Convergent algorithm based on Carleman estimates for coefficient inverse problems.

Maya de Buhan

CNRS - Université Paris Descartes

Collaboration with Lucie Baudouin and Sylvain Ervedoza

Journées Contrôle, Problème Inverse et Applications
Université Clermont-Auvergne, 25 septembre 2017

Motivations

We are interested in coefficient inverse problems for evolutionary partial differential equations (wave equation, heat equation, elasticity system...).

Motivations

We are interested in coefficient inverse problems for evolutionary partial differential equations (wave equation, heat equation, elasticity system...).

In these cases, the solution u of the PDE depends nonlinearly of the coefficient p. A classical method for finding p consists in minimizing a cost functional

$$\mathcal{F}(q) = \|u[q] - u[p]\|.$$

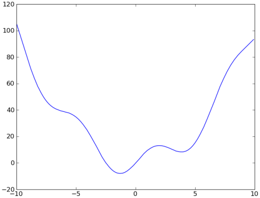
Motivations

We are interested in coefficient inverse problems for evolutionary partial differential equations (wave equation, heat equation, elasticity system...).

In these cases, the solution u of the PDE depends nonlinearly of the coefficient p. A classical method for finding p consists in minimizing a cost functional

$$\mathcal{F}(q) = \|u[q] - u[p]\|.$$

Unfortunately, $\mathcal{F}$ is generally not convex and may have several local minima. Classical minimization algorithms are not guaranteed to converge toward the global minimum of $\mathcal{F}$.



Motivations

We propose a new algorithm to solve some coefficient inverse problems.

- At each iteration, it simply consists in the minimization of a strictly convex and coercive quadratic functional.
- It is based on Carleman estimates.
- We prove its global convergence for any initial guess.

Motivations

We propose a new algorithm to solve some coefficient inverse problems.

- At each iteration, it simply consists in the minimization of a strictly convex and coercive quadratic functional.
- It is based on Carleman estimates.
- We prove its global convergence for any initial guess.

Until now, we applied the idea to three cases :

1. the recovery of a potential in the wave equation.

   Global Carleman estimates for waves and applications, Communications in PDE, 2013.

   Convergent algorithm for the recovery of a potential in the wave equation, SIAM Numerical Analysis, 2017.

2. the recovery of the wave speed in the wave equation.

   With Lucie Baudouin, Sylvain Ervedoza and Axel Osses.

3. the recovery of a source term in a non linear heat equation.

   With Erica Schwindt and Muriel Boulakia.

Part I: Recovery of a potentiel in the wave equation

1. Statement of the problem
    - The wave equation with potential
    - The coefficient inverse problem for a single boundary measurement
    - Classical uniqueness and stability result
    - Classical numerical method
2. The algorithm
3. An improved version
4. Numerical results

Statement of the problem

The wave equation with potential

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with potential

$$\begin{cases} \partial_t^2 u - \Delta u + p(x)u = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega. \end{cases} \qquad (1)$$

- u denotes the amplitude of the waves,
- p is a potential supposed to be in $L^\infty(\Omega)$,
- f and g are source terms in $L^1(0,T;L^2(\Omega))$ and $H^1(\partial\Omega \times (0,T))$,
- (u_0, u_1) are the initial data in $H^1(\Omega) \times L^2(\Omega)$ with $u_0(x) = g(x,0)$.
- D'Alembert operator: $\Box = \partial_t^2 - \Delta$.

Statement of the problem

The wave equation with potential

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with potential

$$\begin{cases} \partial_t^2 u - \Delta u + p(x)u = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega. \end{cases} \tag{1}$$

- u denotes the amplitude of the waves,
- p is a potential supposed to be in $L^\infty(\Omega)$,
- f and g are source terms in $L^1(0,T;L^2(\Omega))$ and $H^1(\partial\Omega \times (0,T))$,
- (u_0, u_1) are the initial data in $H^1(\Omega) \times L^2(\Omega)$ with $u_0(x) = g(x,0)$.
- D'Alembert operator: $\Box = \partial_t^2 - \Delta$.

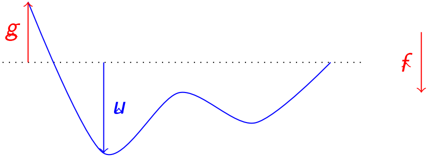
Theorem (Lasiecka-Lions-Triggiani 86)

$\exists!\ u \in C^0(0,T;H^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ *and* $\partial_\nu u \in L^2(\partial\Omega \times (0,T))$.

Statement of the problem

The wave equation with potential

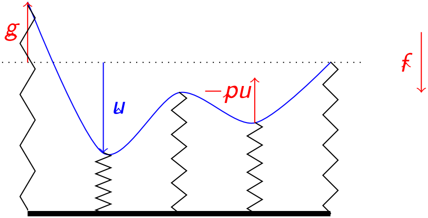
System (1) modelizes the transversal oscillations of a string/membrane and the term *pu* represents the acting force of a spring with stiffness *p*.



Statement of the problem

The wave equation with potential

System (1) modelizes the transversal oscillations of a string/membrane and the term pu represents the acting force of a spring with stiffness p.



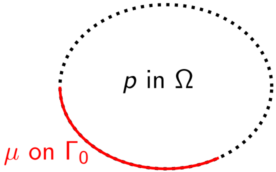
Statement of the problem

The coefficient inverse problem for a single boundary measurement

Given the source terms (f, g) and the initial data (u_0, u_1), can we determine the unknown potential $p(x)$, $\forall x \in \Omega$, from the additional knowledge of the flux

$$\mu = \partial_\nu u[p], \quad \text{on } \Gamma_0 \times (0, T),$$

where Γ_0 is a part of $\partial\Omega$?



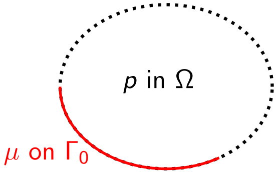
Statement of the problem

The coefficient inverse problem for a single boundary measurement

Given the source terms (f, g) and the initial data (u_0, u_1), can we determine the unknown potential $p(x)$, $\forall x \in \Omega$, from the additional knowledge of the flux

$$\mu = \partial_\nu u[p], \qquad \text{on } \Gamma_0 \times (0, T),$$

where Γ_0 is a part of $\partial\Omega$?



- Uniqueness: Given two potentials $p \neq q$, can we guarantee that $\partial_\nu u[p] \neq \partial_\nu u[q]$?
- Stability: if $\partial_\nu u[p] \approx \partial_\nu u[q]$, can we guarantee that $p \approx q$?
- Numerical resolution: how to compute p from $\partial_\nu u[p]$?

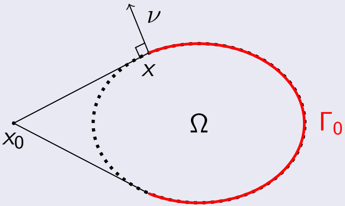
Statement of the problem

Classical uniqueness and stability result

Theorem (Yamamoto 99, Baudouin-Puel 01)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$$

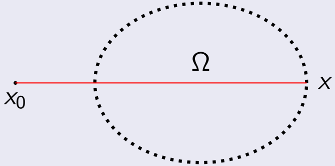


Theorem (Yamamoto 99, Baudouin-Puel 01)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) \geq 0\},$$

- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0|$,



Statement of the problem

Classical uniqueness and stability result

Theorem (Yamamoto 99, Baudouin-Puel 01)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \textit{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$$

- **Time condition:** $T > \sup_{x \in \Omega} |x - x_0|$,
- **Regularity assumption:** $u \in H^1((0,T); L^\infty(\Omega))$,

Statement of the problem

Classical uniqueness and stability result

Theorem (Yamamoto 99, Baudouin-Puel 01)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \text{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$$

- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0|$,
- **Regularity assumption:** $u \in H^1((0, T); L^\infty(\Omega))$,
- **Positivity condition:** $\exists \alpha > 0$ *such that* $|u_0| \geq \alpha$ *in* Ω.

Statement of the problem

Classical uniqueness and stability result

Theorem (Yamamoto 99, Baudouin-Puel 01)

- **Geometric condition:**

$$\exists\, x_0 \notin \overline{\Omega} \textit{ such that } \Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) \geq 0\},$$

- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0|$,
- **Regularity assumption:** $u \in H^1((0,T); L^\infty(\Omega))$,
- **Positivity condition:** $\exists\alpha > 0$ *such that* $|u_0| \geq \alpha$ *in* Ω.

