delta = h(-)1_A = (h \otimes \mathrm{id}_A) \circ \Delta$.

(3) A is generated, as a $*$-algebra, by a family of unitary multiplicative matrices $u_\lambda \in M_{n_\lambda}(A)$, $\lambda \in \Lambda$, such that $\overline{u_\lambda}$ is unitarizable for any λ.

The equivalences between these conditions is from [32], where unfortunately, the proof of (3) $\Rightarrow$ (1) has a mistake. We refer to [46] for a complete proof.

Example 1.12. If Γ is a discrete group, the group algebra $\mathbb{C}\Gamma$ is compact. Conversely, if A is a cocommutative compact Hopf algebra (for any $a \in A$, $\Delta(a) = \tau\Delta(a)$, where $\tau : A \otimes A \to A \otimes A$ is the canonical flip), then $A \simeq \mathbb{C}\Gamma$, for $\Gamma = \mathrm{Gr}(A)$.

Example 1.13. If G is a compact group, then $\mathcal{O}(G)$ is a compact Hopf algebra (the functional h in Theorem 1.11 is the usual Haar integral on G). Conversely, if A is a commutative compact Hopf algebra, then $A \simeq \mathcal{O}(G)$, where $G = G_{\mathbb{R}}(A)$: this is the Hopf algebraic version of the **Tannaka-Krein duality** theorem, see [21].

Example 1.14. If A is a commutative and cocommutative compact Hopf algebra, then $A \simeq \mathcal{O}(G)$ and $A \simeq \mathbb{C}\Gamma$, for an abelian compact group G and an abelian discrete group Γ such that $G \simeq \widehat{\Gamma}$ and $\Gamma \simeq \widehat{G}$ (the two groups are Pontryagin dual of each other).

In view of these examples, if A is an arbitrary compact Hopf algebra, we may formally write

$$A = \mathcal{O}(G), \quad A = \mathbb{C}\Gamma, \quad G = \widehat{\Gamma}, \quad G = \widehat{\Gamma}$$

for a **compact quantum group** G and a **discrete quantum group** Γ that are dual of each other.

Remark 1.15. Let A is a compact Hopf algebra. If $B \subset A$ is a Hopf $*$-subalgebra, then B is a compact Hopf algebra as well. If $f : A \to B$ is a surjective Hopf $*$-algebra map, then B is a compact Hopf algebra. These facts are consequences of the various equivalent definitions in Theorem-Definition 1.11.

To construct examples outside classical compact and discrete groups, we can use Lemma 1.7.

Example 1.16 (SU(2) and its quantum q-deformation $\mathrm{SU}_q(2)$, [87]). Let $q \in \mathbb{R}^*$. The $*$-algebra $\mathcal{O}(\mathrm{SU}_q(2))$ is, as an algebra, presented by generators $u_{11}, u_{12}, u_{21}, u_{22}$, submitted to the relations

$$u_{12}u_{11} = qu_{11}u_{12}, \quad u_{21}u_{11} = qu_{11}u_{21}, \quad u_{22}u_{12} = qu_{12}u_{22}, \quad u_{22}u_{21} = qu_{21}u_{22}$$
$$u_{12}u_{21} = u_{21}u_{12}, \quad u_{11}u_{22} - u_{22}u_{11} = (q^{-1} - q)u_{12}u_{21}$$
$$u_{11}u_{22} - q^{-1}u_{12}u_{21} = 1$$

Its $*$-structure is defined by

$$u_{11}^* = u_{22},\ u_{22}^* = u_{11},\ u_{12}^* = -q^{-1}u_{21},\ u_{21}^* = -qu_{12}$$

and $\mathcal{O}(\mathrm{SU}_q(2))$ is a Hopf $*$-algebra with

$$\Delta(u_{ij}) = \sum_{k=1}^{2} u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}$$

and

$$S(u_{11}) = u_{22},\ S(u_{12}) = -qu_{12},\ S(u_{21}) = -q^{-1}u_{21},\ S(u_{22}) = u_{11}$$

See Lemma 1.7. Moreover it is easily checked that the matrix $u = (u_{ij}) \in M_2(\mathcal{O}(\mathrm{SU}_q(2)))$ is unitary and hence $\mathcal{O}(\mathrm{SU}_q(2))$ is a compact Hopf algebra by Theorem 1.11.

If $q = 1$, then $\mathcal{O}(\mathrm{SU}_1(2))$ is commutative with $G_{\mathbb{R}}(\mathcal{O}(\mathrm{SU}_1(2))) \simeq \mathrm{SU}(2)$ and thus by the Tannaka-Krein Theorem, we have $\mathcal{O}(\mathrm{SU}_1(2)) \simeq \mathcal{O}(\mathrm{SU}(2))$.

If $q \neq 1$, then $\mathcal{O}(\mathrm{SU}_q(2))$ is noncommutative and noncommutative (this is not completely obvious). The corresponding compact quantum group, of great historical importance, was introduced by Woronowicz [87], and is called the quantum group $\mathrm{SU}_q(2)$. There are also quantum groups $\mathrm{SU}_q(n)$ [89] and q-deformations for other classical groups [71], that we do not discuss here.

We now present the series of "free" examples O_n^+, U_n^+, S_n^+, introduced by Wang in the nineties.

Example 1.17 (O_n and its free version O_n^+, [33], [83]). Let $n \geq 1$. Consider the commutative algebra

$$A = \mathbb{C}[x_{ij},\ 1 \leq i, j \leq n \mid xx^t = I_n = x^t x]$$

where $x = (x_{ij})$, and where the notation means that we exactly have all the relations making the written matrix relations written hold. There are algebra maps

$$\Delta : A \longrightarrow A \otimes A, \quad \varepsilon : A \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A \longrightarrow A$$

defined by $(1 \leq i, j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

that endow A with a Hopf algebra structure (Lemma 1.7. Moreover A has a $*$-algebra structure, with $x_{ij}^* = x_{ij}$, and is a Hopf $*$-algebra. The matrix x is then unitary, and Theorem 1.11 ensures that A is a compact Hopf algebra. We have $G_{\mathbb{R}}(A) \simeq O_n$, and the Tannaka-Krein duality theorem ensures that $A \simeq \mathcal{O}(O_n)$.

The free version "O_n^+" of O_n is obtained by removing the commutativity relations in the above presentation of $\mathcal{O}(O_n)$. More precisely let

$$A_o(n) = \mathbb{C}\langle x_{ij},\ 1 \leq i,j \leq n \mid xx^t = I_n = x^t x\rangle$$

where $x = (x_{ij})$. We have algebra maps

$$\Delta : A_o(n) \longrightarrow A_o(n) \otimes A_o(n), \quad \varepsilon : A_o(n) \longrightarrow \mathbb{C}, \quad \text{et} \quad S : A_o(n) \longrightarrow A_o(n)^{\rm op}$$

defined by $(1 \leq i,j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

othat endow $A_o(n)$ with a Hopf algebra structure. Moreover $A_o(n)$ has a $*$-algebra structure, with $x_{ij}^* = x_{ij}$, and is a Hopf $*$-algebra, non commutative and non-cocommutative if $n \geq 2$ (since it has $\mathbb{C}\mathbb{Z}_2^{*n}$ and $\mathcal{O}(O_n)$ as Hopf $*$-algebra quotients). The matrix x is then unitary, and Theorem 1.11 ensures that $A_o(n)$ is a compact Hopf algebra. We put $A_o(n) = \mathcal{O}(O_n^+)$, and call O_n^+ the free orthogonal quantum group (one can show that $A_o(2) \simeq \mathcal{O}(\mathrm{SU}_{-1}(2))$.

Example 1.18 (U_n ant its free version U_n^+, [83]). Let $n \geq 1$. Consider the commutative $*$-algebra

$$A = \mathbb{C}[u_{ij}, u_{ij}^*,\ 1 \leq i,j \leq n \mid uu^* = I_n = u^* u]$$

where $u = (u_{ij})$, and where the notation means that we exactly have all the relations making the written matrix relations written hold. There are $*$-algebra maps

$$\Delta : A \longrightarrow A \otimes A, \quad \varepsilon : A \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A \longrightarrow A$$

defined by $(1 \leq i,j \leq n)$

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^*$$

that endow A with a Hopf $*$-algebra structure (Lemma 1.7). The matrix u and $\overline{u}$ are then unitary, and Theorem 1.11 ensures that A is a compact Hopf algebra. We have $G_{\mathbb{R}}(A) \simeq U_n$, and the Tannaka-Krein duality theorem ensures that $A \simeq \mathcal{O}(U_n)$.

The "free" version U_n^+ of U_n is obtained by removing the commutativity relations in the above presentation of $\mathcal{O}(U_n)$ (in a more subtle manner than for O_n). More precisely consider the $*$-algebra

$$A_u(n) = \mathbb{C}\langle u_{ij}, u_{ij}^*,\ 1 \leq i,j \leq n \mid uu^* = I_n = u^* u,\ \overline{u}u^t = I_n = u^t\overline{u}\rangle$$

where $u = (u_{ij})$. We have $*$-algebra maps

$$\Delta : A_u(n) \longrightarrow A_u(n) \otimes A_u(n), \quad \varepsilon : A_u(n) \longrightarrow \mathbb{C}, \quad \text{et} \quad S : A_u(n) \longrightarrow A_u(n)^{\rm op}$$

defined by $(1 \leq i,j \leq n)$

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^*$$

that endow $A_u(n)$ with a Hopf $*$-algebra structure (Lemma 1.7), noncommutative and noncocommutative if $n \geq 2$ (since it has $\mathbb{C}\mathbb{F}_n$ and $\mathcal{O}(U_n)$ as Hopf $*$-algebra quotients). Moreover, the matrices u and $\overline{u}$ are unitary, and Theorem 1.11 ensures that A is a compact Hopf algebra. We put $A_u(n) = \mathcal{O}(U_n^+)$, and call U_n^+ the free unitary quantum group.

Example 1.19 (S_n ant its free version S_n^+, [84]). Let $n \geq 1$. Consider the commutative algebra A presented by generators x_{ij}, $1 \leq i,j \leq n$, submitted to the relations of permutation matrices $(1 \leq i,j,k \leq n)$

$$\sum_{l=1}^n x_{il} = 1 = \sum_{l=1}^n x_{li}, \quad x_{ij}x_{ik} = \delta_{jk}x_{ij},\ x_{ji}x_{ki} = \delta_{jk}x_{ji}$$

We have algebra maps

$$\Delta : A \longrightarrow A \otimes A, \quad \varepsilon : A \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A \longrightarrow A$$

defined by $(1 \leq i, j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

that endow A with a Hopf algebra structure. Moreover A has a $*$-algebra structure, with $x_{ij}^* = x_{ij}$, and is a Hopf $*$-algebra. The matrix x is then unitary, and Theorem 1.11 ensures that A is a compact Hopf algebra. We have $G_{\mathbb{R}}(A) \simeq S_n$, and the Tannaka-Krein duality theorem ensures that $A \simeq \mathcal{O}(S_n)$.

