Zariski decompositions on arithmetic varieties

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First recall the definition of Zariski decompositions on an algebraic surface. Let S be a non-singular projective surface over an algebraically closed field. Let $D \in \mathrm{Div}(S) \otimes \mathbb{R}$ be a pseudo-effective $\mathbb{R}$-divisor on S. A decomposition $D = P + N$ ($P, N \in \mathrm{Div}(S) \otimes \mathbb{R}$) is called a Zariski decomposition of D if the following conditions are satisfied:

1. P is nef and N is effective.
2. $(P \cdot N) = 0$.
3. If $N = a_1 E_1 + \cdots + a_n E_n$ is the irreducible decomposition of N, then the matrix $\{(E_i \cdot E_j)\}_{1 \le i,j \le n}$ is negative definite.

We have the following characterization of the Zariski decomposition.

Proposition

1. *(Bauer) The positive part P of the Zariski decomposition of D is characterized by the greatest element of*

$$\{M \mid M \text{ is a nef } \mathbb{R}\text{-divisor on } S \text{ and } M \leq D\}.$$

2. *If D is big, P is nef, N is effective and* $\mathrm{vol}(P) = \mathrm{vol}(D)$, *then $D = P + N$ yields a Zariski decomposition of D.*

Let M be an n-equidimensional smooth projective variety over $\mathbb{C}$. Let $\mathrm{Div}(M)$ be the group of Cartier divisors on M and let $\mathrm{Div}(M)_{\mathbb{R}} := \mathrm{Div}(M) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor.
Let us fix $D \in \mathrm{Div}(M)_{\mathbb{R}}$. We set $D = a_1 D_1 + \cdots + a_l D_l$, where $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are prime divisors on M.
Let $g : M \to \mathbb{R} \cup \{\pm\infty\}$ be a locally integrable function on M. We say g is a D-Green function of C^∞-type (resp. C^0-type) if, for each point $x \in M$, there are an open neighborhood U_x of x, local equations $f_1, \ldots, f_l$ of $D_1, \ldots, D_l$ respectively and a C^∞ (resp. C^0) function u_x over U_x such that

$$g = u_x + \sum_{i=1}^{l} (-a_i) \log |f_i| \quad (a.e.)$$

over U_x. The above equation is called a local expression of g.

Moreover, we say g is a D-Green function of PSH-type if u_x is a plurisubharmonic function on U_x.
Let g be a D-Green function of C^0-type on M. Let

$$g = u + \sum (-a_i) \log |f_i| = u' + \sum (-a_i) \log |f_i'| \quad (a.e.)$$

be two local expressions of g. Then, as $\sum (-a_i) \log |f_i / f_i'|$ is dd^c-closed, we have $2dd^c(u) = 2dd^c(u')$ as currents, so that it can be defined globally. We denote it by $c_1(D, g)$. Note that $c_1(D, g)$ is a closed $(1, 1)$-current on M. If g is of C^∞-type, then $c_1(D, g)$ is represented by a C^∞-form. Moreover, if g is of PSH-type, then $c_1(D, g)$ is a positive current.

Let X be a d-dimensional, generically smooth normal projective arithmetic variety, that is,

1. X is projective flat integral scheme over $\mathbb{Z}$.
2. If $X_{\mathbb{Q}} = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q})$ is the generic fiber of $X \to \mathrm{Spec}(\mathbb{Z})$, then $X_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$.
3. The Krull dimension of X is d, that is, $\dim X_{\mathbb{Q}} = d - 1$.
4. X is normal.

Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X and $\mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor on X. For $D \in \mathrm{Div}(X)_{\mathbb{R}}$, we set $D = \sum_i a_i D_i$, where $a_i \in \mathbb{R}$ and D_i's are reduced and irreducible subschemes of codimension one. We say D is effective if $a_i \geq 0$ for all i. Moreover, for $D, E \in \mathrm{Div}(X)_{\mathbb{R}}$,

$$D \leq E \text{ (or } E \geq D) \quad \Longleftrightarrow \quad E - D \text{ is effective}$$

Let D be an $\mathbb{R}$-divisor on X and let g be a locally integrable function on $X(\mathbb{C})$. We say a pair $\overline{D} = (D, g)$ is an arithmetic $\mathbb{R}$-divisor on X if $F_\infty^*(g) = g$ $(a.e.)$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map, i.e. for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition $\mathrm{Spec}(\mathbb{C}) \xrightarrow{-} \mathrm{Spec}(\mathbb{C}) \xrightarrow{x} X$. Moreover, we say $\overline{D}$ is of C^∞-type (resp. C^0-type, PSH-type) if g is a D-Green function of C^∞-type (resp. C^0-type, PSH-type). For arithmetic divisors $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$, we define $\overline{D}_1 = \overline{D}_2$ and $\overline{D}_1 \leq \overline{D}_2$ to be

$$\overline{D}_1 = \overline{D}_2 \iff D_1 = D_2 \text{ and } g_1 = g_2 \ (a.e.),$$
$$\overline{D}_1 \leq \overline{D}_2 \iff D_1 \leq D_2 \text{ and } g_1 \leq g_2 \ (a.e.).$$

We say $\overline{D}$ is effective if $\overline{D} \geq (0, 0)$.

Let $\mathrm{Rat}(X)$ be the field of rational functions on X. For $\phi \in \mathrm{Rat}(X)^\times$, we set

$$(\phi) := \sum_{\Gamma} \mathrm{ord}_\Gamma(\phi)\Gamma \quad \text{and} \quad \widehat{(\phi)} := ((\phi), -\log|\phi|).$$

Note that $\widehat{(\phi)}$ is an arithmetic divisor of C^∞-type.

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X.

- $H^0(X, D) := \{\phi \in \mathrm{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}$. Note that $H^0(X, D)$ is finitely generated $\mathbb{Z}$-module.
- $\hat{H}^0(X, \overline{D}) := \{\phi \in \mathrm{Rat}(X)^\times \mid \overline{D} + \widehat{(\phi)} \geq (0,0)\} \cup \{0\}$. Note that $\hat{H}^0(X, \overline{D})$ is a finite set.
- $\hat{h}^0(X, \overline{D}) := \log \#\hat{H}^0(X, \overline{D})$.
- $\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n\to\infty} \dfrac{\log \#\hat{H}^0(X, n\overline{D})}{n^d/d!}$.

We assume that $X = \mathrm{Spec}(O_K)$, where K is a number field and O_K is the ring of integers in K.Moreover, $\overline{D}$ is given by

$$\left(\sum_{P \in \mathrm{Spec}(O_K) \setminus \{0\}} n_P P, \sum_{\sigma \in K(\mathbb{C})} \xi_\sigma \sigma \right),$$

where $K(\mathbb{C})$ is the set of all embeddings K into $\mathbb{C}$.Then the arithmetic degree $\widehat{\deg}(\overline{D})$ of $\overline{D}$ is defined by

$$\sum_P n_P \log(\#(O_K/P)) + \sum_\sigma \xi_\sigma.$$

Note that $\widehat{\deg}(\widehat{(\phi)}) = 0$ for $\phi \in K^\times$ (the product formula).

Let C be a reduced and irreducible 1-dimensional closed subscheme of X. We would like to define $\widehat{\deg}(\overline{D}|_C)$. It is characterized by the following properties:

1. $\widehat{\deg}(\overline{D}|_C)$ is linear with respect to $\overline{D}$.
2. If $\phi \in \mathrm{Rat}(X)^\times_{\mathbb{R}} := \mathrm{Rat}(X)^\times \otimes \mathbb{R}$, then $\widehat{\deg}(\widehat{(\phi)}\Big|_C) = 0$.
3. If $C \not\subseteq \mathrm{Supp}(D)$ and C is vertical, then $\widehat{\deg}(\overline{D}|_C) = \log(p) \deg(D|_C)$, where C is contained in the fiber over a prime p.
4. If $C \not\subseteq \mathrm{Supp}(D)$ and C is horizontal, then $\widehat{\deg}(\overline{D}|_C) = \widehat{\deg}(\overline{D}|_{\widetilde{C}})$, where $\widetilde{C}$ is the normalization of C. Note that $\widetilde{C} = \mathrm{Spec}(O_K)$ for some number field K.