Then for $m > 0$*, there exists a positive constant* $M = M(\Omega, T, x_0, m)$ *such that for all* p *and* q *in* $L^\infty_m(\Omega) = \{p \in L^\infty(\Omega), \|p\|_{L^\infty(\Omega)} \leq m\}$*:*

$$\|q - p\|_{L^2(\Omega)} \leq M\|\partial_\nu u[q] - \partial_\nu u[p]\|_{H^1(0,T;L^2(\Gamma_0))},$$

where $u[p]$ *and* $u[q]$ *denote the corresponding solutions of* (1).

Statement of the problem

Classical numerical method

A classical method for finding p consists in minimizing the non-convex cost functional:

$$\mathcal{F}(q) = \|\partial_\nu u[q] - \partial_\nu u[p]\|^2_{H^1(0,T;L^2(\Gamma_0))}.$$

This can be achieved for example thanks to a BFGS algorithm. The gradient of $\mathcal{F}$ is easily computed by an adjoint method.

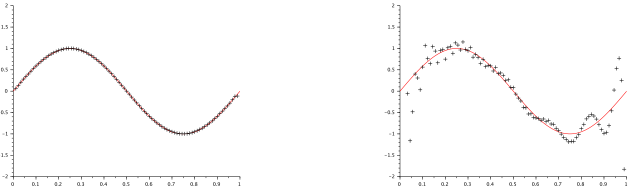
Statement of the problem

Classical numerical method

A classical method for finding p consists in minimizing the non-convex cost functional:

$$\mathcal{F}(q) = \|\partial_\nu u[q] - \partial_\nu u[p]\|^2_{H^1(0,T;L^2(\Gamma_0))}.$$

This can be achieved for example thanks to a BFGS algorithm. The gradient of $\mathcal{F}$ is easily computed by an adjoint method.



(a) initialization by $q^0 = 0$ (b) initialization by $q^0 = 10$

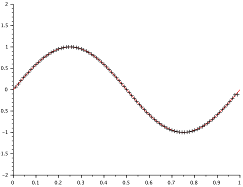
Statement of the problem

Classical numerical method

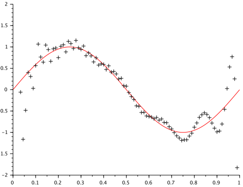
A classical method for finding p consists in minimizing the non-convex cost functional:

$$\mathcal{F}(q) = \|\partial_\nu u[q] - \partial_\nu u[p]\|^2_{H^1(0,T;L^2(\Gamma_0))}.$$

This can be achieved for example thanks to a BFGS algorithm. The gradient of $\mathcal{F}$ is easily computed by an adjoint method.



(a) initialization by $q^0 = 0$



(b) initialization by $q^0 = 10$

There exist many strategies to overcome partially this problem (Tikhonov regularization, stochastic optimization, ..., see also works by Beilina and Klibanov). The algorithm we propose converges to the global minimum from any initial guess.

Part I: Recovery of a potentiel in the wave equation

1. Statement of the problem
2. The algorithm
    - A Carleman estimate with pointwise term in time 0
    - Iterative loop
    - The convergence result
    - Proof of the convergence result
    - Nice in theory, useless in practice
3. An improved version
4. Numerical results

Theorem (Baudouin-deBuhan-Ervedoza 13)

Assume the **geometric and time conditions**. *Suppose* $\beta \in (0,1)$ *and* $\lambda > 0$ *such that*

$$\beta T > \sup_{x\in\Omega} |x - x_0| \quad \textit{and} \quad \varphi(x,t) = e^{\lambda(|x-x_0|^2-\beta t^2)}.$$

The algorithm

A Carleman estimate with pointwise term in time 0

Theorem (Baudouin-deBuhan-Ervedoza 13)

Assume the **geometric and time conditions**. *Suppose* $\beta \in (0,1)$ *and* $\lambda > 0$ *such that*

$$\beta T > \sup_{x\in\Omega} |x - x_0| \quad \text{and} \quad \varphi(x,t) = e^{\lambda(|x-x_0|^2 - \beta t^2)}.$$

Then with $m > 0$, *there exists a constant* $M > 0$ *such that for all* s *and* λ *large enough, for all* $q \in L^\infty_m(\Omega)$ *and for all* $z \in L^2(0,T; H^1_0(\Omega))$ *satisfying* $\Box z + qz \in L^2(\Omega \times (0,T))$, $\partial_\nu z \in L^2(\Gamma_0 \times (0,T))$ *and* $z(0) = 0$ *in* Ω *:*

$$\underbrace{s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx}_{\text{initial energy}} + s \int_0^T \int_\Omega e^{2s\varphi} (|\partial_t z|^2 + |\nabla z|^2 + s^2 |z|^2) \, dxdt$$

$$\leq M \underbrace{\int_0^T \int_\Omega e^{2s\varphi} |\Box z + qz|^2 \, dxdt}_{\text{source}} + M \underbrace{s \int_0^T \int_{\Gamma_0} e^{2s\varphi} \left|\partial_\nu z\right|^2 \, d\gamma dt}_{\text{observations}}.$$

The algorithm
Iterative loop

Initialization: Take any $q^0 \in L^\infty_m(\Omega)$.

Iteration: Given q^k,

The algorithm

Iterative loop

Initialization: Take any $q^0 \in L^\infty_m(\Omega)$.

Iteration: Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + q^k u = f, & \text{in } \Omega \times (0, T), \\ u = g, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu u[q^k] - \mu \right)$ on $\Gamma_0 \times (0, T)$.

The algorithm

Iterative loop

<u>Initialization:</u> Take any $q^0 \in L^\infty_m(\Omega)$.

<u>Iteration:</u> Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + q^k u = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu u[q^k] - \mu \right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

$$J_0(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,$$

on the space $\mathcal{T} = \{z \in L^2(0,T; H^1_0(\Omega)), z(0) = 0, \Box z + q^k z \in L^2(\Omega \times (0,T)), \partial_\nu z \in L^2(\Gamma_0 \times (0,T))\}$.

The algorithm

Iterative loop

Lemma

Assume the **geometric and time conditions***. Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0 is continuous, strictly convex and coercive on $\mathcal{T}$ endowed with a suitable weighted norm.*

The algorithm

Iterative loop

Lemma

Assume the **geometric and time conditions***. Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0 is continuous, strictly convex and coercive on $\mathcal{T}$ endowed with a suitable weighted norm.*

3 - Let Z be the unique minimizer of the functional J_0, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z(0)}{u_0},$$

where u_0 is the initial condition of (1).

The algorithm
Iterative loop

Lemma

Assume the **geometric and time conditions**. *Then, for all $s > 0$ and $k \in \mathbb{N}$, the functional J_0 is continuous, strictly convex and coercive on $\mathcal{T}$ endowed with a suitable weighted norm.*

3 - Let Z be the unique minimizer of the functional J_0, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z(0)}{u_0},$$

where u_0 is the initial condition of (1).

4 - Finally, set

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| > m. \end{cases}$$

The algorithm

The convergence result

Theorem (Baudouin-deBuhan-Ervedoza 13)

Assume **the geometric and time conditions, the regularity assumption and the positivity condition**. *Let $p \in L^\infty_m(\Omega)$. There exists a constant $M > 0$ such that for all s large enough and for all $k \in \mathbb{N}$,*

$$\int_\Omega e^{2s\varphi(0)} |q^k - p|^2 \, dx \leq \left(\frac{M}{\sqrt{s}} \right)^k \int_\Omega e^{2s\varphi(0)} |p - q^0|^2 \, dx.$$

In particular, if s is large enough, q^k converges toward p when k goes to infinity.