The free version "S_n^+" of S_n is obtained by removing the commutativity relations in the above presentation of $\mathcal{O}(S_n)$. More precisely let $A_s(n)$ be the algebra presented by generators x_{ij}, $1 \leq i, j \leq n$, submitted to the relations of permutation matrices $(1 \leq i, j, k \leq n)$

$$\sum_{l=1}^{n} x_{il} = 1 = \sum_{l=1}^{n} x_{li}, \quad x_{ij}x_{ik} = \delta_{jk}x_{ij}, \ x_{ji}x_{ki} = \delta_{jk}x_{ji}$$

We have algebra maps

$$\Delta : A_s(n) \longrightarrow A_s(n) \otimes A_s(n), \quad \varepsilon : A_s(n) \longrightarrow \mathbb{C}, \quad \text{and} \quad S : A_s(n) \longrightarrow A_s(n)^{\mathrm{op}}$$

defined by $(1 \leq i, j \leq n)$

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

that endow $A_s(n)$ with a Hopf algebra structure. Moreover $A_s(n)$ has a $*$-algebra structure, with $x_{ij}^* = x_{ij}$, and is a Hopf $*$-algebra. The matrix x is then unitary, and Theorem 1.11 ensures that $A_s(n)$ is a compact Hopf algebra, noncommutative and noncocommutative if $n \geq 4$. We put $A_s(n) = \mathcal{O}(S_n^+)$, and call S_n^+ the quantum permutation group on n points (an alternative terminology could be the free permutation quantum group S_n^+, but this terminology is not too much in use). The quantum permutation group S_n^+ is the largest compact quantum group acting on the classical set formed by n points, whence his name, see [84].

1.3. **Comodules.** We now discuss comodules over a (compact) Hopf algebra, which correspond to representations of the corresponding (compact) quantum group, and are crucial in analysing its structure.

Definition 1.20. Let A be a Hopf algebra. A (right) A-**comodule** is a vector space V endowed with a linear map $\alpha : V \longrightarrow V \otimes A$ (called coaction) such that the following conditions are satisfied:

(1) $(\alpha \otimes \mathrm{id}_A) \circ \alpha = (\mathrm{id}_V \otimes \Delta) \circ \alpha$;
(2) $(\mathrm{id}_V \otimes \varepsilon) \circ \alpha = \mathrm{id}_V$.

Example 1.21. The comultiplication $\Delta : A \longrightarrow A \otimes A$ endows A with a right A-comodule structure, called the regular A-comodule.

Example 1.22. Let Γ be a group and V be a vector space. A $\mathbb{C}\Gamma$-comodule structure on V is the same as a Γ-grading on V, i.e. a direct sum decomposition $V = \oplus_{g \in \Gamma} V_g$.

Example 1.23. Let G be a compact group. An $\mathcal{O}(G)$-comodule structure on a finite-dimensional vector space precisely corresponds to a (continuous) representation $G \to \mathrm{GL}(V)$, see Proposition 1.24.

One defines morphisms of comodules in a straightforward manner: if A is a Hopf algebra and $V = (V, \alpha_V)$ and $W = (W, \alpha_W)$ are A-comodules, an A-comodule morphism $V \longrightarrow W$ is a linear map $f : V \longrightarrow W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"] & W \arrow[d, "\alpha_W"] \\ V \otimes A \arrow[r, "f\otimes \mathrm{id}_A"] & W \otimes A \end{tikzcd}$$

One also says that an A-comodule morphism is an A-colinar map.

The category of A-comodules is denoted $\mathsf{Comod}(A)$, while the full subcategory of finite-dimensional A-comodules is denoted $\mathsf{Comod}_f(A)$. Both are abelian subcategories of Vect, the category of vector spaces, which means that the standard operations in linear algebra such as direct sums, kernels, cokernels can be performed inside these categories.

Finite-dimensional comodules can be described by by means of multiplicative matrices, as shown by the following result, whose verification is an easy exercise.

Proposition 1.24. *Let A be a Hopf algebra and let V be a finite-dimensional vector space.*

(1) Assume that V has an A-comodule structure with coaction $\alpha : V \longrightarrow V \otimes A$. Let $v_1, \dots, v_n$ be a basis of V and let $x = (x_{ij}) \in M_n(A)$ be the matrix such that $\forall i$,

$$\alpha_V(v_i) = \sum_{j=1}^{n} v_j \otimes x_{ji}$$

Then $x = (x_{ij})$ is a multiplicative matrix.

(2) Conversely, if $x = (x_{ij}) \in M_n(A)$ is a multiplicative matrix, for each basis of V, the above formula defines an A-comodule structure on V.

(3) If V is an A-comodule with corresponding multiplicative $x = (x_{ij}) \in M_n(A)$ associated to the choice of a basis of V, then $A(V) = \mathrm{Span}(x_{ij}, 1 \leq i, j \leq \dim(V))$ does not depend on the choice of the basis, and is a subcoalgebra of A (i.e. $\Delta(A(V)) \subset A(V) \otimes A(V)$). Moreover if W is a comodule isomorphic to V, then $A(V) = A(W)$.

The last part of the proposition, together with the fact that a finite-dimensional comodule V is simple (i.e. has no non-trivial subcomodule) if and only if $\dim(A(V)) = \dim(V)^2$ (this follows from a classical result of Burnside), enables us to formulate the Peter-Weyl decomposition in the compact Hopf algebra framework.

Theorem 1.25. *Let A be a compact Hopf algebra.*

(1) The category $\mathsf{Comod}_f(A)$ is semisimple: every object is a direct sum of simple comodules.

(2) Let λ be the set of isomorphism classes of simple A-comodules. We have a (Peter-Weyl) decomposition

$$A = \bigoplus_{\lambda \in \Lambda} A(\lambda)$$

where each $A(\lambda)$ is the subcoalgebra of coefficients of a simple comodule associated to λ.

The orthogonality relations [56, 88] even make the Peter-Weyl decomposition more precise: see [46]. We will not need their precise form.

We end the subsection by briefly discussing the tensor category structure on the category of comodules over a Hopf algebra.

If $V = (V, \alpha_V)$, $W = (W, \alpha_W)$ are comodules over A, their tensor product has a natural A-comodule structure defined by

$$V \otimes W \overset{\alpha_V \otimes \alpha_W}{\longrightarrow} V \otimes A \otimes W \otimes A \overset{\mathrm{id}\otimes\tau\otimes\mathrm{id}}{\longrightarrow} V \otimes W \otimes A \otimes A \overset{\mathrm{id}\otimes m}{\longrightarrow} V \otimes W \otimes A$$

The natural associativity isomorphisms $(V \otimes W) \otimes Z \simeq V \otimes (W \otimes Z)$ are morphisms of comodules, and together with the trivial comodule $\mathbb{C}$ (defined by $1 \mapsto 1 \otimes 1_A$)), these structures make the category $\mathsf{Comod}_f(A)$ into a **tensor category**, see [35, 45, 66].

If now A is compact Hopf algebra, the set of isomorphism classes of simple A-comodules together with the decompositions of tensor products of simple comodules into direct sums of simple comodules produces a combinatorial data called the **fusion rules** of A (see e.g. [8]). The fusion rules of the examples of quantum groups in the previous paragraph have been determined in classic papers of Woronowicz ([87], for $\mathrm{SU}_q(2)$, the fusion rules are the same as those of the classical $\mathrm{SU}(2)$) and Banica (([5] for O_n^+ which has the same fusion rules as those of $\mathrm{SU}(2)$, [6] for U_n^+ which has noncommutative fusion rules, [7] for S_n^+ which has the same fusion rules as $\mathrm{SO}(3) \simeq \mathrm{PU}(2)$).

There is also a stronger relation than the one of having the same fusion rules, the relation of monoidal equivalence. We say that two Hopf algebras A and B are **monoidally equivalent** if there exists a tensor category equivalence $\mathsf{Comod}_f(A) \simeq^{\otimes} \mathsf{Comod}_f(B)$ (i.e an equivalence of categories that preserves the tensor products up to isomorphism in a coherent way, we refer to [35, 45, 66] for the precise definition). Among the previous quantum groups, let us mention the monoidal equivalences

$$\mathsf{Comod}_f(A_o(n)) \simeq^{\otimes} \mathsf{Comod}_f(\mathcal{O}(\mathrm{SU}_q(2)), \text{ for } q+q^{-1}=-n, \text{ [12, 17]}$$

and

$$\mathsf{Comod}_f(A_s(n)) \simeq^{\otimes} \mathsf{Comod}_f(\mathcal{O}(\mathrm{PU}_q(2)), \text{ for } q+q^{-1}=\sqrt{n}, \text{ [28, 65]}$$

where $\mathrm{PU}_q(2)$ is defined in Example 2.7.

1.4. **Operator algebras associated compact Hopf algebras.** Let A be compact Hopf algebra, with Haar state $h : A \to \mathbb{C}$. The formula

$$\langle a, b\rangle = h(b^*a)$$

defines a faithful, positive definite and sesquilinear form on A. The pair $(A, \langle,\rangle)$ is therefore a pre-Hilbert space and we denote by $L^2(A)$ its Hilbert space completion. It follows from the orthogonality relations that for any $a \in A$, there exists a constant $C_a > 0$ such that for any $b \in A$

$$\langle ab, ab\rangle = h(b^*a^*ab) \leq C_a h(b^*b) = C_a\langle b, b\rangle$$

This precisely means that the elements of A, acting by left multiplication on A, act continuously and hence extend to bounded operators on $L^2(A)$. We get an injective $*$-algebra map

$$\pi_h : A \longrightarrow \mathcal{B}(L^2(A))$$

We get a C^*-algebra and a von Neumann algebra

$$C^*_{\mathrm{red}}(A) = \overline{\pi_h(A)}^{||\cdot||}, \quad \mathcal{L}(A) = \pi_h(A)''$$

When $A = \mathbb{C}\Gamma$ is the group algebra of a discrete group we have

$$C^*_{\mathrm{red}}(\mathbb{C}\Gamma) = C^*_{\mathrm{red}}(\Gamma), \quad \mathcal{L}(\mathbb{C}\Gamma) = \mathcal{L}(\Gamma)$$

and when $A = \mathcal{O}(G)$ for a compact group G, we have

$$C^*_{\mathrm{red}}(\mathcal{O}(G)) = C(G), \quad \mathcal{L}(\mathcal{O}(G)) = L^\infty(G)$$

1.5. **Free products.** Let A, B be algebras, and let $A * B$ be their free product (coproduct in the category of unital algebras). For algebras defined by generators and relations, the free product $A * B$ is constructed as the algebra presented by the generators of A and B, and the only relations in $A * B$ are those coming from those of A and B.

For the reader's convenience, let us recall one other possible construction of $A * B$ (see [67]). First, let us say that a subspace X of an algebra A is an **augmentation subspace of** A if $A = \mathbb{C}1 \oplus X$. Now if $X = Z_1$ and $Y = Z_2$ are augmentation subspaces of A and B respectively, we have

$$A * B = \mathbb{C}1 \oplus \left(\bigoplus_{m=1}^{\infty} \bigoplus_{i_1 \neq i_2 \neq \cdots \neq i_m} Z_{i_1} \otimes \cdots \otimes Z_{i_m} \right).$$

The right-handed term is denoted $X*Y$; this is an augmentation subspace of $A*B$. If $\{a_i, i\in I\}$, $\{b_j,\ j\in J\}$ denote respective bases of X and Y, then the elements

$$\begin{aligned}
&a_{i_1}b_{j_1}\cdots a_{i_m}b_{j_m}a_{i_{m+1}},\ i_1,\dots,i_{m+1}\in I,\ j_1,\dots,j_m\in J,\ m\geq 0\\
&b_{j_1}a_{i_1}\cdots b_{j_m}a_{i_m}b_{j_{m+1}},\ i_1,\dots,i_m\in I,\ j_1,\dots,\ j_{m+1}\in J, m\geq 0\\
&a_{i_1}b_{j_1}\cdots a_{i_m}b_{j_m},\ i_1,\dots,i_m\in I,\ j_1,\dots,j_m\in J,\ m\geq 1\\
&b_{j_1}a_{i_1}\cdots b_{j_m}a_{i_m},\ i_1,\dots,i_m\in I,\ j_1,\dots,j_m\in J,\ m\geq 1
\end{aligned}\tag{1.1}$$

form a basis of $X*Y$.