- $\overline{D}$ is big $\overset{\text{def}}{\iff}$ $\widehat{\mathrm{vol}}(\overline{D}) > 0$.
- $\overline{D} = (D, g)$ is nef $\overset{\text{def}}{\iff}$
  1. $\widehat{\deg}(\overline{D}\big|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed subschemes C of X.
  2. g is of $(C^0 \cap \mathrm{PSH})$-type, that is, $c_1(D, g)$ is a positive current.

Theorem (Zariski decomposition on arithmetic surfaces)

(M.) We assume that $d = 2$ *and* X *is regular. Let* $\overline{D}$ *be an arithmetic* $\mathbb{R}$*-divisor of* C^0*-type on* X *such that the set*

$$\Upsilon(\overline{D}) = \{\overline{M} \mid \overline{M} \text{ is a nef arithmetic } \mathbb{R}\text{-divisor on } X \text{ and } \overline{M} \leq \overline{D}\}$$

is not empty. Then there is a nef arithmetic $\mathbb{R}$*-divisor* $\overline{P}$ *such that* $\overline{P}$ *gives the greatest element of* $\Upsilon(\overline{D})$*, that is,* $\overline{P} \in \Upsilon(\overline{D})$ *and* $\overline{M} \leq \overline{P}$ *for all* $\overline{M} \in \Upsilon(\overline{D})$*.*

Theorem (Continued)

Moreover, if we set $\overline{N} = \overline{D} - \overline{P}$*, then the following properties hold:*

1. $\hat{H}^0(X, n\overline{P}) = \hat{H}^0(X, n\overline{D})$ *for all* $n \geq 0$.
2. $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2)$.
3. *If* $\overline{B}$ *is an arithmetic* $\mathbb{R}$*-Cartier divisor of* C^0*-type such that* $(0,0) \lneqq \overline{B} \leq \overline{N}$ *and* $c_1(\overline{P} + \overline{B})$ *is a positive current, then* $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$ *and* $\widehat{\deg}(\overline{B}^2) < 0$.

For example, if $X = \mathrm{Proj}(\mathbb{Z}[T_0, T_1])$, $D = \{T_0 = 0\}$ and $g = \log(a_0 + a_1 z^2)/2$, where $a_0, a_1 \in \mathbb{R}_{>0}$ and $z = T_1/T_0$. The Zariski decomposition of $\overline{D}$ exists if and only if $a_0 + a_1 \geq 1$. Moreover, the positive part $\overline{P}$ of $\overline{D}$ is given by $(\theta H_0 - \theta' H_1, p)$, where $H_0 = D = \{T_0 = 0\}$, $H_1 = \{T_1 = 0\}$,

$$\begin{aligned}\Theta = \{x \in [0,1] \mid -(1-x)\log(1-x) - x\log x \\ + (1-x)\log a_0 + x \log a_1 \geq 0\},\end{aligned}$$

$\theta' = \inf \Theta$, $\theta = \sup \Theta$ and

$$p(z) = \begin{cases} \theta' \log |z| & \text{if } |z| < \sqrt{\frac{a_0\theta'}{a_1(1-\theta')}}, \\ \log(a_0 + a_1|z|^2)/2 & \text{if } \sqrt{\frac{a_0\theta'}{a_1(1-\theta')}} \leq |z| \leq \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}, \\ \theta \log |z| & \text{if } |z| > \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}. \end{cases}$$

Let X be a d-dimensional, generically smooth normal projective arithmetic variety and let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on X. A decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a Zariski decomposition of $\overline{D}$ if

1. $\overline{P}$ is a nef arithmetic $\mathbb{R}$-divisor on X.
2. $\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of C^0-type on X.
3. $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P})$.