The algorithm

Proof of the convergence result

The algorithm is based on the method of Bukhgeim and Klibanov and use the fact that $v = \partial_t \left(u[q^k] - u[p] \right)$ solves

$$\begin{cases} \partial_t^2 v - \Delta v + q^k v = h, & \text{in } \Omega \times (0, T), \\ v = 0, & \text{on } \partial\Omega \times (0, T), \\ v(0) = 0, \quad \partial_t v(0) = (p - q^k) u_0, & \text{in } \Omega, \end{cases}$$

where

$$h = (p - q^k) \partial_t u[p].$$

The algorithm

Proof of the convergence result

The algorithm is based on the method of Bukhgeim and Klibanov and use the fact that $v = \partial_t \left(u[q^k] - u[p] \right)$ solves

$$\left\{ \begin{array}{ll} \partial_t^2 v - \Delta v + q^k v = h, & \text{in } \Omega \times (0, T), \\ v = 0, & \text{on } \partial\Omega \times (0, T), \\ v(0) = 0, \quad \partial_t v(0) = (p - q^k) u_0, & \text{in } \Omega, \end{array} \right.$$

where

$$h = (p - q^k) \partial_t u[p].$$

Moreover, by definition,

$$\mu^k = \partial_\nu v \text{ on } \Gamma_0 \times (0, T),$$

and we notice that v is the unique minimizer of the functional:

$$J_h(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z - h|^2 \, dx dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt,$$

on the space $\mathcal{T} = \{ z \in L^2(0, T; H_0^1(\Omega)), \partial_\nu z \in L^2(\Gamma_0 \times (0, T)), \Box z + q^k z \in L^2(\Omega \times (0, T)), z(0) = 0 \}$

The algorithm
Proof of the convergence result

Let us write the Euler Lagrange equations satisfied by:

- Z minimizer of J_0:

$$\mathcal{D}J_0(Z,z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box Z + q^k Z)(\Box z + q^k z)\,dxdt \\ + s\int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z - \mu^k)\partial_\nu z\,d\gamma dt = 0,$$

Let us write the Euler Lagrange equations satisfied by:

- Z minimizer of J_0:

$$\mathcal{D}J_0(Z,z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box Z + q^k Z)(\Box z + q^k z)\, dxdt \\ + s\int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z - \mu^k)\partial_\nu z\, d\gamma dt = 0,$$

- and v minimizer of J_h:

$$\mathcal{D}J_h(v,z) = \int_0^T \int_\Omega e^{2s\varphi}(\Box v + q^k v - h)(\Box z + q^k z)\, dxdt \\ + s\int_0^T \int_{\Gamma_0} e^{2s\varphi}(\partial_\nu v - \mu^k)\partial_\nu z\, d\gamma dt = 0,$$

for all $z \in \mathcal{T}$.

The algorithm

Proof of the convergence result

Applying these equations to $z = Z - v$ and subtracting the two identities, we obtain:

$$\int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt$$
$$= \int_0^T \int_\Omega e^{2s\varphi} h(\Box z + q^k z) \, dxdt.$$

The algorithm

Proof of the convergence result

Applying these equations to $z = Z - v$ and subtracting the two identities, we obtain:

$$\int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt$$
$$= \int_0^T \int_\Omega e^{2s\varphi} h(\Box z + q^k z) \, dxdt.$$

This implies ($2ab \leq a^2 + b^2$) that

$$\frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z|^2 \, d\gamma dt$$
$$\leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |h|^2 \, dxdt.$$

The algorithm

Proof of the convergence result

The left hand side precisely is the right hand side of the Carleman estimate. Hence, we deduce:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t z(0)|^2\,dx \leq M\int_0^T\int_\Omega e^{2s\varphi}|h|^2\,dxdt,$$

where

$$\partial_t z(0) = \partial_t Z(0) - \partial_t v(0).$$

The algorithm

Proof of the convergence result

The left hand side precisely is the right hand side of the Carleman estimate. Hence, we deduce:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx \leq M \int_0^T \int_\Omega e^{2s\varphi} |h|^2 \, dxdt,$$

where

$$\partial_t z(0) = \partial_t Z(0) - \partial_t v(0).$$

Moreover

$$\begin{aligned} &\partial_t Z(0) = (\tilde{q}^{k+1} - q^k) u_0, \text{ by definition of } \tilde{q}^{k+1}, \\ &\partial_t v(0) = (p - q^k) u_0, \\ &h = (p - q^k) \partial_t u[p]. \end{aligned}$$

The algorithm

Proof of the convergence result

Therefore, since $\varphi(t) \leq \varphi(0)$ for all $t \in (0, T)$ we have:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |u_0|^2 (\tilde{q}^{k+1} - p)^2 \, dx$$
$$\leq M \|\partial_t u[p]\|^2_{L^2(0,T;L^\infty(\Omega))} \int_\Omega e^{2s\varphi(0)} (q^k - p)^2 \, dx.$$

The algorithm

Proof of the convergence result

Therefore, since $\varphi(t) \leq \varphi(0)$ for all $t \in (0, T)$ we have:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |u_0|^2 (\tilde{q}^{k+1} - p)^2 \, dx$$
$$\leq M \|\partial_t u[p]\|^2_{L^2(0,T;L^\infty(\Omega))} \int_\Omega e^{2s\varphi(0)} (q^k - p)^2 \, dx.$$

Using the positivity condition on u_0 and the fact that

$$|q^{k+1} - p| = |T_m(\tilde{q}^{k+1}) - T_m(p)| \leq |\tilde{q}^{k+1} - p|$$

because T_m is Lipschitz and $T_m(p) = p$, we immediately deduce

$$\int_\Omega e^{2s\varphi(0)} (q^{k+1} - p)^2 \, dx \leq \left(\frac{M}{\sqrt{s}} \right)^{k+1} \int_\Omega e^{2s\varphi(0)} (q^0 - p)^2 \, dx.$$

□

The algorithm

Nice in theory, useless in practice

The parameters s and λ should be large to ensure the convergence of the algorithm.

The algorithm

Nice in theory, useless in practice

The parameters s and λ should be large to ensure the convergence of the algorithm.
But for $\lambda = 1$ and $s = 3$, $\max(\exp(2s\varphi))/\min(\exp(2s\varphi)) = 10^{110}$!

$$J_0(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dx dt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt$$

The algorithm

Nice in theory, useless in practice

The parameters s and λ should be large to ensure the convergence of the algorithm.
But for $\lambda = 1$ and $s = 3$, $\max(\exp(2s\varphi))/\min(\exp(2s\varphi)) = 10^{110}$!

$$J_0(z) = \int_0^T \int_\Omega e^{2s\varphi} |\Box z + q^k z|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2 \, d\gamma dt$$

This version of the algorithm is useless in practice. We propose some improvements to be able to implement it numerically.

Part I: Recovery of a potentiel in the wave equation

1. Statement of the problem
2. The algorithm
3. An improved version
    - Step 1: Use a Carleman estimate with a single weight
    - Step 2: Conjugate the wave operator
    - Step 3: Use cut-off functions
    - Some modifications in the algorithm...
    - ...but the same convergence result
4. Numerical results

An improved version

Step 1: Use a Carleman estimate with a single weight

Theorem (LB-MdB-SE 15, from Imanuvilov-Yamamoto 01)

Assume the **geometric and time conditions***. Suppose* $\beta \in (0,1)$,

$$\beta T > \sup_{x \in \Omega} |x - x_0| \quad \textit{and} \quad \varphi(x,t) = |x - x_0|^2 - \beta t^2.$$

Then with $m > 0$*, there exists a constant* $M > 0$ *such that for all* s *large enough, for all* $q \in L^\infty_m(\Omega)$ *and for all* $z \in \mathcal{T}$ *:*

$$\underbrace{s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx}_{\textit{initial energy}} \leq \underbrace{M \int_0^T \int_\Omega e^{2s\varphi} |\Box z + qz|^2 \, dxdt}_{\textit{source}}$$

$$+ M \, s \underbrace{\int_0^T \int_{\Gamma_0} e^{2s\varphi} \left|\partial_\nu z\right|^2 \, d\gamma dt}_{\textit{observations}} + Ms^3 \iint_{\{\varphi < 0\}} e^{2s\varphi} z^2 \, dxdt.$$

We remove some exponential factors by introducing the conjugate variable $y = e^{s\varphi} z$ and minimizing the new functional

$$\widetilde{J}_0(y) = \int_0^T \int_\Omega |\mathcal{L}y|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu^k e^{s\varphi}|^2 \; d\sigma dt$$
$$+ s^3 \iint_{\{\varphi<0\}} |y|^2 \, dxdt,$$

where the conjugate operator is

$$\begin{aligned}\mathcal{L}y =& e^{s\varphi}(\partial_t^2 - \Delta + q^k)(e^{-s\varphi} y) \\ =& \partial_t^2 y - \Delta y + q^k y + 4\beta s t \partial_t y + 4s(x - x0) \cdot \nabla y \\ & + 2s(\beta + n)y + 4s^2(|\beta t|^2 - |x - x0|^2)y.\end{aligned}$$

We remove some exponential factors by introducing the conjugate variable $y = e^{s\varphi} z$ and minimizing the new functional

$$\widetilde{J}_0(y) = \int_0^T \int_\Omega |\mathcal{L}y|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu^k e^{s\varphi}|^2 \; d\sigma dt$$
$$+ s^3 \iint_{\{\varphi<0\}} |y|^2 \, dxdt,$$

where the conjugate operator is

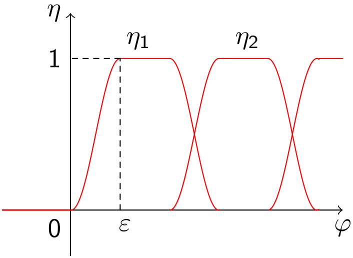
$$\begin{aligned} \mathcal{L}y =& e^{s\varphi}(\partial_t^2 - \Delta + q^k)(e^{-s\varphi} y) \\ =& \partial_t^2 y - \Delta y + q^k y + 4\beta s t \partial_t y + 4s(x - x0) \cdot \nabla y \\ &+ 2s(\beta + n)y + 4s^2(|\beta t|^2 - |x - x0|^2)y. \end{aligned}$$

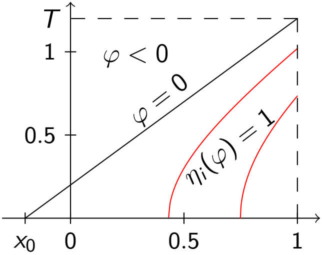
Nevertheless, there are still exponential factors appearing in the measurements.