Now let A,B be compact Hopf algebras. Recall [83] that the free product algebra $A*B$ has a unique compact Hopf algebra structure such that the canonical morphisms $A\to A*B$ and $B\to A*B$ are Hopf $*$-algebra maps. An $A*B$-comodule is said to be a simple alternated $A*B$-comodule if it has the form $V_1\otimes\cdots\otimes V_n$,where each V_i is a simple non-trivial A-comodule or B-comodule, and if V_i is an A-comodule, then V_{i+1} is an B-comodule, and conversely. It is proved in [83, Theorem 3.10] that the simple $A*B$-comodules are exactly the simple alternated comodules. This result is obtained using the Peter-Weyl decompositions of A and B, together with the above description of $A*B$.

1.6. **Notations and further premiminaries.** Let A be a Hopf algebra. The very convenient Sweedler notation is, for $a\in A$,

$$\Delta(a)=a_{(1)}\otimes a_{(2)}$$

With this notation, the Hopf algebra axioms become

$$(\Delta\otimes\mathrm{id}_A)\Delta(a)=a_{(1)}\otimes a_{(2)}\otimes a_{(3)}=(\mathrm{id}_A\otimes\Delta)\Delta(a)$$

$$\varepsilon(a_{(1)})a_{(2)}=a=a_{(1)}\varepsilon(a_{(2)}),\quad S(a_{(1)})a_{(2)}=\varepsilon(a)1=a_{(1)}S(a_{(2)})$$

If V is an A-comodule with coaction $\alpha:V\to V\otimes A$, the Sweedler notation is

$$\alpha(v)=v_{(0)}\otimes v_{(1)}$$

and the comodule axioms are

$$(\alpha\otimes\mathrm{id}_A)\alpha(v)=v_{(0)}\otimes v_{(1)}\otimes v_{(2)}=(\mathrm{id}_V\otimes\Delta)\alpha(v),\ (\mathrm{id}_V\otimes\varepsilon)\alpha(v)=v_{(0)}\varepsilon(v_{(1)})=v$$

We will consider module over A as well, and most often right A-modules. If $\alpha:A\to\mathbb{C}$ is an algebra map, we will denote by $\mathbb{C}_\alpha$ the right one-dimensional A-module defined by $1\cdot a=\alpha(a)1$.

If M, N are (right) A-modules, we denote by $\mathrm{Hom}_A(M,N)$ the space of A-linear maps from M to N.

2. Exact sequences of compact Hopf algebras

In this section we discuss exact sequences of compact Hopf algebras. This relies on some classical but technical and non-trivial Hopf algebra works, e.g. [3, 75, 76, 79]. Fortunately the situation has been simplified thanks to a recent result by Chirvasitu [26].

2.1. **Crossed product.** We begin with a simple but important construction. Let Γ be a discrete group acting on a Hopf $*$-algebra A, via a group morphism $\alpha:\Gamma\to\mathrm{Aut}(A)$ (where $\mathrm{Aut}(A)$ means the group of Hopf $*$-algebra automorphisms of A). To this data, we associate, as usual, the crossed product $*$-algebra $A\rtimes\Gamma$, which has $A\otimes\mathbb{C}\Gamma$ as underlying vector space, and product and involution defined by

$$a\otimes g\cdot b\otimes h=a\alpha_g(b)\otimes gh,\ (a\otimes g)^*=\alpha_{g^{-1}}(a^*)\otimes g^{-1}\quad a,b\in A,\ g,h\in G$$

Then $A\rtimes\Gamma$ has a natural Hopf $*$-algebra structure defined by

$$\Delta(a\otimes g)=a_{(1)}\otimes g\otimes a_{(2)}\otimes g,\ \varepsilon(a\otimes g)=\varepsilon(a),\ S(a\otimes g)=\alpha_{g^{-1}}(S(a))\otimes g^{-1}$$

and is compact if A is.

2.2. **Exact sequences.** We now define exact sequences of compact Hopf algebras. We begin with the following preliminary notation and results.

- Let $B \subset A$ be a compact Hopf subalgebra (this means that A is a compact Hopf algebra and that B is a Hopf subalgebra: B is then automatically a Hopf $*$-subalgebra and is compact). Let $B^+ = \mathrm{Ker}(\varepsilon) \cap B$ and let B^+A (resp. AB^+) be the right (resp. left) sub-A-module of A generated by B^+. When $B^+A = AB^+$, then this space is a Hopf $*$-ideal, and hence the quotient A/B^+A has a compact Hopf algebra structure such that the canonical map $p : A \to A/B^+A$ is a Hopf $*$-algebra map.
- Let $p : A \to L$ be a surjective morphism of compact Hopf algebras, and let
$$A^{\mathrm{co}L} = \{a \in A : (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}, \quad {}^{\mathrm{co}L}A = \{a \in A : (p \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}$$
Both are $*$-subalgebras of A, and when $A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$, this is a compact Hopf subalgebra of A.

Theorem-Definition 2.1. *A sequence of compact Hopf algebra maps*
$$\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$$
with i injective and p surjective, is said to be ***exact*** *if the following equivalent conditions hold.*

(1) $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$ *and* $i(B) = A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$.
(2) $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$.
(3) $i(B) = A^{\mathrm{co}L} = {}^{\mathrm{co}L}A$.

Comments on the proof. Clearly (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). To prove (2) $\Rightarrow$ (1), one can combine results of Chirvasitu [26] (faithful flatness of A as an B-module) and of Takeuchi [79, Theorem 1] (regarding this last reference, the reader will find [64, Proposition 3.4.3] of easier access). For (3) $\Rightarrow$ (1), [79, Theorem 2] does the job, since L is compact, and hence cosemisimple (regarding the proof of (3) $\Rightarrow$ (1), the reader might like to consult [85] for a more direct argument in the cosemisimple case). $\square$

Example 2.2. If Γ is a discrete group acting on a compact Hopf $*$-algebra A, then
$$\mathbb{C} \to A \xrightarrow{i} A \rtimes \Gamma \xrightarrow{\varepsilon \otimes \mathrm{id}} \mathbb{C}\Gamma \to \mathbb{C}$$
is an exact sequence of compact Hopf algebras.

Example 2.3. A sequence $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ of morphisms of discrete groups is exact if and only if the corresponding sequence of compact Hopf algebras $\mathbb{C} \to \mathbb{C}\Gamma_1 \to \mathbb{C}\Gamma_2 \to \mathbb{C}\Gamma_3 \to \mathbb{C}$ is exact.

Example 2.4. A sequence $1 \to N \to G \to H \to 1$ of morphisms of compact groups is exact if and only if the corresponding sequence of compact Hopf algebras $\mathbb{C} \to \mathcal{O}(H) \to \mathcal{O}(G) \to \mathcal{O}(N) \to \mathbb{C}$ is exact.

2.3. **Cocentral exact sequences.** The main example of exact sequence of Hopf algebras we will use is of a special type, that we discuss now. One advantage is that in this restricted setting, we can prove quite simply and directly exactness in all the possible senses of Theorem-Definition 2.1.

A Hopf algebra map $f : A \to B$ is said to be **cocentral** if $f(a_{(1)}) \otimes a_{(2)} = f(a_{(2)}) \otimes a_{(1)}$ for any $a \in A$.

Example 2.5. Let $H \subset G$ be a closed subgroup of classical compact group. Then the restriction map
$$\begin{aligned} p : \mathcal{O}(G) &\longrightarrow \mathcal{O}(H) \\ f &\longmapsto f_{|H} \end{aligned}$$
is cocentral if and only if $H \subset Z(G)$. Indeed, p is cocentral if and only if
$$f(xh) = f(hx), \ \forall f \in \mathcal{O}(G), \ \forall x \in G, \ \forall h \in H$$
and the conclusion follows since $\mathcal{O}(G)$ separates the points of G.

Proposition 2.6. *Let $p : A \to \mathbb{C}\Gamma$ be surjective cocentral morphism of compact Hopf algebras. Then $A^{\mathrm{co}\mathbb{C}\Gamma} = {}^{\mathrm{co}\mathbb{C}\Gamma}A$, and the sequence*

$$\mathbb{C} \to A^{\mathrm{co}\mathbb{C}\Gamma} \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\Gamma \to \mathbb{C}$$

is exact.

Proof. The cocentrality condition clearly ensures that $A^{\mathrm{co}\mathbb{C}\Gamma} = {}^{\mathrm{co}\mathbb{C}\Gamma}A$, so we can use the previous theorem to conclude that the sequence is exact. We give direct proof of exactness in this particular setting. We have to show that letting $B = A^{\mathrm{co}\mathbb{C}\Gamma}$, we have $\mathrm{Ker}(p) = AB^+ = B^+A$.

The map $(\mathrm{id} \otimes p)\Delta : A \to A \otimes \mathbb{C}\Gamma$ endows A with a $\mathbb{C}\Gamma$-comodule structure, so the structure of comodules of a group algebra (Example 1.22) gives a decomposition $A = \bigoplus_{g\in\Gamma} A_g$ with

$$A_g = \{a \in A \mid a_{(1)} \otimes p(a_{(2)}) = a \otimes g\}$$

and $A_1 = A^{\mathrm{co}\mathbb{C}\Gamma} = B$, $p_{|A_g} = \varepsilon(-)g$, $A_gA_h \subset A_{gh}$, $\Delta(A_g) \subset A_g \otimes A_g$, $S(A_g) \subset A_{g^{-1}}$. For fixed $g, h \in \Gamma$ take $b \in A_{h^{-1}}$ such that $\varepsilon(b) = 1$ (such a b exists by surjectivity of p). Then for any $a \in A_{gh}$ we have

$$a = ab_{(1)}S(b_{(2)}) \in A_gA_h.$$

The same argument shows that if $a \in A^+_{gh}$, then $a \in A^+_gA_h$, hence $A^+_gA_h = A^+_{gh}$. Similarly one checks that $A^+_{gh} = A_gA^+_h$. Now let $a \in \mathrm{Ker}(p)$, and write $a = \sum_{g\in\Gamma} a_g$, with $a_g \in A_g$. Since $a \in \mathrm{Ker}(p)$, each a_g belongs to $A^+_g = A^+_{gg^{-1}g} = A^+_1A_g \in B^+A$, and to $A^+_g = A^+_{gg^{-1}g} = A_gA^+_1 \subset AB^+$. This finishes the proof. $\square$

Example 2.7. It is an immediate verification that the Hopf $*$-algebra map

$$\begin{aligned} \mathcal{O}(\mathrm{SU}_q(2)) &\longrightarrow \mathbb{C}\mathbb{Z}_2 \\ u_{ij} &\longmapsto \delta_{ij}g \end{aligned}$$

where g denotes the generator of the cyclic group of order 2, is cocentral. We thus get a cocentral exact sequence

$$\mathbb{C} \to \mathcal{O}(\mathrm{PU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

where $\mathcal{O}(\mathrm{PU}_q(2)) = \mathcal{O}(\mathrm{SU}_q(2)^{\mathrm{co}\mathbb{C}\mathbb{Z}_2}$ is the subalgebra generated by the elements $u_{ij}u_{kl}$, $1 \leq i, j, k, l \leq 2$.

2.4. **Graded twisting.** To finish the section, we discuss a construction of a new Hopf compact Hopf algebra from an old one, called graded twisting [20], combining crossed products and cocentral Hopf algebra maps.

