Lower bound for the Néron-Tate height

Éric Gaudron
(joint work with Vincent Bosser)

We propose a totally explicit lower bound for the Néron-Tate height of algebraic points of infinite order of abelian varieties.

Let k be a number field of degree $D = [k : \mathbb{Q}]$. Let A be an abelian variety defined over a fixed subfield of k, of dimension g. Let L be a polarization of A. We denote by $\widehat{h}_L$ the Néron-Tate height on $A(k)$ relative to L. It is well-known that, for $p \in A(k)$, we have $\widehat{h}_L(p) = 0$ if and only if p is a torsion point, i.e. $np = 0$ for some positive integer n. The general problem of bounding from below $\widehat{h}_L(p)$ when $p \in A(k)$ is not a torsion point has been often tackled in the literature, overall from the point of view of the dependence on D (Lehmer's problem) or on the Faltings height $h_F(A)$ of A (Lang-Silverman conjecture). Moreover most of results concern elliptic curves or abelian varieties with complex multiplication. Let us cite two emblematic results due to David Masser, valid in great generality (A_{tors} is the set of torsion points) [3, 4].

Theorem (Masser, 1985-86) *In the above setting, there exist positive constants $c(A, \varepsilon)$ and $c(k, g)$, depending only on A, ε and on k, g respectively, such that, for all $\varepsilon > 0$ and all $p \in A(k) \setminus A_{\mathrm{tors}}$, one has*

$$\widehat{h}_L(p)^{-1} \leq c(A, \varepsilon) D^{2g+1+\varepsilon} \quad \textit{and} \quad \widehat{h}_L(p)^{-1} \leq c(k, g) \max\left(1, h_F(A)\right)^{2g+1}.$$

Unfortunately, there is no bound which takes into account both degree and Faltings height (at this level of generality). Up to now, there is only one published bound for $\widehat{h}_L(p)^{-1}$ which is totally explicit in all parameters. It is due to Bruno Winckler (PhD thesis, 2015, [8]) valid for a CM elliptic curve A. His bound looks like Dobrowolski-Laurent's one: $c(A) D(\max(1, \log D)/\max(1, \log\log D))^3$ with a constant $c(A)$ quite complicated (but explicit). We propose here the following much simpler bound.

Theorem 1. *Let (A, L) be a polarized abelian variety over k and $p \in A(k) \setminus A_{\mathrm{tors}}$. Then we have*

$$\widehat{h}_L(p)^{-1} \leq \max\left(D + g^g, h_F(A)\right)^{10^5 g}.$$

Note that the bound does not depend on the polarization L. The proof of this theorem involves two ingredients, namely a generalized period theorem and Minkowski's convex body theorem.

Let us explain the first one. Let $\sigma\colon k \hookrightarrow \mathbb{C}$ be a complex embedding. By extending the scalars we get a complex abelian variety $A_\sigma = A \times_\sigma \operatorname{Spec} \mathbb{C}$ isomorphic to the torus $t_{A_\sigma}/\Omega_{A_\sigma}$ composed with the tangent space at the origin t_{A_σ} and with the period lattice Ω_{A_σ} of A_σ. From the Riemann form associated to L_σ, we get an hermitian norm $\|\cdot\|_{L,\sigma}$ on t_{A_σ} (see for instance [1, § 2.4]). For $\omega \in \Omega_{A_\sigma}$, let A_ω be the smallest abelian subvariety of A_σ such that $\omega \in t_{A_\omega}$. Actually A_ω is an abelian variety defined over a number field K/k of relative degree $\leq 2(9g)^{2g}$

(Silverberg [7]). A *period theorem* consists of bounding from above the geometrical degree $\deg_L A_\omega$ in terms of $g, D, \|\omega\|_{L,\sigma}$ and $h_F(A)$. Such a theorem is useful to bound the minimal isogeny degree between two isogeneous abelian varieties ([1, 2, 5, 6]). A *generalized period theorem* consists of replacing ω by a logarithm $u \in t_{A_\sigma}$ of a k-rational point $p \in A(k)$ (we have $\sigma(p) = \exp_{A_\sigma}(u)$). In this setting we have the following bound (written in a very simplified form).

Theorem 2. *If $u \neq 0$ then*

$$(\deg_L A_u)^{1/(2\dim A_u)} \leq \left(D\widehat{h}_L(p) + \|u\|^2_{L,\sigma}\right) \max\left(D + g^g, h_F(A)\right)^{50}.$$

The proof of Theorem 2 extends that of the period theorem [1] using Gel'fond-Baker's method with Philippon-Waldschmidt's approach and some adelic geometry. Since it is long enough, we shall only explain in the rest of the exposition how to deduce Theorem 1 from Theorem 2. The very classical argument is to use the pigeonhole principle. Here we replace it by the more convenient Minkowski's first theorem. Let E be the $\mathbb{R}$-vector space $\mathbb{R} \times t_{A_\sigma}$ endowed with the Euclidean norm

$$\|(a, x)\|^2 := a^2 D\widehat{h}_L(p) + \|a.u + x\|^2_{L,\sigma}.$$

In $(E, \|\cdot\|)$ stands the lattice $\mathbb{Z} \times \Omega_{A_\sigma}$ whose determinant is $D\widehat{h}_L(p)\mathrm{h}^0(A, L)^2$. So, by Minkowski, there exists $(\ell, \omega) \in \mathbb{Z} \times \Omega_{A_\sigma} \setminus \{0\}$ such that

$$(\star) \qquad D\widehat{h}_L(\ell p) + \|\ell u + \omega\|^2_{L,\sigma} \leq \gamma_{2g+1} \left(D\widehat{h}_L(p)\mathrm{h}^0(A, L)^2\right)^{1/(2g+1)}$$

where $\gamma_{2g+1} \leq g + 1$ is the Hermite constant. Since p is assumed to be non-torsion, the logarithm $\ell u + \omega$ of $\sigma(\ell p)$ is not 0 and Theorem 2 gives a lower bound for the left-hand side of inequality $(\star)$, involving a lower bound for $\widehat{h}_L(p)$. Nevertheless, at this stage, the dimension $\mathrm{h}^0(A, L)$ of the global sections space of the polarization is still in the bound. To remove it, we use Zarhin's trick by replacing A with $(A \times \widehat{A})^4$ (here $\widehat{A}$ is the dual abelian variety), endowed with a principal polarization compatible to L. Then the Néron-Tate height of p remains unchanged whereas Faltings height and dimension of A are multiplied by 8, ruining the numerical constant but also making $\mathrm{h}^0(A, L)$ disappear.

References

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