An improved version

Step 3: Use cut-off functions

We split the observations in several parts and consider intervals in which the weight function does not significantly change. To do that, we use cut-off functions η_i.





and notice that:

$$\forall i, Y_i \text{ minimizer of } \widetilde{J}_0[\eta_i] \quad \Longleftrightarrow \quad Y = \sum_i Y_i \text{ minimizer of } \widetilde{J}_0[\sum_i \eta_i].$$

An improved version

Some modifications in the algorithm...

Initialization: Any $q \in L^\infty_m(\Omega)$.

Iteration: Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + q^k u = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega, \end{cases}$$

and for each i, set $\mu_i^k = \eta_i(\varphi)\partial_t \left(\partial_\nu u[q^k] - \mu\right)$ on $\Gamma_0 \times (0,T)$.

An improved version

Some modifications in the algorithm...

Initialization: Any $q \in L^\infty_m(\Omega)$.

Iteration: Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + q^k u = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega, \end{cases}$$

and for each i, set $\mu_i^k = \eta_i(\varphi)\partial_t \left(\partial_\nu u[q^k] - \mu\right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional

$$\widetilde{J}_0[\mu_i^k](y) = \int_0^T \int_\Omega |\mathcal{L}y|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu_i^k e^{s\varphi}|^2 \, d\sigma dt + s^3 \iint_{\{\varphi<0\}} |y|^2 \, dxdt.$$

An improved version

Some modifications in the algorithm...

3 - For each i, let Y_i be the unique minimizer of the functional $\tilde{J}_0[\mu_i^k]$, and then set

$$\widetilde{q}^{k+1} = q^k + \sum_i \frac{\partial_t Y_i(0)}{u_0 e^{s\varphi(0)}},$$

where u_0 is the initial condition of (1).

An improved version

Some modifications in the algorithm...

3 - For each i, let Y_i be the unique minimizer of the functional $\tilde{J}_0[\mu_i^k]$, and then set

$$\widetilde{q}^{k+1} = q^k + \sum_i \frac{\partial_t Y_i(0)}{u_0 e^{s\varphi(0)}},$$

where u_0 is the initial condition of (1).

4 - Finally, set

$$q^{k+1} = T_m(\widetilde{q}^{k+1}), \quad \text{where } T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q)m, & \text{if } |q| > m. \end{cases}$$

An improved version

...but the same convergence result

Theorem (Baudouin-deBuhan-Ervedoza 15)

Assume **the geometric and time conditions, the regularity assumption and the positivity condition***. Let $p \in L^\infty_m(\Omega)$. There exists a constant $M > 0$ such that for all s large enough and for all $k \in \mathbb{N}$,*

$$\int_\Omega e^{2s\varphi(0)}|q^k - p|^2 \, dx \leq \left(\frac{M}{\sqrt{s}} \right)^k \int_\Omega e^{2s\varphi(0)}|p|^2 \, dx.$$

In particular, if s is large enough, q^k converges toward p when k goes to infinity.

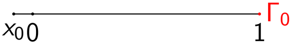
Part I: Recovery of a potentiel in the wave equation

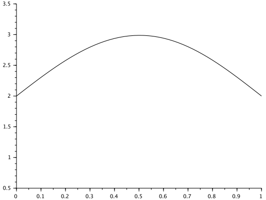
1. Statement of the problem
2. The algorithm
3. An improved version
4. Numerical results
    - Discretization of the problem
    - Illustration of the convergence of the algorithm
    - Numerical results in 1D
    - Wrong choices of the parameters
    - Numerical results in 2D

Numerical results

Discretization of the problem

- $\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 100$, $f = 0$, $g = 2$, $u_0(x) = 2 + \sin(x\pi)$ and $u_1 = 0$.

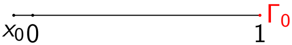




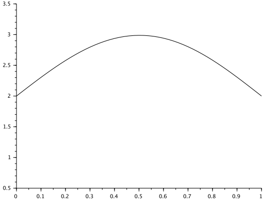
Numerical results

Discretization of the problem

- $\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 100$, $f = 0$, $g = 2$, $u_0(x) = 2 + \sin(x\pi)$ and $u_1 = 0$.



- To avoid the inverse crime, we use neither the same schemes nor the same meshes in the direct and the inverse problems:
  - direct problem: finite differences in space $h = 0.00025$, implicit theta scheme in time $\tau = 0.00033$;
  - inverse problem: finite differences in space $h = 0.05$, explicit Euler scheme in time $\tau = 0.05$, that is $CFL = 1$.



The minimizer Y of $\widetilde{J}_0[\mu_i^k]$ satisfies the variational formulation

$$\int_0^T \int_\Omega \mathcal{L}Y \mathcal{L}y \, dxdt + s \int_0^T \int_{\Gamma_0} \partial_\nu Y \partial_\nu y \; d\sigma dt + s^3 \iint_{\{\varphi<0\}} Yy \, dxdt$$
$$= s \int_0^T \int_{\Gamma_0} \mu_i^k e^{s\varphi} \partial_\nu y \; d\sigma dt.$$

where $\mathcal{L}$ is a differential operator of order 2.

Numerical results

Discretization of the problem

The minimizer Y of $\widetilde{J}_0[\mu_i^k]$ satisfies the variational formulation

$$\int_0^T \int_\Omega \mathcal{L}Y \mathcal{L}y \, dxdt + s \int_0^T \int_{\Gamma_0} \partial_\nu Y \partial_\nu y \; d\sigma dt + s^3 \iint_{\{\varphi<0\}} Yy \, dxdt$$
$$= s \int_0^T \int_{\Gamma_0} \mu_i^k e^{s\varphi} \partial_\nu y \; d\sigma dt.$$

where $\mathcal{L}$ is a differential operator of order 2.
Two options to discretize it:

- Use high order finite elements (Cîndea-FernándezCara-Münch 13) to guarantee a conformal approximation.

Numerical results

Discretization of the problem

The minimizer Y of $\widetilde{J}_0[\mu_i^k]$ satisfies the variational formulation

$$\int_0^T \int_\Omega \mathcal{L}Y\mathcal{L}y\,dxdt + s\int_0^T \int_{\Gamma_0} \partial_\nu Y \partial_\nu y\ d\sigma dt + s^3 \iint_{\{\varphi<0\}} Yy\,dxdt$$
$$= s\int_0^T \int_{\Gamma_0} \mu_i^k e^{s\varphi}\partial_\nu y\ d\sigma dt.$$

where $\mathcal{L}$ is a differential operator of order 2.
Two options to discretize it:

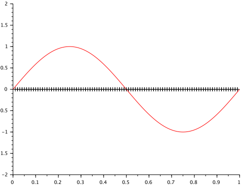
- Use high order finite elements (Cîndea-FernándezCara-Münch 13) to guarantee a conformal approximation.
- Add a viscous term (Baudouin-Ervedoza-Osses 14, Baudouin-Ervedoza 13) in the functional to penalize the high frequencies:

$$+s\int_0^T \int_\Omega |h\nabla\partial_t y|^2\,dxdt.$$

This makes the Carleman estimates uniform with respect to the discretization parameter h but requires that sh is bounded.