Definition 2.8. Let A be a compact Hopf algebra and let Γ be a discrete group. An **invariant cocentral action** of Γ on A is a pair (p, α) where

(1) $p\colon A \to \mathbb{C}\Gamma$ is a cocentral surjective Hopf $*$-algebra map,
(2) $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ is an action of Γ by Hopf $*$-algebra automorphisms on A, with $p\alpha_g = p$ for all $g \in \Gamma$.

Recall from the proof of Proposition 2.6 that $p\colon A \to \mathbb{C}\Gamma$ as above gives a decomposition (a Γ-grading)

$$A = \bigoplus_{g\in\Gamma} A_g \text{ with } A_g = \{a \in A \mid a_{(1)} \otimes p(a_{(2)}) = a \otimes g\}$$

and $A_1 = A^{\mathrm{co}\mathbb{C}\Gamma} = B$, $p_{|A_g} = \varepsilon(-)g$, $A_gA_h \subset A_{gh}$, $\Delta(A_g) \subset A_g \otimes A_g$, $S(A_g) \subset A_{g^{-1}}$.

In terms of this Γ-grading, the last condition is equivalent to $\alpha_g(A_h) = A_h$ for all $g, h \in \Gamma$.

Definition 2.9. Given an invariant cocentral action (p, α) of a discrete group Γ on a compact Hopf algebra A, the **graded twisting** $A^{t,\alpha}$ of A is the Hopf $*$-subalgebra

$$A^{t,\alpha} = \sum_{g\in\Gamma} A_g \otimes g \subset A \rtimes \Gamma,$$

A graded twisting of a compact Hopf algebra is clearly again a compact Hopf algebra. Notice A and $A^{t,\alpha}$ are isomorphic as *coalgebras*, via

$$j\colon A \to A^{t,\alpha}, \quad \sum_g a_g \mapsto \sum_g a_g \otimes g.$$

Proposition 2.10. *Let A be compact Hopf algebra, and let $A^{t,\alpha}$ be a graded twisting of A by a discrete group Γ. Then we have cocentral exact sequences*

$$\mathbb{C} \to B \to A \to \mathbb{C}\Gamma \to \mathbb{C}, \quad \mathbb{C} \to B \to A^{t,\alpha} \to \mathbb{C}\Gamma \to \mathbb{C}$$

for the same compact Hopf algebra B

Proof. The first exact sequence arises, by Proposition 2.6, from the given invariant cocentral action (p, α), with $B = A_1 = A^{\mathrm{co}\mathbb{C}\Gamma}$. For the second one, let $\tilde{p}$ be the restriction of $\varepsilon \otimes \mathrm{id} : A \rtimes \Gamma \to \mathbb{C}\Gamma$ to $A^{t,\alpha}$. It is then a direct verification to check that $\tilde{p}$ is a cocentral surjective Hopf $*$-algebra map, with $(A^{t,\alpha})^{\mathrm{co}\mathbb{C}\Gamma} = A^{\mathrm{co}\mathbb{C}\Gamma} \otimes \mathbb{C} \simeq A^{\mathrm{co}\mathbb{C}\Gamma}$, and hence Proposition 2.6 again furnishes the cocentral exact sequence. □

We now will use the graded twisting to relate the quantum groups O_n^+ and U_n^+.

Consider the compact Hopf algebra $A_o(n) * A_o(n)$ (the free product of the compact Hopf algebra $A_o(n)$ by itself): $A_o(n) * A_o(n)$ is the algebra generated by two copies of $A_o(n)$ without futher relations than those of $A_o(n)$. In other words it is the algebra presented by generators x_{ij}, y_{ij}, $1 \leq i, j \leq n$, submitted to the relations

$$xx^t = x^t x = I_n = y^t y = yy^t$$

and whose Hopf $*$-algebra structure is induced by those of the two copies of $A_o(n)$.

We then have:

- A cocentral Hopf $*$-algebra map
$$p : A_o(n) * A_o(n) \to \mathbb{C}\mathbb{Z}_2,\ x_{ij}, y_{ij} \mapsto \delta_{ij} g$$
- An action α of $\mathbb{Z}_2$ on $A_o(n) * A_o(n)$ given by the automorphism that exchanges the two copies.

It is then straighforward to check that we have in this way an invariant cocentral action of $\mathbb{Z}_2$ on $A_o(n) * A_o(n)$, so that we can form the graded twisting $(A_o(n) * A_o(n))^{t,\alpha}$.

Proposition 2.11. *The Hopf $*$-algebra map*

$$\theta\colon A_u(n) \longrightarrow (A_o(n) * A_o(n))^{t,\alpha}$$
$$u_{ij}, u_{ij}^* \longmapsto x_{ij} \otimes g, y_{ij} \otimes g$$

is an isomorphism.

Proof. Of course, the first thing to do is to check that this algebra map is indeed well-defined: this is straightforward. To construct the inverse, notice that we have:

- A cocentral Hopf $*$-algebra map
$$q : A_u(n) \to \mathbb{C}\mathbb{Z}_2,\ u_{ij} \mapsto \delta_{ij} g$$
- An action β of $\mathbb{Z}_2$ on $A_o(n) * A_o(n)$ given by the automorphism $u_{ij} \mapsto u_{ij}^*$.

It is straighforward to check that we have in this way an invariant cocentral action of $\mathbb{Z}_2$ on $A_u(n)$, and we can form the graded twisting $A_u(n)^{t,\beta}$. We then have a Hopf $*$-algebra map

$$\pi\colon A_o(n) * A_o(n) \longrightarrow A_u(n)^{t,\beta} \subset A_u(n) \rtimes \mathbb{Z}_2$$
$$x_{ij}, y_{ij} \longmapsto u_{ij} \otimes g, u_{ij}^* \otimes g$$

We have a $\mathbb{Z}_2$-action on $A_u(n) \rtimes \mathbb{Z}_2$ given by $\tilde{\beta}_g(a \otimes t) = \beta_g(a) \otimes t$, and π above is equivariant with respect to these actions (the first one is the given one on $A_o(n) * A_o(n)$), so π induces a $*$-algebra map

$$\pi \otimes \mathrm{id}\colon (A_o(n) * A_o(n)) \rtimes \mathbb{Z}_2 \to (A_u(n) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$$

We then have the $*$-algebra map

$$(\pi \otimes \mathrm{id})\theta\colon A_u(n) \longrightarrow (A_u(n) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$$
$$u_{ij},\ u_{ij}^* \longmapsto u_{ij} \otimes g \otimes g,\ u_{ij}^* \otimes g \otimes g$$

which thus satisfies $(\pi \otimes \mathrm{id})\theta(a) = a \otimes h \otimes h$ for any $a \in A_u(n)_h$, $h \in \{1, g\}$. Hence θ is injective and π is surjective. A similar reasoning, exchanging the roles of θ and π, shows that π is injective and θ is surjective, and this finishes the proof. $\square$

3. Homological algebra

We now present the necessary homological algebra background to define the homological invariants we are interested in.

3.1. **Projective modules.** Let A be an algebra and let M be an A-module. The functor $\mathrm{Hom}_A(M, -)$ from A-modules to vector spaces is left exact: if

$$0 \to X \stackrel{i}{\to} Y \stackrel{p}{\to} Z \to 0$$

is an exact sequence of A-modules (in the usual sense: i is injective, p is surjective, and $\mathrm{Im}(i) = \mathrm{Ker}(p)$), then the sequence

$$0 \to \mathrm{Hom}_A(P, X) \stackrel{i\circ -}{\to} \mathrm{Hom}_A(P, Y) \stackrel{p\circ -}{\to} \mathrm{Hom}_A(P, Z)$$

is exact. Projective modules are precisely those for which this functor is exact.

Proposition-Definition 3.1. *A (right) A-module P is said to be* ***projective*** *if one of the equivalent following conditions holds.*

1. *The functor $\mathrm{Hom}_A(P, -)$ is exact.*
2. *For any surjective A-linear $p : M \to N$ and any A-linear map $\phi : P \to N$, there exists an A-linear map $\psi : P \to N$ such that $p\psi = \phi$.*
3. *Any surjective A-linear map $f : M \to P$ admits a section, i.e. there exists an A-linear map $s : P \to M$ such that $fs = \mathrm{id}_P$.*
4. *There exists a free A-module F and an A-module Q such that $F \simeq P \oplus Q$ as A-modules.*

The proof, left as an exercise, can be found in any algebra textbook. Notice that if $M = \oplus_{i\in I} M_i$ is a direct sum of A-modules , then M is projective if and only if each M_i is.

For a Hopf algebra A, projectivity of the trivial A-module $\mathbb{C}_\varepsilon$ has very important consequences on the structure of A.

Proposition 3.2. *Let A be a Hopf algebra. The following properties are equivalent.*

1. *The trivial A-module $\mathbb{C}_\varepsilon$ is projective.*
2. *There exists $t \in A$ such that $ta = \varepsilon(a)t$, for any $a \in A$, and $\varepsilon(t) = 1$.*
3. *The algebra A is semisimple and finite-dimensional.*

Proof. (1) $\Rightarrow$ (2): the counit can be interpreted as a surjective A-linear map $\varepsilon : A \to \mathbb{C}_\varepsilon$. Hence if $\mathbb{C}_\varepsilon$ is projective, the previous proposition furnishes a section to ε, and hence the announced t. An algebra is semisimple precisely when all its modules are projective, so (3) $\Rightarrow$ (1) is trivial.

It remains to prove that (2) $\Rightarrow$ (3). Assume that such a t exists. Given an A-module M, we denote by M^A the space of A-invariants:

$$M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x, \forall a \in A\}$$

It is not difficult to check that for t as in (2), one has $M^A = M \cdot t$.

Now if M, N are A-modules, then $\mathrm{Hom}(M, N)$ admits a right A-module structure defined by

$$f \cdot a(x) = f(x \cdot S(a_{(1)})) \cdot a_{(2)}$$

and we have then $\mathrm{Hom}_A(M, N) = \mathrm{Hom}(M, N)^A$ (check this). So for $f \in \mathrm{Hom}(M, N)$, we have $f \cdot t\ \mathrm{Hom}_A(M, N)$. If $N \subset M$ is a sub-A-module, let $p : M \to N$ be a $\mathbb{C}$-linear map such that

$p_{|N} = \mathrm{id}_N$. One sees easily that still $p \cdot t_{|N} = \mathrm{id}_N$, so we have the direct sum of A-modules $M = N \oplus \mathrm{Ker}(p \cdot t)$, and A is indeed semisimple.

To conclude that A is finite-dimensional, we will show that the linear map

$$\begin{aligned} A^* &\longrightarrow A \\ \omega &\longmapsto \omega(t_{(1)})t_{(2)} \end{aligned}$$

is injective (see Lemma 1.2 in [81] for a left-handed version), which will force A to be finite-dimensional.

For $a \in A$, we have

$$\begin{aligned} ta = \varepsilon(a)t \Rightarrow ta_{(1)} \otimes a_{(2)} = t \otimes a \Rightarrow\ t_{(1)}a_{(1)} \otimes t_{(2)}a_{(2)} \otimes a_{(3)} = t_{(1)} \otimes t_{(2)} \otimes a \\ \Rightarrow t_{(1)}a_{(1)} \otimes t_{(2)}a_{(2)}S(a_{(3)}) = t_{(1)} \otimes t_{(2)}S(a) \Rightarrow\ t_{(1)}a \otimes t_{(2)} = t_{(1)} \otimes t_{(2)}S(a) \end{aligned}$$

Hence if ω is in the kernel of the above map, we have $\omega(t_{(1)}a)t_{(2)} = 0$ for any $a \in A$. Writing $\Delta(t) = \sum_{i=1}^m a_i \otimes b_i$ with $b_1, \ldots, b_m$ linearly independent, we thus have $\omega(a_i a) = 0$ for any i and any a. Hence $\omega(a_i S(b_i)a) = 0$ for any i, and

$$0 = \sum_{i=1}^m \omega(a_i S(b_i)a) = \omega(t_{(1)}S(t_{(2)})a) = \varepsilon(t)\omega(a) = \omega(a)$$

Hence $\omega = 0$, as needed. □

The question whether a Hopf algebra is projective as a module over a its Hopf subalgebras is a crucial one in Hopf algebra theory, and has a negative answer in full generality [74]. Here is a positive basic result.