Let us consider a Zariski decomposition on an arithmetic toric variety.
Let N be a free $\mathbb{Z}$-module of rank d and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.
Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and let

$$\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$$

be the natural pairing.
Let Σ be a non-singular and complete fan in $N_{\mathbb{R}}$. The toric variety over $\mathbb{Z}$ associated with Σ is denoted by $X(\Sigma)_{\mathbb{Z}}$, that is,

$$X(\Sigma)_{\mathbb{Z}} := \bigcup_{\sigma \in \Sigma} \mathrm{Spec}(\mathbb{Z}[\sigma^{\vee} \cap M]).$$

A function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ is called a supporting function with respect to Σ if it is linear on each cone in Σ.
By the definition, we can find $m_\sigma \in M_{\mathbb{R}}$ such that

$$\varphi(x) = \langle m_\sigma, x \rangle \quad (x \in \sigma)$$

for each $\sigma \in \Sigma$.
The $\mathbb{R}$-Cartier divisor given by (χ^{-m_σ}) on each U_σ $(\sigma \in \Sigma)$ is denoted by D_φ.
Moreover, we set

$$\Delta_\varphi := \{ m \in M_{\mathbb{R}} \mid \langle m, x \rangle \geq \varphi(x) \ \forall x \in N_{\mathbb{R}} \} .$$

Note that $m \in \Delta_\varphi$ if and only if $D_\varphi + (\chi^m) \geq 0$.

A D_φ-Green function g of C^0-type on $X(\Sigma)_\mathbb{C}$ is said to be torus-invariant if g is invariant under the action of $\mathbb{T}_N(S^1) := N \otimes_\mathbb{Z} \{z \in \mathbb{C} \mid |z| = 1\}$.
A pair $\overline{D}_\varphi = (D_\varphi, g)$ of a torus-invariant $\mathbb{R}$-Cartier divisor D_φ and a torus-invariant D_φ-Green function g of C^0-type is called a torus-invariant arithmetic $\mathbb{R}$-Cartier divisor of C^0-type.
We define the function ξ_g on $N_\mathbb{R}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{T}_N(\mathbb{C}^\times) & = & N \otimes_\mathbb{Z} \mathbb{C}^\times \\
{\scriptstyle \mathrm{val}}\downarrow & & \downarrow{\scriptstyle -g} \\
N_\mathbb{R} & \xrightarrow{\xi_g} & \mathbb{R}
\end{array}
$$

where $\mathrm{val}(n \otimes \lambda) = n \otimes (-\log|\lambda|)$.

The Legendre transform $\vartheta_g = \mathcal{L}(\xi_g)$ of ξ_g is called the roof function of g, that is,

$$\vartheta_g(m) = \mathcal{L}(\xi_g)(m) = \inf\{\langle m, x\rangle - \xi_g(x) \mid x \in N_{\mathbb{R}}\}.$$

Moreover, we set

$$\Theta_g := \{m \in M_{\mathbb{R}} \mid \vartheta_g(m) \geq 0\}.$$

Then we have the following theorem.

Theorem (Burgos Gil, M., Philippon and Sombra)

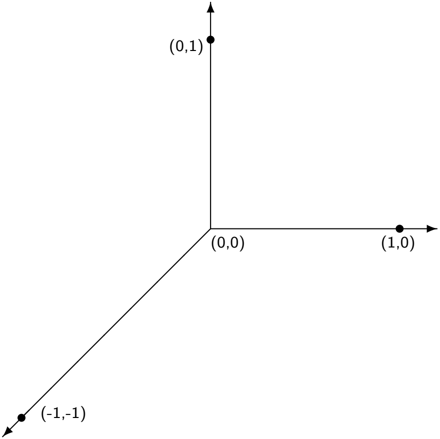
We assume that $\overline{D}$ is big. Then the following conditions are equivalent:

1. *There is a birational morphism $f : Y \to X(\Sigma)_{\mathbb{Z}}$ of generically smooth, normal projective arithmetic varieties such that $f^*(\overline{D}_\varphi)$ has a Zariski decomposition.*
2. *Θ_g is a quasi-rational convex polyhedron, that is, there are $\gamma_1, \dots, \gamma_l \in N_{\mathbb{Q}}$ and $a_1, \dots, a_l \in \mathbb{R}$ such that*

$$\Theta_g = \{x \in M_{\mathbb{R}} \mid \langle x, \gamma_i \rangle \geq a_i \ \forall i = 1, \dots, l\},$$

where $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let us consider examples.
We assume that Σ is given by



Then $X(\Sigma)_{\mathbb{Z}} = \mathbb{P}^2_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, T_1, T_2])$. We set $z_i = T_i/T_0$ for $i = 1, 2$. Let φ be a supporting function given by

$$\varphi(1,0) = \varphi(0,1) = 0, \quad \varphi(-1,-1) = -1.$$

Note that $D_\varphi = \{T_0 = 0\}$. Let $\overline{D}_\varphi = (D_\varphi, g)$ be a torus-invariant big arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on $\mathbb{P}^2_{\mathbb{Z}}$, that is,

$$g(\exp(2\pi\sqrt{-1}\theta_1)z_1, \exp(2\pi\sqrt{-1}\theta_2)z_2) = g(z_1, z_2)$$

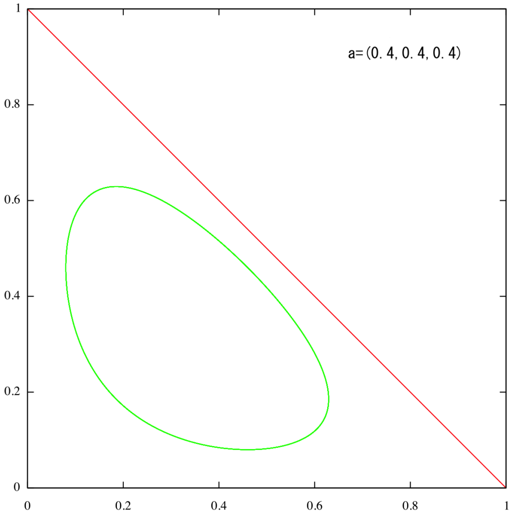
for all $\theta_1, \theta_2 \in [0,1]$.

Let us fix a sequence $\boldsymbol{a} = (a_0, a_1, a_2)$ of positive numbers. Let us consider a D_φ-Green function g of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^2(\mathbb{C})$ and a torus-invariant arithmetic divisor $\overline{D}_\varphi$ of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^2_{\mathbb{Z}}$ given by

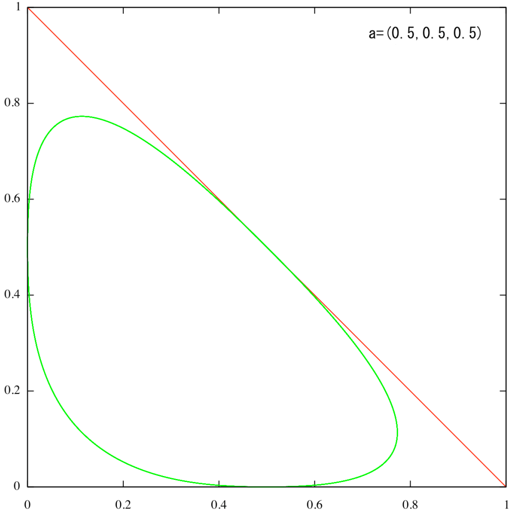
$$g := \frac{1}{2}\log(a_0 + a_1|z_1|^2 + a_2|z_2|^2) \quad \text{and} \quad \overline{D}_\varphi := (D_\varphi, g).$$