Numerical results

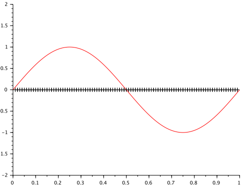
Illustration of the convergence of the algorithm



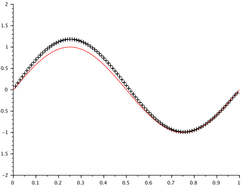
(a) $q^0 = 0$

Numerical results

Illustration of the convergence of the algorithm



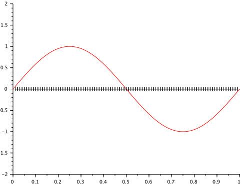
(a) $q^0 = 0$



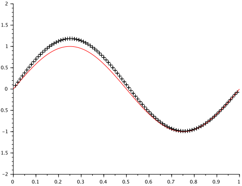
(b) q^1

Numerical results

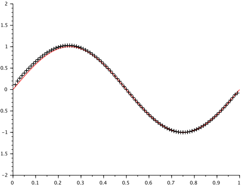
Illustration of the convergence of the algorithm



(a) $q^0 = 0$



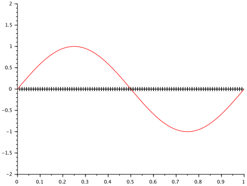
(b) q^1



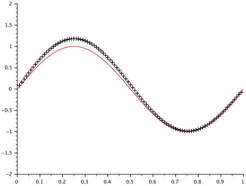
(c) q^2

Numerical results

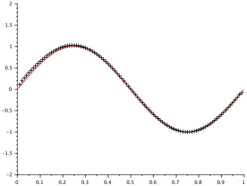
Illustration of the convergence of the algorithm



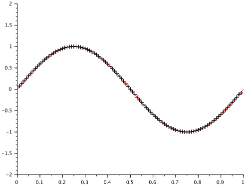
(a) $q^0 = 0$



(b) q^1



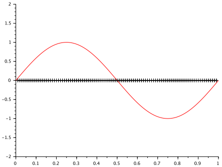
(c) q^2



(d) q^3

Numerical results

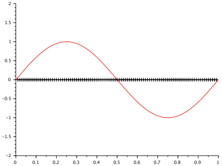
Illustration of the convergence of the algorithm



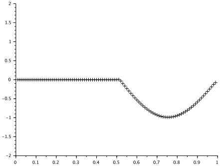
(a) $q_0^0 = q^0$

Numerical results

Illustration of the convergence of the algorithm



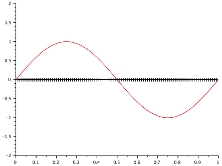
(a) $q_0^0 = q^0$



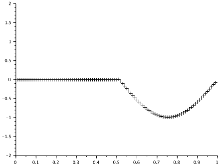
(b) q_1^0

Numerical results

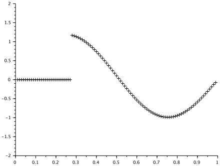
Illustration of the convergence of the algorithm



(a) $q_0^0 = q^0$



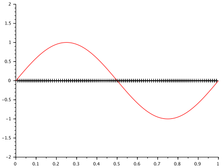
(b) q_1^0



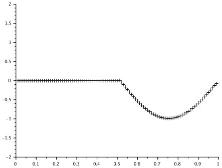
(c) q_2^0

Numerical results

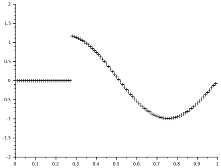
Illustration of the convergence of the algorithm



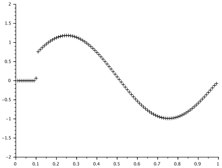
(a) $q_0^0 = q^0$



(b) q_1^0



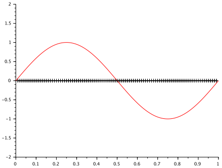
(c) q_2^0



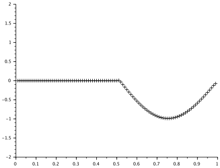
(d) q_3^0

Numerical results

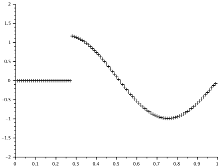
Illustration of the convergence of the algorithm



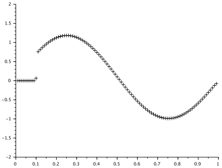
(a) $q_0^0 = q^0$



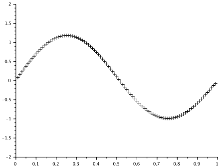
(b) q_1^0



(c) q_2^0



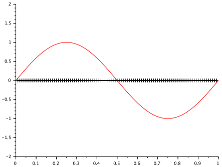
(d) q_3^0



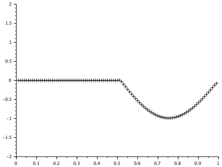
(e) q_4^0

Numerical results

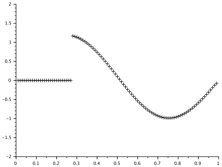
Illustration of the convergence of the algorithm



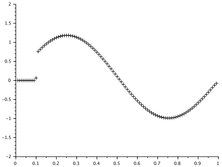
(a) $q_0^0 = q^0$



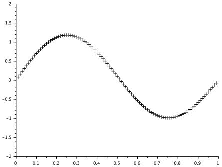
(b) q_1^0



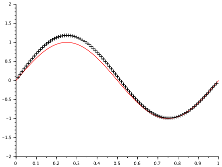
(c) q_2^0



(d) q_3^0



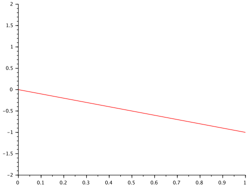
(e) q_4^0



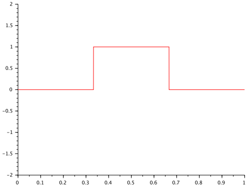
(f) $q_5^0 = q^1$

Numerical results

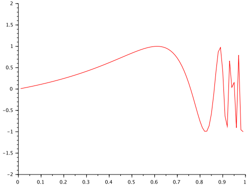
Numerical results in 1D



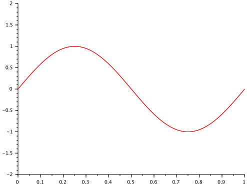
(a) $p = -x$



(b) p heaviside



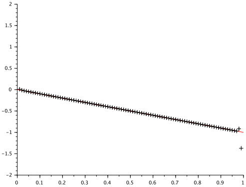
(c) $p(x) = \sin(\frac{x}{1-x})$



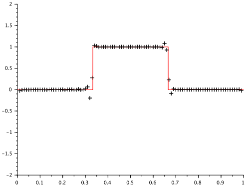
(d) $p(x) = \sin(2\pi x)$, with $q^0 = 10$

Numerical results

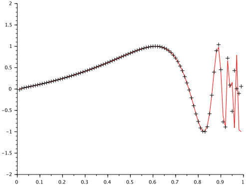
Numerical results in 1D



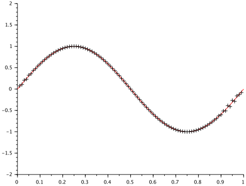
(a) $p = -x$



(b) p heaviside

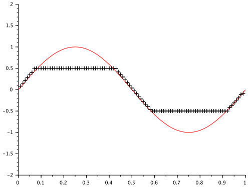


(c) $p(x) = \sin(\frac{x}{1-x})$



(d) $p(x) = \sin(2\pi x)$, with $q^0 = 10$

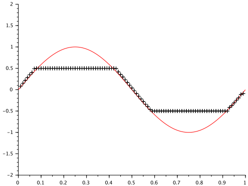
Numerical results
Wrong choices of the parameters



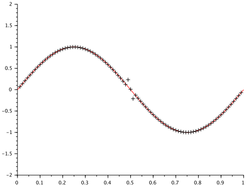
(a) Wrong choice of m

Numerical results

Wrong choices of the parameters



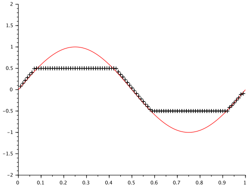
(a) Wrong choice of m



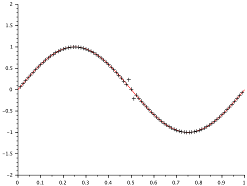
(b) u_0 vanishes at $x = 0.5$

Numerical results

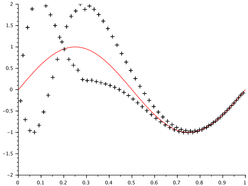
Wrong choices of the parameters



(a) Wrong choice of m



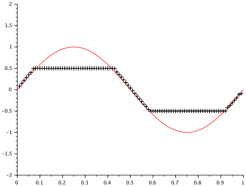
(b) u_0 vanishes at $x = 0.5$



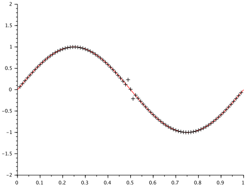
(c) No viscous term or *sh* too large

Numerical results

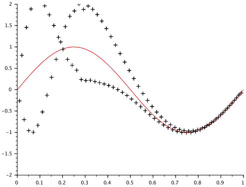
Wrong choices of the parameters



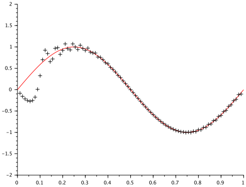
(a) Wrong choice of m



(b) u_0 vanishes at $x = 0.5$



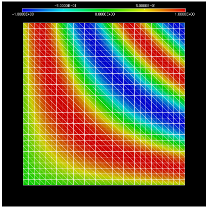
(c) No viscous term or *sh* too large



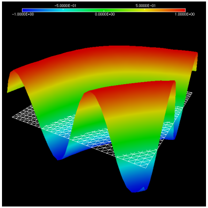
(d) $T = 0.9$

Numerical results
Numerical results in 2D

$\Omega = [0,1]^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma_0 = \{x = 1\} \cup \{y = 1\}$