Proposition 3.3. *Let $p : A \to \mathbb{C}\Gamma$ be surjective cocentral morphism of compact Hopf algebras, with Γ a discrete group. Let $B = A^{\mathrm{co}\mathbb{C}\Gamma}$. Then A is projective as (left or right) B-module.*

Proof. We retain the notation in the proof of Proposition 2.6, where we have shown that $A_1 = B = A_g A_{g^{-1}}$ for any $g \in \Gamma$. If we choose $x_i \in A_g$ and $y_i \in A_{g^{-1}}$ such that $\sum_{i=1}^n x_i y_i = 1$, then we can define a right A_1-module map $A_g \to B^n$ by $a \mapsto (y_i a)_{i=1}^n$ and its left inverse $B^n \to A_g$ by $(a_i)_{i=1}^n \mapsto \sum_i x_i a_i$. Hence each A_g is B-projective and so is A. □

In fact, projectivity over Hopf subalgebras holds generally in the compact case.

Theorem 3.4. *Let $B \subset A$ be a compact Hopf subalgebra. Then A is projective as a left and right B-module.*

About the proof. The proof follows by combining [26, Theorem 2.1] (faithful flatness of A as a B-module) and [75, Corollary 1.8]. The proofs of these results require some work and material that we do not wish to develop here. □

3.2. **Projective dimension of a module.** We now define the projective dimension of a module, which measures how far it is from being projective. It is the key step towards the definition of the cohomological dimension of a Hopf algebra in the next section.

Definition 3.5. Let M be an A-module. A **resolution** of M is an exact sequence of A-modules

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \overset{\partial_n}{\to} P_{n-1} \to \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} M \to 0$$

The resolution $P_* \to M$ is said to be

1. **finite** if there exists $n \geq 0$ such that for any $k > n$, $P_k = 0$, the smallest such n being called the **length** of the resolution;
2. **projective** if the P_i's are projective A-modules;
3. **free** is the P_i's are free A-modules.

Of course we make the convention that the 0-module is free. It is not difficult to prove that any A-module admits a free (and hence projective) resolution, that one can construct step by step.

Definition 3.6. The **projective dimension** of a non-zero A-module M is defined to be

$$\mathrm{pd}_A(M) = \min\{n : M \text{ admits a projective resolution of length } n\} \in \mathbb{N} \cup \{\infty\}$$

and we make the convention that the projective dimension of the zero module is zero.

Examples 3.7. (1) An A-module M is projective if and only if $\mathrm{pd}_A(M) = 0$.

(2) Let $A = \mathbb{C}\mathbb{Z} = \mathbb{C}[t, t^{-1}]$ be the group algebra of $\mathbb{Z}$. Then A^+ is easily seen to be free as an A-module (freely generated by $t-1$), so we have a free resolution of $\mathbb{C}_\varepsilon$

$$0 \to A^+ \to A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

and hence $\mathrm{pd}(\mathbb{C}_\varepsilon) \leq 1$. Since A is infinite-dimensional, we have $\mathrm{pd}_A(\mathbb{C}_\varepsilon) > 0$, so $\mathrm{pd}_A(\mathbb{C}_\varepsilon) = 1$. More generally, one can show that if $A = \mathbb{C}\mathbb{F}_n$ is the group algebra of the free group on $n \geq 1$ generators, then $\mathrm{pd}_A(\mathbb{C}_\varepsilon) = 1$ (see e.g. [86, Chapter 6])

(3) If A is a Hopf algebra, the standard resolution of the trivial object $\mathbb{C}_\varepsilon$ is the free resolution (A acting by multiplication on the left)

$$\cdots \longrightarrow A^{\otimes n+1} \longrightarrow A^{\otimes n} \longrightarrow \cdots \longrightarrow A \otimes A \longrightarrow A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

where each map $A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$a_1 \otimes \cdots \otimes a_{n+1} \mapsto \varepsilon(a_1) a_2 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

To checks exactness, one shows that the identity of this complex is *homotopic* to the zero map, see e.g. [86].

3.3. Ext **spaces.** We now provide another interpretation of projective dimension, in terms of certain cohomology spaces.

Definition 3.8. A **cochain complex** $C_* = (C_*, d_*)$ consists of a sequence of complex vector spaces and linear maps

$$0 \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \to \cdots \to C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} C_{n+2} \to \cdots$$

such that for any $n \geq 0$, we have $d_{n+1} d_n = 0$. For $n \geq 0$, the n-th **cohomology** space of the complex C_* is then defined by

$$H^n(C_*) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1})$$

making the convention that $d_{-1} = 0$.

Definition 3.9. A **chain complex** $C_* = (C_*, d_*)$ consists of a sequence of complex vector spaces and linear maps

$$\cdots \to C_{n+2} \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \to \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

such that for any $n \geq 0$, we have $d_n d_{n+1} = 0$. For $n \geq 0$, the n-th **homology** space of the complex C_* is then defined by

$$H_n(C_*) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$$

making the convention that $d_0 = 0$.

Remark 3.10. If M is an A-module, a resolution of M

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \to 0$$

as in Definition 3.5 can be seen as a chain complex with trivial homology. Forgetting M, we get a chain complex P_*

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to 0$$

with $H_0(P_*) \simeq M$ and $H_n(P_*) = 0$ is $n \geq 1$

We can now define the Ext-spaces between two A-modules.

Theorem-Definition 3.11. *Let M, N be right A-modules. Let $P_* \to M \to 0$ be a projective resolution of M*

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \to \cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \to 0$$

and consider the associated complex $\mathrm{Hom}_A(P_*, N)$

$$0 \to \mathrm{Hom}_A(P_0, N) \xrightarrow{-\circ\partial_1} \mathrm{Hom}_A(P_1, N) \xrightarrow{-\circ\partial_2} \mathrm{Hom}_A(P_2, N) \xrightarrow{-\circ\partial_3} \cdots$$

Then the cohomology spaces $H^*(\mathrm{Hom}_A(P_*, N))$ do not depend on the choice of the projective resolution P_*, and are denoted $\mathrm{Ext}^*_A(M, N)$.

For the proof, see any homological algebra textbook, for example [86]. It is easy to see that $\mathrm{Ext}^0_A(M, N) \simeq \mathrm{Hom}_A(M, N)$, and in fact the equivalences classes of elements in $\mathrm{Ext}^n_A(M, N)$ truly correspond to equivalences classes of exact sequences of A-modules of length $n+2$ starting at N and finishing at M, see [86] again.

The Ext-spaces and the projective dimension are related as follows.

Proposition 3.12. *Let M be an A-module. The following assertions are equivalent.*

(1) $\mathrm{pd}_A(M) \leq n$.
(2) $\mathrm{Ext}^i_A(M, -) = 0$ *for* $i > n$.
(3) $\mathrm{Ext}^{n+1}_A(M, -) = 0$.
(4) For any exact sequence of A-modules $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, then K is projective.
(5) For any exact sequence of A-modules $0 \to L \xrightarrow{i} P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, there exists $r \in \mathrm{Hom}_A(P_n, L)$ such that $ri = \mathrm{id}_L$.

Proof. (2) $\Rightarrow$ (3) is obvious, and so are (4) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2), just by writing the definitions. Assume that (3) holds, and let $0 \to K \xrightarrow{i} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence of A-modules with each P_i projective. Complete this exact sequence to a projective resolution

$$\begin{tikzcd}
{} \arrow[r] & \cdots P_{n+2} \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[dr, "q"'] & P_n \arrow[r, "d_n"] \arrow[dr, "p"'] & P_{n-1} \arrow[r, "d_{n-1}"] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & & L \arrow[u, "j"'] & K \arrow[u, "i"'] & & & &
\end{tikzcd}$$

We are going to show that $P_n \simeq K \oplus L$ as A-modules, so that K, being a direct summand of a projective module, will be projective, and this will prove that (3) $\Rightarrow$ (4).

We have an exact sequence $0 \to L \xrightarrow{j} P_n \xrightarrow{p} K \to 0$, and hence to show that $P_n \simeq K \oplus L$, it is enough to show that there exists an A-linear map $r : P_n \to L$ such that $rj = \mathrm{id}_L$. Consider $q \in \mathrm{Hom}_A(P_{n+1}, L)$. We have $qd_{n+2} = 0$ since $jqd_{n+2} = d_{n+1}d_{n+2} = 0$ and j is injective. Hence since $\mathrm{Ext}^{n+1}_A(M, L) = 0$, there exists $r \in \mathrm{Hom}_A(P_n, L)$ such that $q = rd_{n+1}$. Then $rjq = q$, and since q is surjective, we have $rj = \mathrm{id}_L$, as needed.

Assume now that (4) holds, and let $0 \to L \xrightarrow{i} P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be exact with each P_i projective. We then have an exact sequence

$$0 \to P_n/\mathrm{Im}(i) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

By (4), we have that $P_n/\mathrm{Im}(i)$ is a projective A-module, so $P_n \simeq P_n/\mathrm{Im}(i) \oplus L$ as A-modules and (5) follows. The proof of (5) $\Rightarrow$ (1) is left as an exercise. $\square$

Corollary 3.13. *We have, for any A-module M*

$$\mathrm{pd}_A(M) = \sup\{n : \ \mathrm{Ext}^n_A(M, N) \neq 0 \text{ for some } A\text{-module } N\}$$

3.4. **Cohomology of a Hopf algebra.** Let A be Hopf algebra. If M is a right A-module, the Ext-spaces

$$\mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, M)$$

serve as cohomology spaces for A. It is thus tempting to denote them $H^*(A, M)$, but we will not do exactly that, since it is contrary to some more usual notations. Indeed, in general, if A is an algebra and M is an A-bimodule, then $H^*(A, M)$ denotes usually the Hochschild cohomology of A with coefficients in M.

Definition 3.14. The **cohomology of a Hopf algebra with coefficients in a right A-module** M, denoted $H^*(A, {}_\varepsilon M)$, is defined by

$$H^*(A, {}_\varepsilon M) = \mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, M)$$

Remark 3.15. Given a right A-module as above, the cohomology $H^*(A, {}_\varepsilon M)$ as above coincides with the Hochschild cohomology $H^*(A, {}_\varepsilon M)$, where ${}_\varepsilon M$ is the A-bimodule having M as underlying right A-module, and trivial left A-module structure given by $a \cdot x = \varepsilon(a)x$. So our notation is consistent with the usual one in the literature.

Remark 3.16. The cohomology of a discrete group Γ is defined similarly as above, but using the *integral* group ring $\mathbb{Z}\Gamma$. Since we cannot seriously impose that the Hopf algebras we are interest in are defined over $\mathbb{Z}$, our definition is *not* a full generalization of ordinary group cohomology, but rather of group cohomology with coefficients into $\mathbb{C}\Gamma$-modules.

Remark 3.17. The cohomology of a Hopf algebra only depends of the underlying augmented algebra.

Using the standard resolution of the trivial module (Examples 3.7) together with Theorem 3.11, we get, after some identifications, a more concrete definition for cohomology.