The roof function $\vartheta_g : \Delta_n \to \mathbb{R}$ is given by

$$\vartheta_g(x_1, x_2) := \frac{1}{2}\left((1 - x_1 - x_2)\log\frac{a_0}{1 - x_1 - x_2} + x_1 \log\frac{a_1}{x_1} + x_2 \log\frac{a_2}{x_2}\right).$$



a=(0. 4, 0. 4, 0. 4)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1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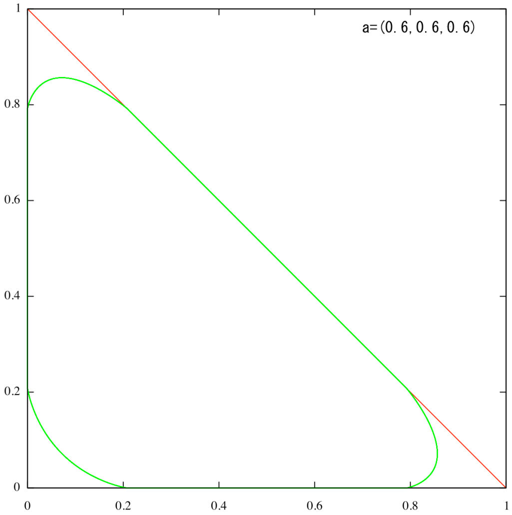
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1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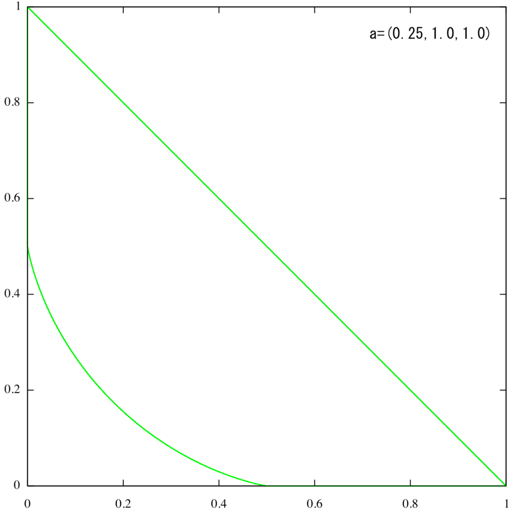
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1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0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

By the theorem due to Burgos Gil, M., Philippon and Sombra, we have the following:
(1) $\overline{D}_\varphi$ is nef $\iff$ $a_0 \geq 1, a_1 \geq 1, a_2 \geq 1$.
(2) $\overline{D}_\varphi$ is big $\iff$ $a_0 + a_1 + a_2 > 1$.
(3) If $\overline{D}_\varphi$ is big and not nef, then there is no birational morphism $f : Y \to \mathbb{P}^2_{\mathbb{Z}}$ of generically smooth and projective normal arithmetic varieties such that a Zariski decomposition of $f^*(\overline{D}_\varphi)$ exists on X.

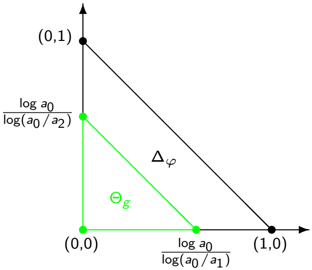
Let us consider another example, that is,

$$g = \frac{1}{2}\log\max\{a_0, a_1|z_1|^2, a_2|z_2|^2\}.$$

Note that

$$\vartheta_g(x_1, x_2) = \frac{1}{2}\left((1 - x_1 - x_2)\log a_0 + x_1 \log a_1 + x_2 \log a_2\right).$$

In the case where $a_0 > 1$, $a_1 < 1$ and $a_2 < 1$,



By the theorem due to Burgos Gil, M., Philippon and Sombra, we have the following:

(1) $\overline{D}_{\varphi}$ is nef $\iff$ $a_0 \geq 1, a_1 \geq 1, a_2 \geq 1$.

(2) $\overline{D}_{\varphi}$ is big $\iff$ $\max\{a_0, a_1, a_2\} > 1$.