(a) Exact potentiel

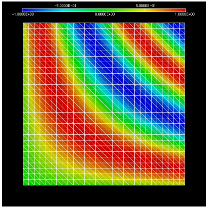


(b) 3D view

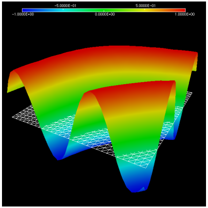
Numerical results

Numerical results in 2D

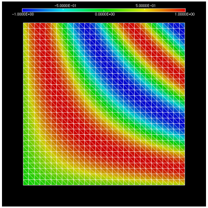
$\Omega = [0,1]^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma_0 = \{x = 1\} \cup \{y = 1\}$



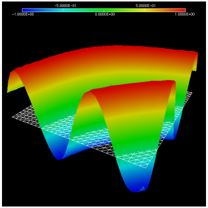
(a) Exact potentiel



(b) 3D view

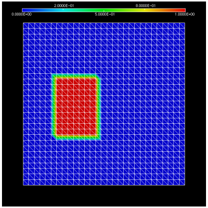


(c) Numerical

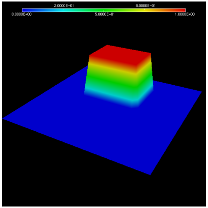


(d) 3D view

Numerical results
Numerical results in 2D



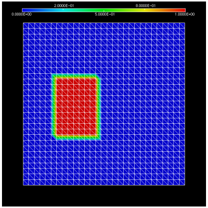
(a) Exact potential



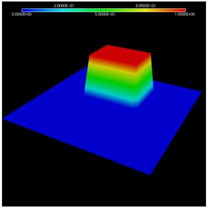
(b) 3D view

Numerical results

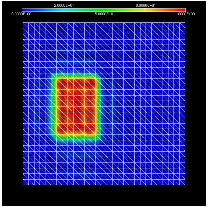
Numerical results in 2D



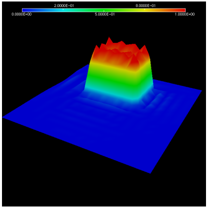
(a) Exact potential



(b) 3D view



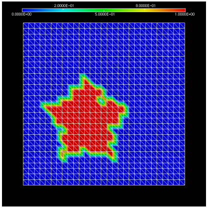
(c) Numerical



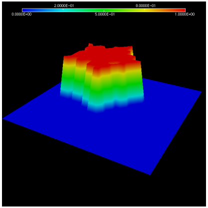
(d) 3D view

Numerical results

Numerical results in 2D



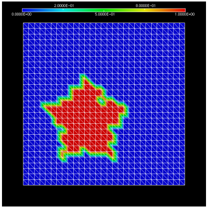
(a) Exact potential



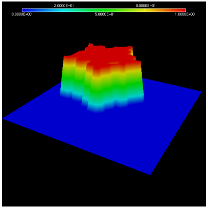
(b) 3D view

Numerical results

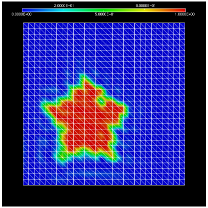
Numerical results in 2D



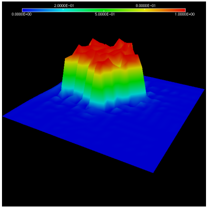
(a) Exact potential



(b) 3D view



(c) Numerical



(d) 3D view

Part II: Recovery of the wave speed

5. Statement of the problem
    - An inverse problem for the wave equation
    - Classical uniqueness and stability result
6. The algorithm
7. Numerical results

Statement of the problem

An inverse problem for the wave equation

Direct problem

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with non constant speed

$$\begin{cases} \partial_t^2 u - \nabla \cdot (p(x)\nabla u) = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega. \end{cases} \tag{2}$$

- u denotes the amplitude of the waves,
- $p = c^2 > 0$ is the square of the wave speed in $C^1(\Omega)$,
- f and g are source terms in $L^1(0,T;L^2(\Omega))$ and $H^1(\partial\Omega \times (0,T))$,
- (u_0, u_1) are the initial data in $H^1(\Omega) \times L^2(\Omega)$ with $u_0(x) = g(x,0)$.

Statement of the problem

An inverse problem for the wave equation

Direct problem

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We consider the wave equation with non constant speed

$$\begin{cases} \partial_t^2 u - \nabla \cdot (p(x)\nabla u) = f, & \text{in } \Omega \times (0, T), \\ u = g, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, \quad \partial_t u(0) = u_1, & \text{in } \Omega. \end{cases} \tag{2}$$

- u denotes the amplitude of the waves,
- $p = c^2 > 0$ is the square of the wave speed in $C^1(\Omega)$,
- f and g are source terms in $L^1(0, T; L^2(\Omega))$ and $H^1(\partial\Omega \times (0, T))$,
- (u_0, u_1) are the initial data in $H^1(\Omega) \times L^2(\Omega)$ with $u_0(x) = g(x, 0)$.

Inverse problem

Given the source terms (f, g) and the initial data (u_0, u_1), can we determine the unknown coefficient $p(x)$, $\forall x \in \Omega$, from the knowledge of the flux

$$\mu = \partial_\nu u[p], \qquad \text{on } \Gamma_0 \times (0, T)?$$

Statement of the problem

Classical uniqueness and stability result

Theorem (Imanuvilov-Yamamoto 03)

- **Admissible set:** $\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \text{ on } \partial\Omega,$ $0 < a_0 \leq p(x) \leq a_1 \text{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \text{ in } \Omega\},$

Statement of the problem

Classical uniqueness and stability result

Theorem (Imanuvilov-Yamamoto 03)

- **Admissible set:** $\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \textit{ on } \partial\Omega,$ $0 < a_0 \leq p(x) \leq a_1 \textit{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \textit{ in } \Omega\},$
- **Geometric condition:** $\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$
- **Time condition:** $T > \sup_{x \in \Omega} |x - x_0| / \sqrt{a_0},$

Statement of the problem

Classical uniqueness and stability result

Theorem (Imanuvilov-Yamamoto 03)

- **Admissible set:** $\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \text{ on } \partial\Omega,$ $0 < a_0 \leq p(x) \leq a_1 \text{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \text{ in } \Omega\},$
- **Geometric condition:** $\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$
- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0| / \sqrt{a_0},$
- **Regularity assumption:** $u \in W^{2,\infty}(\Omega \times (0, T)),$

Statement of the problem

Classical uniqueness and stability result

Theorem (Imanuvilov-Yamamoto 03)

- **Admissible set:** $\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \text{ on } \partial\Omega,$ $0 < a_0 \leq p(x) \leq a_1 \text{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \text{ in } \Omega\},$
- **Geometric condition:** $\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$
- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0| / \sqrt{a_0},$
- **Regularity assumption:** $u \in W^{2,\infty}(\Omega \times (0, T)),$
- **Initial conditions:** $|\nabla u_0 \cdot (x - x_0)| \geq \alpha > 0 \text{ and } u_1 = 0 \text{ in } \Omega.$

Statement of the problem

Classical uniqueness and stability result

Theorem (Imanuvilov-Yamamoto 03)

- **Admissible set:** $\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \text{ on } \partial\Omega,$ $0 < a_0 \leq p(x) \leq a_1 \text{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \text{ in } \Omega\},$
- **Geometric condition:** $\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$
- **Time condition:** $T > \sup_{x\in\Omega} |x - x_0|/\sqrt{a_0},$
- **Regularity assumption:** $u \in W^{2,\infty}(\Omega \times (0, T)),$
- **Initial conditions:** $|\nabla u_0 \cdot (x - x_0)| \geq \alpha > 0 \text{ and } u_1 = 0 \text{ in } \Omega.$

Then there exists a positive constant $M = M(\Omega, T, x_0, \alpha, a, a_0, a_1)$ *such that for all* p *and* q *in* $\mathcal{V}$*:*

$$\|q - p\|_{H^1(\Omega)} \leq M\|\partial_\nu u[q] - \partial_\nu u[p]\|_{H^2(0,T;L^2(\Gamma_0))},$$

where $u[p]$ *and* $u[q]$ *denote the corresponding solutions of* (2).