Proposition 3.18. *Let A be Hopf algebra and let M be a right A-module. Then the cohomology $H^*(A, {}_\varepsilon M)$ is the cohomology of the complex*

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M) \xrightarrow{\delta} \mathrm{Hom}(A, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta\colon \mathrm{Hom}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$

We thus have in particular

$$H^0(A, {}_\varepsilon M) = M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x, \ \forall a \in A\}$$

and

$$H^1(A, {}_\varepsilon M) = \mathrm{Der}(A, {}_\varepsilon M)/\mathrm{InnDer}(A, {}_\varepsilon M)$$

where $\mathrm{Der}(A, {}_\varepsilon M)$ is the vector space of derivations $d : A \to M$, i.e. $d(ab) = \varepsilon(a)d(b) + d(a) \cdot b$ for any a, b, and $\mathrm{InnDer}(A, {}_\varepsilon M)$ is the subspace of inner derivations, i.e. those of type defined by $d(a) = \varepsilon(a)x - x \cdot a$ for some x in M.

In higher degrees, the concrete description is rarely useful to proceed with concrete computations, and the best is often to search for short of simple resolutions of the trivial module $\mathbb{C}_\varepsilon$.

Example 3.19. Let G be a classical compact Lie group. Then

$$H^*(\mathcal{O}(G), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \Lambda^*(\mathfrak{g})$$

where $\mathfrak{g}$ is the (complexification of the) Lie algebra of G. This follows from the HKR (Hochschild-Kostant-Rosenberg) theorem [44], with some other considerations. All this involves some standard but quite non-trivial commutative algebra material, that we do not wish to discuss here. See [86, 58] (the reader might also like to read [47])

Remark 3.20. The second cohomology space $H^2(A, {}_\varepsilon\mathbb{C}_\varepsilon)$ has some interest in quantum probability and the study of Lévy processes on quantum groups, because its vanishing implies that A has the property called AC in [38]: all cocycles can be completed to a Schürmann triple. See [38] for details.

4. Cohomological dimension of a Hopf algebra

We now define and study the cohomological dimension of a (compact) Hopf algebra, and study the examples presented in Section 1.

4.1. **Definition, basic results and examples.**

The cohomological dimension of a Hopf algebra is defined using the trivial module:

Definition 4.1. The **cohomological dimension** of a Hopf algebra A is defined by

$$\mathrm{cd}(A) = \mathrm{pd}_A(\mathbb{C}_\varepsilon) \in \mathbb{N} \cup \{\infty\}$$

We thus have, by Proposition 3.12,

$$\begin{aligned}\mathrm{cd}(A) &= \sup\{n : \ H^n(A, {}_\varepsilon M) \neq 0 \text{ for some } A\text{-module } M\}\\ &= \min\{n : \ H^{n+1}(A, {}_\varepsilon M) = 0 \text{ for any } A\text{-module M}\}\end{aligned}$$

Example 4.2. If Γ is a discrete group, then $\mathrm{cd}(\mathbb{C}\Gamma) = \mathrm{cd}_{\mathbb{C}}(\Gamma)$, the cohomological dimension of Γ with coefficients $\mathbb{C}$. We have $\mathrm{cd}(\mathbb{C}\Gamma) = 0$ if and only if Γ is finite (see Proposition 4.5). If Γ is finitely generated, then $\mathrm{cd}(\mathbb{C}\Gamma) = 1$ if and only if Γ contains a free normal subgroup of finite index, see [34, 30, 31]. If Γ is the fundamental group of an aspherical manifold of dimension n, then $\mathrm{cd}(\mathbb{C}\Gamma) = n$, see [25].

Example 4.3. If $A = \mathcal{O}(G)$, the algebra of representative functions on a compact Lie group G, then $\mathrm{cd}(\mathcal{O}(G)) = \dim G$, the usual dimension of G, i.e. the linear dimension of the Lie algebra of G.

Since the trivial module $\mathbb{C}_\varepsilon$ is a distinguished one, the above definition of the cohomological dimension of a Hopf algebra is perfectly natural, and two isomorphic Hopf algebras have the same cohomological dimension. In fact the following result shows that the cohomological dimension does not even depend on the choice of special module.

Proposition 4.4. *Let A be a Hopf algebra. Then*

$$\mathrm{cd}(A) = \mathrm{Sup}\{\mathrm{pd}_A(M), \ M \in \mathrm{Mod}(A)\}$$

Proof. It is clear that $\mathrm{cd}(A)$ is smaller than the quantity on the right, and to prove the equality, we can assume that $n = \mathrm{cd}(A)$ is finite. Before proceeding, following the argument in [59], we need some preliminaries. If M, N are A-modules, then their tensor product $M \otimes N$ has an A-module structure defined by

$$(x \otimes y) \cdot a = x \cdot a_{(1)} \otimes y \cdot a_{(2)}$$

This defines (exact) functors $- \otimes N$ and $M \otimes -$ on the category of A-modules. The map

$$\begin{aligned}M_t \otimes A &\longrightarrow M \otimes A\\ x \otimes a &\longmapsto x \cdot a_{(1)} \otimes a_{(2)}\end{aligned}$$

is an isomorphism of A-modules (check this, using the antipode), where $M_t \otimes A$ is the free A-module whose A-module structure is given by multiplication on the right. Hence if F is a free A-module, then $N \otimes F$ is a free A-module, and if P is projective, then $N \otimes P$ is a projective A-module.

Consider now a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

For an A-module M, the previous considerations ensure that tensoring this resolution by M on the left yields a projective resolution

$$0 \to M \otimes P_n \to M \otimes P_{n-1} \to \cdots \to M \otimes P_1 \to M \otimes P_0 \to M \otimes \mathbb{C}_\varepsilon \simeq M \to 0$$

and hence $\mathrm{pd}_A(M) \leq n$, as needed. $\square$

Therefore the cohomological dimension of a Hopf algebra coincides with its right global dimension, one of the most classical homological invariants of an algebra, see [86], and only depends on the algebra structure.

Proposition 4.5. *Let A be compact Hopf algebra. Then $\mathrm{cd}(A) = 0$ if and only if A is finite-dimensional.*

Proof. This follows from Proposition 3.2, since a finite-dimensional compact Hopf algebra, being a C^*-algebra, is a finite product of full matrix algebras, and hence is semisimple. $\square$

Proposition 4.6. *Let $B \subset A$ be a compact Hopf subalgebra. Then $\mathrm{cd}(B) \leq \mathrm{cd}(A)$.*

Proof. We know from Theorem 3.4 that A is projective as a B-module, so the restriction of a projective A-module to a B-module is a projective B-module. The result follows, since the restriction of an A-projective resolution of $\mathbb{C}_\varepsilon$ is a B-projective resolution. $\square$

Proposition 4.7. *Let $\mathbb{C} \to B \to A \to L \to \mathbb{C}$ be an exact sequence of compact Hopf algebras. Then $\mathrm{cd}(A) \leq \mathrm{cd}(B) + \mathrm{cd}(L)$, and if L is finite-dimensional, then $\mathrm{cd}(B) = \mathrm{cd}(A)$.*

This is [15, Proposition 3.2], using Stefan's spectral sequence [77]. We will not use the inequality, and we give an indepent proof of the last equality. We begin with a Lemma.

Lemma 4.8. *Let $\mathbb{C} \to B \to A \xrightarrow{p} L \to \mathbb{C}$ be an exact sequence of compact Hopf algebras as above, with L finite-dimensional. Let $\tau \in L$ be a such that $\tau p(a) = \varepsilon(a)\tau$ for any $a \in A$, with $\varepsilon(\tau) = 1$, and let $t \in A$ be such that $p(t) = \tau$ (Proposition 3.2).*

1. *Let M be a right A-module, and let $M^B = \{x \in M \mid x \cdot b = \varepsilon(b)x, \forall b \in B\}$ be the space of B-invariants. Then the A-module structure on M induces an L-module structure on M^B with $(M^B)^L = M^A$.*
2. *Let V, W be right A-modules and let $f : V \to W$ be a B-linear map. Then the linear map $\tilde{f} : V \to W$ defined by $\tilde{f}(v) = f(v \cdot S(t_{(1)})) \cdot t_{(2)}$ is A-linear. If there exists an A-linear map $j : W \to V$ such that $fj = \mathrm{id}_W$, then $\tilde{f}j = \mathrm{id}_W$ as well.*

Proof. (1) For $x \in M^B$ and $b \in B^+$, we have $x \cdot b = 0$. Moreover, for $x \in M^B$, $a \in A$, one easily sees, using that $AB^+ = B^+A$, that $x \cdot a \in M^B$. Hence the formula $x \cdot p(a) = x \cdot a$ provides a well-defined L-module structure on M^B. The last equality is immediate.

(2) Recall that $\mathrm{Hom}(V, W)$ admits a right A-module structure defined by

$$f \cdot a(v) = f(v \cdot S(a_{(1)})) \cdot a_{(2)}$$

and that

$$\mathrm{Hom}_A(V, W) = \mathrm{Hom}(V, W)^A = (\mathrm{Hom}(V, W)^B)^L$$

Recall also that if M is a right L-module over the semisimple algebra L, then $M^L = M \cdot \tau$. Hence, since $f \in \mathrm{Hom}_B(V, W) = \mathrm{Hom}(V, W)^B$, we have $f \cdot \tau \in (\mathrm{Hom}(V, W)^B)^L = \mathrm{Hom}_A(V, W)$. We now have $f \cdot \tau = f \cdot p(t) = f \cdot t$, and it is clear that $f \cdot t$ is the map $\tilde{f}$ in the statement. The last statement is an immediate verification. $\square$

Proof of the equality in Proposition 4.7. We already know that $\mathrm{cd}(B) \leq \mathrm{cd}(A)$, and to prove the equality we can assume that $m = \mathrm{cd}(B)$ is finite. Consider a resolution of the trivial A-module

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon$$

by projective A-modules. These are in particular projective as B-modules, so since $m = \mathrm{cd}(B)$, Proposition 3.12 yields an exact sequence of B-modules, and of A-modules

$$0 \to K \xrightarrow{i} P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}$$

together with $r : P_m \to K$, a B-linear map such that $ri = \mathrm{id}_K$. The previous lemma yields an A-linear map $\tilde{r} : P_m \to K$ such that $\tilde{r}i = \mathrm{id}_K$. We thus obtain, since a direct summand of a projective is projective, a length m resolution of $\mathbb{C}_\varepsilon$ by projective modules over A, and we conclude that $\mathrm{cd}(A) \leq m$, as required. $\square$

Corollary 4.9. *Let A be compact Hopf algebra, and let $A^{t,\alpha}$ be a graded twisting of A by a finite group Γ. We have $\mathrm{cd}(A) = \mathrm{cd}(A^{t,\alpha})$.*

Proof. This follows from the previous proposition, combined with Proposition 2.10. $\square$

In fact, with some more work, the last part of Proposition 4.7 can be strengthened, as follows [18].

Proposition 4.10. *Let $B \subset A$ be a compact Hopf subalgebra. If A is finitely generated as a B-module, then $\mathrm{cd}(B) = \mathrm{cd}(A)$.*

4.2. **Example: the quantum group $\mathrm{SU}_q(2)$.** The aim of this subsection is to compute the cohomological dimension of $\mathcal{O}(\mathrm{SU}_q(2))$:

Theorem 4.11. *We have, for any $q \in \mathbb{R}^*$, $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2))) = 3 = \mathrm{cd}(\mathcal{O}(\mathrm{PU}_q(2)))$.*

Combining Example 2.7, and Proposition 4.7, we have $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2))) = \mathrm{cd}(\mathcal{O}(\mathrm{PU}_q(2)))$ and hence it remains to compute $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2)))$. The main tool is the following result.