(3) If $\overline{D}_{\varphi}$ is big and not nef, then there is a Zariski decomposition of $f^*(\overline{D}_g)$ for some birational morphism $f : Y \to \mathbb{P}^2_{\mathbb{Z}}$ of generically smooth and projective normal arithmetic varieties if and only if there is $\lambda \in \mathbb{R}_{>0}$ such that

$$\lambda \left(\log\left(\frac{a_1}{a_0}\right), \log\left(\frac{a_2}{a_0}\right) \right) \in \mathbb{Q}^2.$$

Sketch of the proof of the theorem due to Burgos Gil, M., Philippon and Sombra:

"2 $\Longrightarrow$ 1" can be proved by using the following lemma:

Lemma

1. *Let K be a bounded quasi-rational convex polyhedron in $M_{\mathbb{R}}$. Let $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ be a function given by*

$$\varphi(x) = \min\{\langle m, x\rangle \mid m \in K\}.$$

Then there is a complete fan Σ of $N_{\mathbb{R}}$ such that φ is a supporting function with respect to Σ, that is, φ is linear on each cone of Σ. Moreover, φ is concave and $\Delta_{\varphi} = K$.

Lemma (Continued)

2. *Let Σ be a non-singular and complete fan in $N_{\mathbb{R}}$ and let $X(\Sigma)_{\mathbb{C}}$ be the associated toric variety over $\mathbb{C}$. Let φ be a supporting function with respect to Σ and let $\xi : N_{\mathbb{R}} \to \mathbb{R}$ be a function. If φ and ξ are concave, and* $\mathrm{dom}(\mathcal{L}(\xi)) = \Delta_{\varphi}$, *then there is a torus-invariant D_{φ}-Green function g of $(C^0 \cap \mathrm{PSH})$-type $X(\Sigma)_{\mathbb{C}}$ such that $\xi_g = \xi$, where $\mathcal{L}(\xi)$ is the Legendre transform of ξ and*

$$\mathrm{dom}(\mathcal{L}(\xi)) := \{m \in M_{\mathbb{R}} \mid \mathcal{L}(\xi)(m) > -\infty\}.$$

We set $\varphi_+(x) = \min\{\langle m, x\rangle \mid m \in \Theta_g\}$. Then, by 1 in the previous lemma, we can find a non-singular refinement Σ' of Σ such that φ_+ is a supporting function with respect to Σ'. Further, $\Delta_{\varphi_+} = \Theta_g$ and φ_+ is concave. Here we consider a upper-semicontinuous convex function $\mathcal{L}(\xi_g)_+$ given by

$$m \mapsto \begin{cases} \mathcal{L}(\xi_g)(m) & \text{if } m \in \Delta_{\varphi_+}, \\ -\infty & \text{otherwise.} \end{cases}$$

If we set $\xi_+ = \mathcal{L}(\mathcal{L}(\xi_g)_+)$, then we can see that ξ_+ is a concave function on $N_{\mathbb{R}}$ and $\mathrm{dom}(\mathcal{L}(\xi_+)) = \Delta_{\varphi_+}$. Thus, by 2 in the previous lemma, there is a torus-invariant D_{φ_+}-Green function g_+ of $(C^0 \cap \mathrm{PSH})$-type such that $\xi_{g_+} = \xi_+$. Then (D_{φ_+}, g_+) yields the positive part of the Zariski decomposition on $X(\Sigma')_{\mathbb{Z}}$.