Part II: Recovery of the wave speed

The algorithm
Iterative loop

Initialization: Any $q^0 \in \mathcal{V}$.

Iteration: Given q^k,

The algorithm

Iterative loop

Initialization: Any $q^0 \in \mathcal{V}$.

Iteration: Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \nabla \cdot (q^k \nabla u) = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = 0, & \text{in } \Omega, \end{cases}$$

The algorithm

Iterative loop

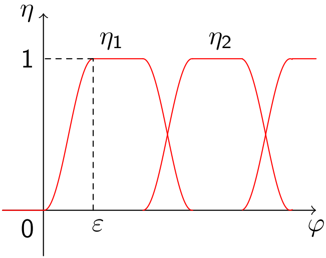
Initialization: Any $q^0 \in \mathcal{V}$.

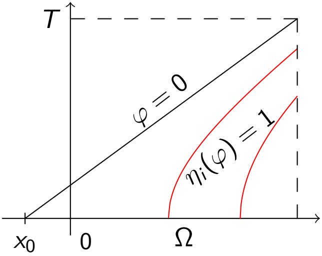
Iteration: Given q^k,

1 - Compute $u[q^k]$ the solution of

$$\begin{cases} \partial_t^2 u - \nabla \cdot (q^k \nabla u) = f, & \text{in } \Omega \times (0,T), \\ u = g, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \partial_t u(0) = 0, & \text{in } \Omega, \end{cases}$$

and for each i, set $\mu_i^k = \eta_i(\varphi)\partial_t^2 \left(\partial_\nu u[q^k] - \mu\right)$ on $\Gamma_0 \times (0,T)$.





The algorithm

Iterative loop

2 - Introduce the functional

$$\widetilde{J}_0[\mu_i^k](y) = \int_0^T \int_\Omega |\mathcal{L}y|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu_i^k e^{s\varphi}|^2 \, d\sigma dt$$
$$+ s^3 \iint_{\{\varphi<0\}} |y|^2 \, dxdt.$$

where $\mathcal{L}y = e^{s\varphi}(\partial_t^2 - \nabla \cdot (q^k \nabla))(e^{-s\varphi} y)$ is the conjugate operator.

The algorithm

Iterative loop

2 - Introduce the functional

$$\widetilde{J}_0[\mu_i^k](y) = \int_0^T \int_\Omega |\mathcal{L}y|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} |\partial_\nu y - \mu_i^k e^{s\varphi}|^2 \, d\sigma dt + s^3 \iint_{\{\varphi<0\}} |y|^2 \, dxdt.$$

where $\mathcal{L}y = e^{s\varphi}(\partial_t^2 - \nabla \cdot (q^k \nabla))(e^{-s\varphi} y)$ is the conjugate operator.

Lemma

Assume the **geometric and time conditions***. Then, for all $s > 0$, the functional $\widetilde{J}_0$ is continuous, strictly convex and coercive on $\mathcal{T}$ endowed with a suitable weighted norm.*

The algorithm

Iterative loop

3 - Let Y_i be the unique minimizer of the functional $\widetilde{J}_0[\mu_i^k]$ on $\mathcal{T}$ and then compute δq_i solution of

$$\begin{cases} \nabla \cdot (\delta q_i(x) \nabla u_0(x)) = \dfrac{Y_i(x,0)}{e^{s\varphi(x,0)}}, & x \in \Omega, \\ \delta q_i(x) = 0, & x \in \partial\Omega, \nabla u_0(x) \cdot \nu(x) > 0, \end{cases}$$

where u_0 is the initial condition of (1).

The algorithm

Iterative loop

3 - Let Y_i be the unique minimizer of the functional $\widetilde{J}_0[\mu_i^k]$ on $\mathcal{T}$ and then compute δq_i solution of

$$\begin{cases} \nabla \cdot (\delta q_i(x) \nabla u_0(x)) = \dfrac{Y_i(x,0)}{e^{s\varphi(x,0)}}, & x \in \Omega, \\ \delta q_i(x) = 0, & x \in \partial\Omega, \nabla u_0(x) \cdot \nu(x) > 0, \end{cases}$$

where u_0 is the initial condition of (1).

4 - Finally, set

$$q^{k+1} = T_{\mathcal{V}}(q^k - \sum_i \delta q_i),$$

where $T_{\mathcal{V}}$ is the projection in the admissible set $\mathcal{V}$.

Theorem (Baudouin-deBuhan-Ervedoza 16)

Assume **the geometric and time conditions, the regularity assumption and the initial condition**. *Let* $p \in \mathcal{V}$. *There exists a constant* $M > 0$ *such that for all* s *large enough and for all* $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)} |q^k - p|^2 \, dx \leq \left(\frac{M}{\sqrt{s}} \right)^k \int_\Omega e^{2s\varphi(0)} |q^0 - p|^2 \, dx.$$

In particular, if s *is large enough,* q^k *converges toward* p *when* k *goes to infinity.*

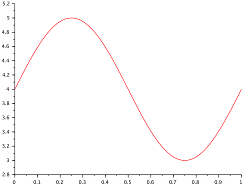
Part II: Recovery of the wave speed

5. Statement of the problem
6. The algorithm
7. Numerical results
    - Numerical results in 1D
    - Noise in the data
    - Numerical problems in 2D

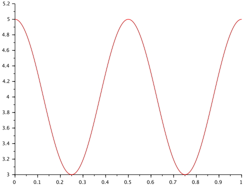
Numerical results

Numerical results in 1D

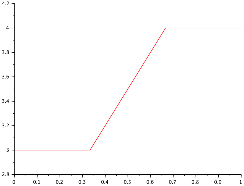
$\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 10$, $f = 0$, $g(0) = 0$, $g(1) = \sin(\pi t)$, $u_0(x) = x$ and $u_1 = 0$.



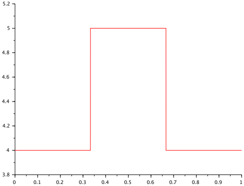
(a) $p(x) = \sin(2\pi x)$



(b) $p(x) = \cos(4\pi x)$



(c) $p \notin \mathcal{V}$

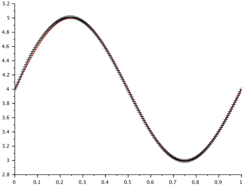


(d) $p \notin \mathcal{V}$

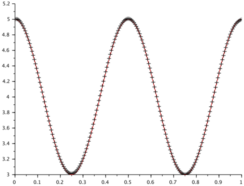
Numerical results

Numerical results in 1D

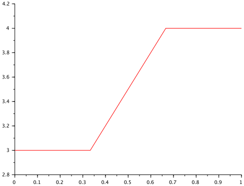
$\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 10$, $f = 0$, $g(0) = 0$, $g(1) = \sin(\pi t)$, $u_0(x) = x$ and $u_1 = 0$.



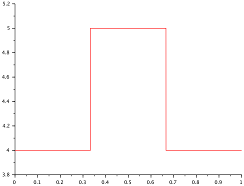
(a) $p = \sin(2\pi x)$



(b) $p = \cos(4\pi x)$



(c) $p \notin \mathcal{V}$

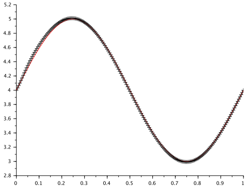


(d) $p \notin \mathcal{V}$

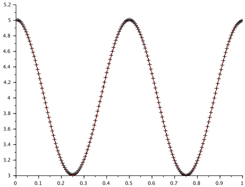
Numerical results

Numerical results in 1D

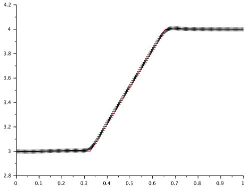
$\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 10$, $f = 0$, $g(0) = 0$, $g(1) = \sin(\pi t)$, $u_0(x) = x$ and $u_1 = 0$.