Theorem 4.12. *Let $A = \mathcal{O}(\mathrm{SU}_q(2))$. There exists a free resolution of A-modules*

$$0 \to A \xrightarrow{\phi_1} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_2} (\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes A \xrightarrow{\phi_3} A \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

and hence $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2))) \leq 3$.

Sketch of proof. For $x \in A$, denoting e_1, e_2 the canonical basis of $\mathbb{C}^2$, and $a = u_{11}$, $b = u_{12}$, $c = u_{21}$, $d = u_{22}$, the maps ϕ_1, ϕ_2, ϕ_3, are defined by

$$\begin{aligned}
\phi_1(x) =& e_1^* \otimes e_1 \otimes ((-q^{-1} + qd)x) + e_1^* \otimes e_2 \otimes (-cx) \\
& + e_2^* \otimes e_1 \otimes (-bx) + e_2^* \otimes e_2 \otimes ((-q + q^{-1}a)x) \\
\phi_2(e_1^* \otimes e_1 \otimes x) =& e_1^* \otimes e_1 \otimes x + e_2^* \otimes e_1 \otimes (-qbx) + e_2^* \otimes e_2 \otimes ax \\
\phi_2(e_1^* \otimes e_2 \otimes x) =& e_1^* \otimes e_1 \otimes bx + e_1^* \otimes e_2 \otimes (1 - q^{-1}a)x \\
\phi_2(e_2^* \otimes e_1 \otimes x) =& e_2^* \otimes e_1 \otimes (1 - qd)x + e_2^* \otimes e_2 \otimes cx \\
\phi_2(e_2^* \otimes e_2 \otimes x) =& e_1^* \otimes e_1 \otimes dx + e_1^* \otimes e_2 \otimes (-q^{-1}cx) + e_2^* \otimes e_2 \otimes x \\
\phi_3(e_1^* \otimes e_1 \otimes x) =& (a-1)x, \quad \phi_3(e_1^* \otimes e_2 \otimes x) = bx, \\
\phi_3(e_2^* \otimes e_1 \otimes x) =& cx, \quad \phi_3(e_2^* \otimes e_2 \otimes x) = (d-1)x
\end{aligned}$$

The exactness, about one page computation, is shown in [14, Lemma 5.6], using well-known ring-theoretic properties of A, that can be found in [23] (in particular that A is an integral domain). $\square$

Corollary 4.13. *We have*

$$H^p(\mathcal{O}(\mathrm{SU}(2)), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ \mathbb{C}^3 & \text{if } p = 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad H^p(\mathcal{O}(\mathrm{SU}_{-1}(2)), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

and for $q \neq \pm 1$,

$$H^p(\mathcal{O}(\mathrm{SU}_q(2)), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad H^p(\mathcal{O}(\mathrm{SU}_q(2)), {}_\varepsilon\mathbb{C}_\psi) \simeq \begin{cases} \mathbb{C} & \text{if } p = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where $\psi : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{C}$ is the algebra map defined by $\psi(a) = q^2$, $\psi(d) = q^{-2}$, $\psi(b) = \psi(c) = 0$.

Proof. Exercise, using the previous resolution. $\square$

Proof of Theorem 4.11. We have $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2))) \leq 3$ by Theorem 4.12, and $\mathrm{cd}(\mathcal{O}(\mathrm{SU}_q(2))) \geq 3$ by Corollary 4.13: the result follows. $\square$

Remark 4.14. The fact that for $q \neq \pm 1$, $H^2(\mathcal{O}(\mathrm{SU}_q(2)), {}_\varepsilon\mathbb{C}_\varepsilon) = 0 = H^3(\mathcal{O}(\mathrm{SU}_q(2)), {}_\varepsilon\mathbb{C}_\varepsilon)$, in contrast with the classical case, is known as the dimension drop, see [43] for this question.

The homological study of $\mathcal{O}(\mathrm{SU}_q(2))$ has been the subject of numerous papers, see [43, 63, 72], for example. For other q-deformations of classical compact Lie groups, we refer the reader to [22, 42, 24], and the references therein.

4.3. **Example : free orthogonal quantum groups.** We now study the case of free orthogonal quantum groups:

Theorem 4.15. *For $n \geq 2$, we have $\mathrm{cd}(A_o(n)) = 3$.*

The reader will notice that the cohomological dimension does not depend on n. This might look surprising, but recall that the cohomological dimension of the free group $\mathbb{F}_n$ also does not depend on n.

Theorem 4.16. *For $n \geq 2$, there exists a free resolution of $A_o(n)$-modules*

$$0 \to A_o(n) \xrightarrow{\phi_1} (\mathbb{C}^n \otimes \mathbb{C}^n) \otimes A_o(n) \xrightarrow{\phi_2} (\mathbb{C}^n \otimes \mathbb{C}^n) \otimes A_o(n) \xrightarrow{\phi_3} A_o(n) \xrightarrow{\varepsilon} \mathbb{C}_\varepsilon \to 0$$

and hence $\mathrm{cd}(A_o(n)) \leq 3$.

About the proof. The resolution, due to Collins-Härtl-Thom, is given in [27], to which we refer for an explicit form. Unfortunately, the verification of exactness is a tedious and long computation, involving computer calculations. Another way to proceed is to use the monoidal equivalence $\mathsf{Comod}_f(A_o(n)) \simeq^{\otimes} \mathsf{Comod}_f(\mathcal{O}(\mathrm{SU}_q(2))$ mentionned in Section 1, to remark that the objects in the resolution in Theorem 4.12 carry a natural comodule structure, so that they are Yetter-Drinfeld modules, and to transport it via the monoidal equivalence. See [14]. $\square$

Corollary 4.17. *For $n \geq 2$, we have*

$$H^p(A_o(n), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ \mathbb{C}^{\frac{n(n-1)}{2}} & \text{if } p = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Exercise, using the previous resolution, to be found in [27]. $\square$

The proof of Theorem 4.15, similarly to the previous section, follows from the combination of the previous two results.

4.4. **Example : free unitary quantum groups.** The computation here is as follows:

Theorem 4.18. *For $n \geq 2$, we have $\mathrm{cd}(A_u(n)) = 3$.*

In view of Proposition 2.11 and of Corollary 4.9, we have $\mathrm{cd}(A_u(n)) = \mathrm{cd}(A_o(n) * A_o(n))$. The proof of Theorem 4.18 is then a consequence of Theorem 4.15 and of the following result.

Proposition 4.19. *Let A, B be non trivial compact Hopf algebras. We have*

$$\mathrm{cd}(A * B) = \begin{cases} 1 & \text{if } \mathrm{cd}(A) = 0 = \mathrm{cd}(B) \\ \max(\mathrm{cd}(A), \mathrm{cd}(B)) & \text{if } \max(\mathrm{cd}(A), \mathrm{cd}(B)) \geq 1 \end{cases}$$

Proof. See [11, Corollary 2.5], or [16]. $\square$

One has $H^0(A_u(n), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \mathbb{C}$, as always, and one can check that $H^1(A_u(n), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq M_n(\mathbb{C})$, but the computation of the full cohomology $H^*(A_u(n), {}_\varepsilon\mathbb{C}_\varepsilon)$ has not been done yet.

Remark 4.20. Another approach to prove Theorem 4.18 would be by using results from [6] and from [7](the proofs require having done the full analysis of the categories of comodules): there are Hopf $*$-algebra embeddings

$$A_u(n) \hookrightarrow A_o(n) * \mathbb{C}[t, t^{-1}], \quad PA_o(n) \hookrightarrow A_u(n)$$

where $PA_o(n)$ is the Hopf $*$-subalgebra of $A_o(n)$ generated by the elements $x_{ij}x_{kl}$ (and with $PA_o(2) = \mathcal{O}(\mathrm{PU}(2))$) and one gets the result by combining Theorem 4.15 together with propositions 4.6 and 4.19.

4.5. **Example : the quantum permutation group.** We finish the section with the quantum permutation group.

Theorem 4.21. *For $n \geq 4$, we have* $\mathrm{cd}(A_s(n)) = 3$.

Idea of the proof. This is proved in [15]. The resolution in Theorem 4.12 induces a length 3 projective resolution for $\mathcal{O}(\mathrm{PU}_q(2))$, that, thanks to appropriate comodule structures, one can transport through the monoidal equivalence $\mathsf{Comod}_f(A_s(n)) \simeq^{\otimes} \mathsf{Comod}_f(\mathcal{O}(\mathrm{PU}_q(2))$, and from this one gets $\mathrm{cd}(A_s(n)) \leq 3$. To check the equality, one uses bialgebra cohomology, see [15]. □

In fact we have:

Theorem 4.22. *We have, for $n \geq 4$,*

$$H^p(A_s(n), {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

Idea of the proof. This is proved in [18]. At $p = 0$ this is clear and at $p = 1$ this is rather immediate. The computation at $p = 2$ is done quite directly, but requires some work, while the computation at $p = 3$ involves a Poincaré type duality between cohomology and homology (that we have not defined). □

5. L^2-Betti-numbers

In this section we survey the recent computation of L^2-Betti numbers of our favourite quantum groups. We begin by recalling the definition of the L^2-Betti numbers.

5.1. **Preliminary remark.** Let A be an algebra and M, N be right A-modules. If N is a left B-module for another algebra B such that N is a B-A-bimodule, then the space of right A-linear maps $\mathrm{Hom}_A(M, N)$ carries a natural left B-module structure defined by

$$(b \cdot f)(x) = b \cdot (f(x))$$

Now if A is a Hopf algebra and M is a B-A-bimodule, this remark ensures that the Ext-spaces

$$\mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, M)$$

carry a natural left B-module structure. Indeed, if $P_* \to \mathbb{C}_\varepsilon \to 0$ is a projective resolution of $\mathbb{C}_\varepsilon$

$$\cdots \to P_{n+1} \overset{\partial_{n+1}}{\to} P_n \to \cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\epsilon}{\to} \mathbb{C}_\varepsilon \to 0$$

the associated complex $\mathrm{Hom}_A(P_*, M)$

$$0 \to \mathrm{Hom}_A(P_0, M) \overset{-\circ\partial_1}{\longrightarrow} \mathrm{Hom}_A(P_1, M) \overset{-\circ\partial_2}{\longrightarrow} \mathrm{Hom}_A(P_2, M) \overset{-\circ\partial_3}{\longrightarrow} \cdots$$

carries a natural left B-module structure, and hence so do the cohomology spaces $\mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, M) = H^*(\mathrm{Hom}_A(P_*, M))$ (the B-module structure does not depend either on the choice of the projective resolution P_*).

5.2. **Lück's dimension function for finite von Neumann algebras.** We now briefly recall how Lück's dimension function for modules over finite von Neumann algebras [60, 61] is constructed. Expositions of the theory can be found in [48, 73].

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra: $\mathcal{M}$ is a von Neumann algebra and $\tau : \mathcal{M} \to \mathbb{C}$ is a normal faithful tracial state. Let P be a finitely generated projective left $\mathcal{M}$-module: there exists an idempotent matrix $a = (a_{ij}) \in M_n(\mathcal{M})$ such that $P \simeq \mathcal{M}^n a$. The Murray-von Neumann dimension of P is defined by

$$\dim_{\mathcal{M}}(P) = \sum_{i=1}^{n} \tau(a_{ii})$$

and is independent of the choice of the matrix a. Now, if M is an arbitrary $\mathcal{M}$-module, the Lück dimension of M is defined by

$$\dim_{\mathcal{M}}(M) = \mathrm{Sup}\{\dim_{\mathcal{M}}(P),\ P \text{ is a finitely generated projective submodule of } M\} \in [0, \infty]$$

The above dimension extends the Murray von Neumann dimension to any module, and has some rather good an natural properties.