$1 \implies 2$: Let X be a generically smooth, projective and normal arithmetic variety. We say that v is a divisorial valuation of $\mathrm{Rat}(X)$ over $\mathbb{Q}$ if there exist a proper birational morphism $\pi : Y \to X_{\mathbb{Q}}$ of normal and complete varieties over $\mathbb{Q}$ and a prime divisor Γ on Y such that $v(\phi) = \mathrm{ord}_{\Gamma}(\phi)$ for $\phi \in \mathrm{Rat}(X)^{\times}$. The set of all divisorial valuations of $\mathrm{Rat}(X)$ over $\mathbb{Q}$ is denoted by $\mathrm{DV}(X_{\mathbb{Q}})$. Let $\mathrm{Div}(X)$ be the group of Cartier divisors of X. Using the divisorial valuation v, we can define a homomorphism

$$\mathrm{mult}_v : \mathrm{Div}(X) \otimes \mathbb{R} \to \mathbb{R}.$$

Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. We set

$$\widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}) = \{\phi \in \mathrm{Rat}(X)^{\times} \otimes \mathbb{R} \mid \overline{D} + \widehat{(\phi)} \geq 0\}.$$

For $v \in \mathrm{DV}(X_{\mathbb{Q}})$, we define $\mu_{\mathbb{R},v}(\overline{D})$ to be

$$\mu_{\mathbb{R},v}(\overline{D}) := \begin{cases} \inf\left\{\mathrm{mult}_v(D + (\phi)) \mid \phi \in \widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D})\right\} & \text{if } \widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Then we have the following lemma:

Lemma

We assme that $\overline{D}_\varphi$ is big.

1. $$\Theta_{\overline{D}_\varphi} = \left\{a \in M_{\mathbb{R}} \mid \mu_{\mathbb{R},v}(\overline{D}_\varphi) \leq \mathrm{mult}_v(D_\varphi + (\chi^a)) \ \forall v \in \mathrm{DV}(X(\Sigma)_{\mathbb{Q}})\right\}.$$

2. *If there is a birational morphism $f : Y \to X(\Sigma)_{\mathbb{Z}}$ of generically smooth, normal projective arithmetic varieties such that $f^*(\overline{D}_\varphi)$ has a Zariski decomposition $f^*(\overline{D}_\varphi) = \overline{P} + \overline{N}$, then $\mu_{\mathbb{R},v}(\overline{D}_\varphi) = \mathrm{mult}_v(N)$ for all $v \in \mathrm{DV}(X_{\mathbb{Q}})$.*

We assume that there is a birational morphism $f : Y \to X(\Sigma)_{\mathbb{Z}}$ of generically smooth, normal projective arithmetic varieties such that $f^*(\overline{D}_\varphi)$ has a Zariski decomposition $f^*(\overline{D}_\varphi) = \overline{P} + \overline{N}$.
Let $\Gamma_1, \dots, \Gamma_l$ be irreducible components of $f^{-1}(X(\Sigma)_{\mathbb{Q}} \setminus \mathbb{T}) \cup \mathrm{Supp}(N)$. By 1 of the previous lemma, it is sufficient to see that if $\mu_{\mathbb{R},\Gamma_i}(\overline{D}_\varphi) \leq \mathrm{mult}_{\Gamma_i}(D_\varphi + (\chi^a))$ for all $i = 1, \dots, l$, then $\mu_{\mathbb{R},v}(\overline{D}_\varphi) \leq \mathrm{mult}_v(D_\varphi + (\chi^a))$ for all $v \in \mathrm{DV}(X(\Sigma)_{\mathbb{Q}})$.

Indeed, by 2 of the previous lemma,

$$\begin{aligned}
\mu_{\mathbb{R},v}(\overline{D}_\varphi) = \mathrm{mult}_v(N) &= \sum_{i=1}^{l} \mathrm{mult}_{\Gamma_i}(N)\,\mathrm{mult}_v(\Gamma_i) \\
&= \sum_{i=1}^{l} \mu_{\mathbb{R},\Gamma_i}(\overline{D}_\varphi)\,\mathrm{mult}_v(\Gamma_i) \\
&\leq \sum_{i=1}^{l} \mathrm{mult}_{\Gamma_i}(D_\varphi + (\chi^a))\,\mathrm{mult}_v(\Gamma_i) \\
&= \mathrm{mult}_v(D_\varphi + (\chi^a)).
\end{aligned}$$

Thank you for your attention.