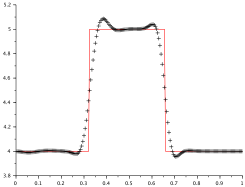
(a) $p = \sin(2\pi x)$



(b) $p = \cos(4\pi x)$



(c) $p \notin \mathcal{V}$



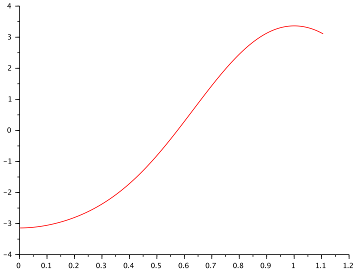
(d) $p \notin \mathcal{V}$

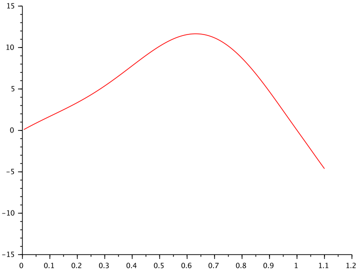
Numerical results
Noise in the data

Additional noise on the observation data:

$$\mu = (1 + \alpha \, \mathsf{Normal}(0, 0.5)) \, \mu, \quad \alpha \geq 0.$$

Problem: In our approach, we derive in time the observations $\partial_t^2 \mu$.


observation at $x = 1$


time derivative

Numerical results

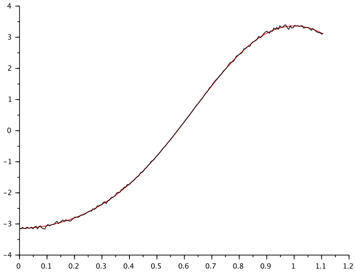
Noise in the data

Additional noise on the observation data:

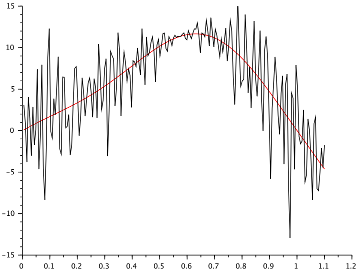
$$\mu = (1 + \alpha \, \mathrm{Normal}(0, 0.5)) \, \mu, \quad \alpha \geq 0.$$

Problem: In our approach, we derive in time the observations $\partial_t^2 \mu$.

observation at $x = 1$



time derivative



Numerical results

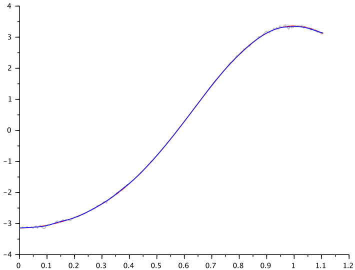
Noise in the data

Additional noise on the observation data:

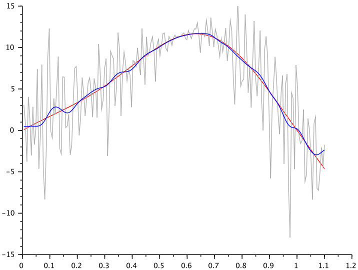
$$\mu = (1 + \alpha \, \mathrm{Normal}(0, 0.5)) \, \mu, \quad \alpha \geq 0.$$

Problem: In our approach, we derive in time the observations $\partial_t^2 \mu$.

observation at $x = 1$



time derivative

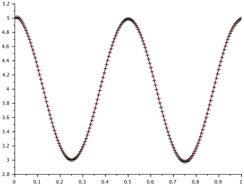


We regularize the signal by convolutions with a gaussian.

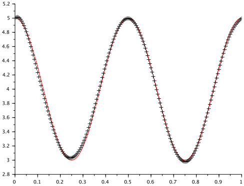
Numerical results

Noise in the data

Numerical results with noisy data



(a) $\alpha = 1\%$

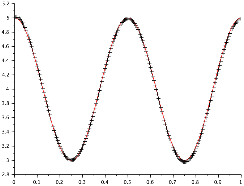


(b) $\alpha = 5\%$

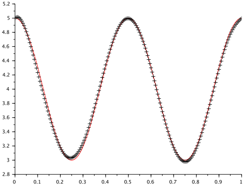
Numerical results

Noise in the data

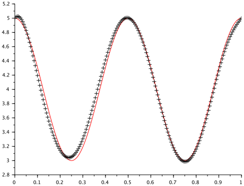
Numerical results with noisy data



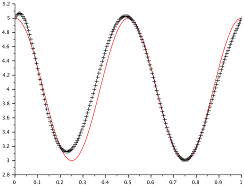
(a) $\alpha = 1\%$



(b) $\alpha = 5\%$



(c) $\alpha = 10\%$



(d) $\alpha = 20\%$

3 - Compute δq solution of

$$\left\{\begin{array}{ll} \nabla\cdot(\delta q(x)\nabla u_0(x)) = F(x), & x\in\Omega, \\ \delta q_i(x) = 0, & x\in\partial\Omega, \nabla u_0(x)\cdot\nu(x) > 0, \end{array}\right.$$

where u_0 is the initial condition of (1).

3 - Compute δq solution of

$$\begin{cases} \nabla \cdot (\delta q(x) \nabla u_0(x)) = F(x), & x \in \Omega, \\ \delta q_i(x) = 0, & x \in \partial\Omega, \nabla u_0(x) \cdot \nu(x) > 0, \end{cases}$$

where u_0 is the initial condition of (1).

Introduce the characteristics

$$\begin{cases} \dfrac{d}{dt} X(x,t) = \nabla u_0(X(x,t)), & t \in (0,T), \\ X(x,t) = x, & x \in \partial\Omega, \nabla u_0(x) \cdot \nu(x) > 0, \end{cases}$$

Then, $\alpha_x(t) = \delta q(X(x,t))$ solves an advection equation

$$\begin{cases} \dfrac{d}{dt} \alpha_x(t) + \alpha_x(t) \Delta u_0(X(x,t)) = F(X(x,t)), & t \in (0,T), \\ \alpha_x(0) = 0, & x \in \partial\Omega, \nabla u_0(x) \cdot \nu(x) > 0, \end{cases}$$

Numerical results
Numerical problems in 2D

4 - Finally, set

$$q^{k+1} = T_{\mathcal{V}}(q^k - \sum_i \delta q_i),$$

where $T_{\mathcal{V}}$ is the projection in the admissible set $\mathcal{V}$.

4 - Finally, set

$$q^{k+1} = T_{\mathcal{V}}(q^k - \sum_i \delta q_i),$$

where $T_{\mathcal{V}}$ is the projection in the admissible set $\mathcal{V}$.

Remember that

$$\mathcal{V} = \{p \in C^1(\overline{\Omega}), \|p\|_{C^1(\overline{\Omega})} \leq m, p = a \text{ on } \partial\Omega,$$

$$0 < a_0 \leq p(x) \leq a_1 \text{ and } p(x) + \frac{1}{2}\nabla p(x) \cdot (x - x_0) \geq \delta > 0 \text{ in } \Omega\}$$

and the associated norm is

$$\|p\|_s^2 = \int_\Omega e^{2s\varphi(0)} \left(s^2|\nabla p|^2 + s^4|p|^2 + |\nabla\left(\nabla \cdot (p\nabla u_0)\right)|^2\right) dx.$$

Conclusion

Would it be possible to design a real imaging system using this strategy ?

Conclusion

Would it be possible to design a real imaging system using this strategy ?

- It could use acoustic, elastic, heat or electromagnetic waves.
- But it should be working in the time domain !

Conclusion

Would it be possible to design a real imaging system using this strategy ?

- It could use acoustic, elastic, heat or electromagnetic waves.
- But it should be working in the time domain !
- It should be able to measure precisely the flux of the solution in a part of the boundary during a small time interval.

Conclusion

Would it be possible to design a real imaging system using this strategy ?

- It could use acoustic, elastic, heat or electromagnetic waves.
- But it should be working in the time domain !
- It should be able to measure precisely the flux of the solution in a part of the boundary during a small time interval.
- How could we in practice impose the condition on u_0 inside the domain without being invasive ?

$$|\nabla u_0 \cdot (x - x_0)| > 0 \text{ in } \Omega.$$

Conclusion

Would it be possible to design a real imaging system using this strategy ?

- It could use acoustic, elastic, heat or electromagnetic waves.
- But it should be working in the time domain !
- It should be able to measure precisely the flux of the solution in a part of the boundary during a small time interval.
- How could we in practice impose the condition on u_0 inside the domain without being invasive ?

$$|\nabla u_0 \cdot (x - x_0)| > 0 \text{ in } \Omega.$$

⋆ ⋆ ⋆