5.3. **Definition of L^2-Betti numbers.** Let A be a compact Hopf algebra of Kac type: the square of the antipode is the identity, or equivalently the Haar integarl $h : A \to \mathbb{C}$ is a trace. Let $\mathcal{L}(A)$ be the von Neumann algebra associated to A and h (see Section 1). Then h extends to a normal faithful state on $\mathcal{L}(A)$, so that $(\mathcal{L}(A), h)$ is a finite von Neumann algebra. The embedding $A \subset \mathcal{L}(A)$ endows $\mathcal{L}(A)$ with a right A-module structure, and hence $\mathcal{L}(A)$ is a $\mathcal{L}(A)$-A-bimodule. The preliminary remark thus ensures that the Ext-spaces

$$\mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, \mathcal{L}(A))$$

carry a natural left $\mathcal{L}(A)$-module structure. We thus have the necessary material to define

Definition 5.1. For $p \geq 0$, the **p-th L^2-Betti number** of a compact Hopf algebra of Kac type is defined by

$$\beta^{(2)}_p(A) = \dim_{\mathcal{L}(A)}(\mathrm{Ext}^*_A(\mathbb{C}_\varepsilon, \mathcal{L}(A))$$

The above L^2-Betti numbers were defined by Kyed [49], using Tor, that we have not defined here. Work of Thom [80] and Reich [70] ensure that the two definitions coincide. Here are some general properties.

- If $A = \mathbb{C}\Gamma$, the group algebra of a discrete group Γ, then $\beta^{(2)}_p(\mathbb{C}\Gamma) = \beta^{(2)}_p(\Gamma)$, the usual L^2-Betti number of Γ, whose theory is developped in [62], where the reader will find the history and connections with geometry and topology. As an example, for the free groups, one has $\beta^{(2)}_p(\mathbb{F}_n) = 0$ is $p \neq 1$, and $\beta^{(2)}_1(\mathbb{F}_n) = n - 1$.
- We have $\beta^{(2)}_0(A) = 0$ if A is infinite-dimensional, and $\beta^{(2)}_0(A) = \frac{1}{\dim(A)}$ otherwise [51].
- If A is coamenable (the counit extends to a bounded operator on the reduced C^*-algebra $C^*_{\mathrm{red}}(A)$, see [66]), then $\beta^{(2)}_p(A) = 0$ if $p \geq 1$ [50]. This holds in particular if A is commutative, so L^2-Betti numbers are truly meaningful if we view compact Hopf algebras as group algebras of discrete quantum groups.
- If $n = \mathrm{cd}(A)$, then $\beta^{(2)}_p(A) = 0$ for any $p > n$.
- For a tensor product of compact Hopf algebras of Kac type, one has [52]

$$\beta^{(2)}_p(A \otimes B) = \sum_{k+l=p} \beta^{(2)}_k(A)\beta^{(2)}_l(B)$$

- For a free product of non-trivial compact Hopf algebras of Kac type, one has [19]

$$\beta^{(2)}_p(A * B) = \begin{cases} 0 & \text{if } p = 0 \\ \beta^{(2)}_1(A) - \beta^{(2)}_0(A) + \beta^{(2)}_1(B) - \beta^{(2)}_0(B) + 1 & \text{if } p = 1 \\ \beta^{(2)}_p(A) + \beta^{(2)}_p(B) & \text{if } p \geq 2 \end{cases}$$

5.4. **Computations for $A_o(n)$, $A_u(n)$ and $A_s(n)$.** We now record the computation of the L^2-Betti numbers for our favourite algebras.

Theorem 5.2. *The L^2-Betti numbers of $A_o(n)$, $A_u(n)$ and $A_s(n)$ are as follows.*

(1) *For $n \geq 2$, $\beta_p^{(2)}(A_o(n)) = 0$ for any $p \geq 0$.*
(2) *For $n \geq 2$, $\beta_p^{(2)}(A_u(n)) = 0$ for any $p \neq 1$, $\beta_1^{(2)}(A_u(n)) = 1$.*
(3) *For $n \geq 4$, $\beta_p^{(2)}(A_s(n)) = 0$ for any $p \geq 0$.*

The vanishing of $\beta_1^{(2)}(A_o(n))$ was proved by Vergnioux [82], while the vanishing of the other L^2-Betti numbers for $A_o(n)$ was proved by Collins-Härtl-Thom using their resolution 4.16.

Vergnioux also proved that $\beta_1^{(2)}(A_u(n)) \neq 0$ in [82], and Kyed-Raum [54] proved that precisely $\beta_1^{(2)}(A_u(n)) = 1$. This has been reproved in [19], where moreover all the other L^2-Betti numbers have been computed, using the above formula for free products, graded twisting (Proposition 2.11) and a formula for L^2-Betti numbers of compact Hopf algebras involved in an exact sequence, which implies that L^2-Betti numbers are invariant under graded twisting by a finite abelian group.

The computation for $A_s(n)$ is from the recent deep paper [55] by Kyed-Raum-Vaes-Valvekens, where much more is done. Using the definition of L^2-Betti numbers for C^*-tensor categories by Popa-Shlyakhtenko-Vaes [69] (which uses the extended dimension function for quasi-finite von Neumann algebras [53]), L^2-Betti numbers are defined for compact Hopf algebras that are not necessarily of Kac type, extending the previous definition, and are shown to be invariant under unitary monoidal equivalence.

6. Bialgebra cohomology

As said earlier, the cohomological dimension of a Hopf algebra depends only of its underlying algebra, while the cohomology spaces depend only the underlying augmented algebra (the algebra together with the counit). It is thus of course desirable to have a cohomology that takes the whole Hopf algebra structure into account. At least one such cohomology exists: the bialgebra cohomology by Gerstenhaber-Schack [41]. We present briefly this cohomology, in the case when A is a compact Hopf algebra.

Before giving the definition, we need the following notation: for a Hopf algebra A, we denote by $\mathrm{Hom}^c(A^{\otimes n}, \mathbb{C})$ the set of linear maps $f : A^{\otimes n} \to \mathbb{C}$ that satisfy

$$f(a_{1(1)} \otimes \cdots \otimes a_{n(1)})a_{1(2)} \otimes \cdots \otimes a_{n(2)} = f(a_{1(2)} \otimes \cdots \otimes a_{n(2)})a_{1(1)} \otimes \cdots \otimes a_{n(1)}$$

for any $a_1, \ldots, a_n \in A$. This is a subspace of $\mathrm{Hom}(A^{\otimes n}, \mathbb{C})$.

Definition 6.1. Let A be a compact Hopf algebra. The **bialgebra cohomology** of A, denoted by $H_b^*(A)$, is defined to be the cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}^c(\mathbb{C}, \mathbb{C}) \xrightarrow{\delta} \mathrm{Hom}^c(A, \mathbb{C}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}^c(A^{\otimes n}, \mathbb{C}) \xrightarrow{\delta} \mathrm{Hom}^c(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta\colon \mathrm{Hom}^c(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}^c(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) =& \varepsilon(a_1)f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n}(-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n)\varepsilon(a_{n+1})\end{aligned}$$

Of course one has to check that $\delta(\mathrm{Hom}^c(A^{\otimes n}, \mathbb{C})) \subset \mathrm{Hom}^c(A^{\otimes n+1}, \mathbb{C})$, and then the above complex is a subcomplex of the complex that defines $H^*(A, {}_\varepsilon\mathbb{C}_\varepsilon)$ in Proposition 3.18, and hence this yields a linear map $H_b^*(A) \to H^*(A, {}_\varepsilon\mathbb{C}_\varepsilon)$ which is not injective in general (e.g. for $\mathcal{O}(\mathrm{SU}_q(2))$ when $q \neq \pm 1$). Here are some general properties of H_b^*.

- $H_b^0(A) \simeq \mathbb{C}$, and if A is cocommutative, then $H_b^*(A) = H^*(A, {}_\varepsilon\mathbb{C}_\varepsilon)$.
- H_b^* is a monoidal invariant: if $\mathsf{Comod}_f(A) \simeq^{\otimes} \mathsf{Comod}_f(B)$, then $H_b^*(A) \simeq H_b^*(B)$, see [15]. This was not true for $H^*(-, {}_\varepsilon\mathbb{C}_\varepsilon)$.

- If A is of Kac type, then the above map $H_b^*(A) \to H^*(A, {}_\varepsilon\mathbb{C}_\varepsilon)$ is injective [15, Proposition 5.9].

Remark 6.2. Of course the above complex makes sense for any Hopf algebra, but is only known to coincide with the bialgebra cohomology of Gerstenhaber-Schack in the cosemisimple case (hence in particular when A is compact), see [15]. This is the same complex as the one defined in [40] in the study of additive deformations of Hopf algebras, which are of interest in quantum probability. This complex is also the complex that defines the so-called Davydov-Yetter cohomology of the tensor category of comodules over A (see [35, Chapter 7] and the references therein).

Theorem 6.3. *For $A = \mathcal{O}(\mathrm{SU}_q(2))$, $A = A_o(n)$ $(n \geq 2)$, $A = A_s(n)(n \geq 4)$, we have*

$$H_b^p(A) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

The result is from [14, 15]. It is not difficult to show that $H_b^1(A_u(n)) \simeq \mathbb{C}$, but the full computation is not known for $A_u(n)$.

Notice that the above computation for $A_s(n)$ was used to show that $\mathrm{cd}(A_s(n)) = 3$ and to compute $H^*(A_s(n), {}_\varepsilon\mathbb{C}_\varepsilon)$ in [15, 18]. Hence bialgebra cohomology can be useful in the study of ordinary cohomology.

To conclude, we wish to point out that bialgebra cohomology is the case of trivial coefficients of Gersthenhaber-Schack cohomology, a cohomology theory whose coefficients are Yetter-Drinfeld modules, see [15], and that Gersthenhaber-Schack cohomology is an Ext-functor on Yetter-Drinfeld modules [78], so that the general principles of homological algebra can be applied to it.

7. Open questions

We conclude these notes by a series of open questions.

Question 7.1. What are the compact Hopf algebras of cohomological dimension one ?

Recall from Example 4.2 that Dunwoody's theorem [34] states that a finitely generated discrete group has cohomological dimension one if only if it constains a free subgroup of finite index. So, is there an analogue of this theorem for compact Hopf algebras ? Notice that it is not difficult to construct examples of noncommutative and noncocommutative compact Hopf algebras of cohomological dimension one, using free product, crossed product or crossed coproduct constructions. Of course, The ring-theoretic analogues of Bass-Serre techniques [29, 36] should play a role here.

Question 7.2. Is is true that if A and B are monoidally equivalent compact Hopf algebras, then $\mathrm{cd}(A) = \mathrm{cd}(B)$?

The answer is known to be positive if A and B are of Kac type [15].

Question 7.3. What is the cohomological dimension of a free wreath product ?

We refer to [13] for the definition of a free wreath product. A positive answer to Question 7.2 together with the results in [57, 37] would allow an answer. L^2-Betti numbers are computed in [55].

Question 7.4. It is known that a finitely generated discrete subgroup of U_n has finite complex cohomological dimension [1]. Is there a generalization for compact Hopf algebras ?

Here the generalization would be in the setting of inner unitary Hopf algebras [9, 2], that necessarily would be compact Hopf algebras of Kac type by [10].

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