Problèmes de contrôle liés aux mouvements de foules

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Contrôle, Problème inverse et Applications
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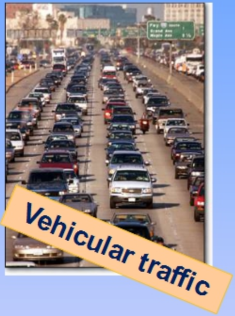
Vehicular traffic


HOTEL
Crowd dynamics


Networked robots


Animal groups

Can we apply a control ?

Outline

1. **Framework**
2. **Controllability**
3. **Minimal time**
4. **Numerical simulation**
5. **Perspectives**

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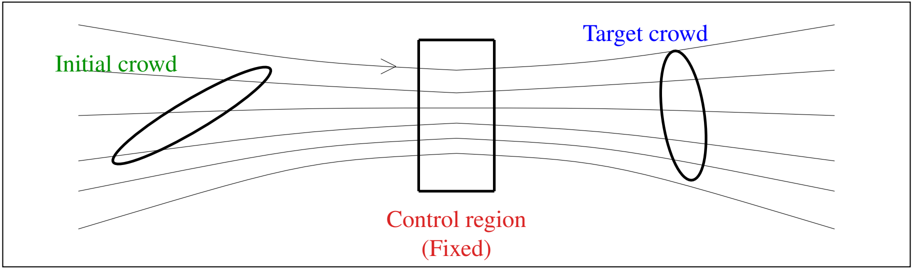
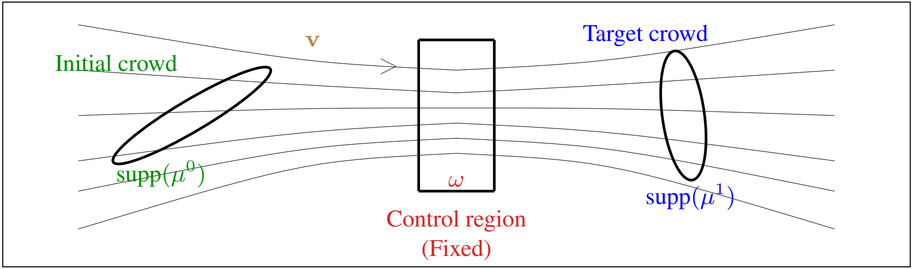
Problematic



Framework : Model

Problematic



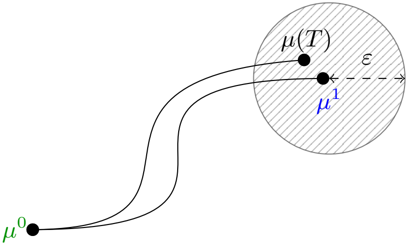
Model

Search u such that

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(0) = \mu^0, \ \mu(T) \approx \mu^1 & \text{in } \mathbb{R}^d, \end{cases} \tag{1}$$

where

- μ : population density
- $\mathbf{v}$: population velocity
- $\mathbb{1}_\omega(x)u(x,t)$: control



Problematic

We search u, called control,
such that the solution y to system

$$\begin{cases} \partial_t \mu + \nabla \cdot ((\mathbf{v} + \mathbb{1}_\omega u)\mu) = 0 \\ \mu(0) = \mu^0 \end{cases}$$

satisfies :

- μ near a given target at time T

$$\forall \varepsilon > 0, \mu^0, \mu^1, \ \exists\, u \quad \text{s.t.} \quad d(\mu(T), \mu^1) \leqslant \varepsilon.$$

➥ **Approximate controllability**

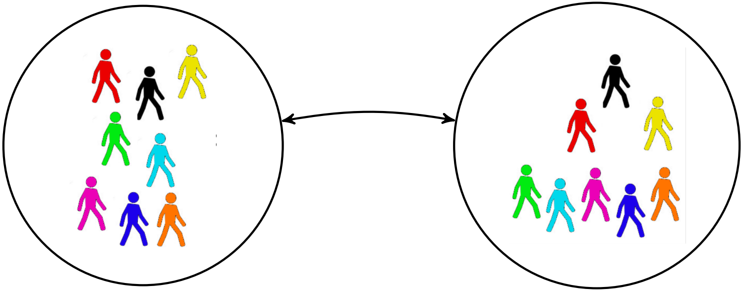
- y reach a target at time T

$$\forall \mu^0, \mu^1, \ \exists\, u \text{ s.t. } \mu(T) = \mu^1.$$

➥ **Exact controllability**

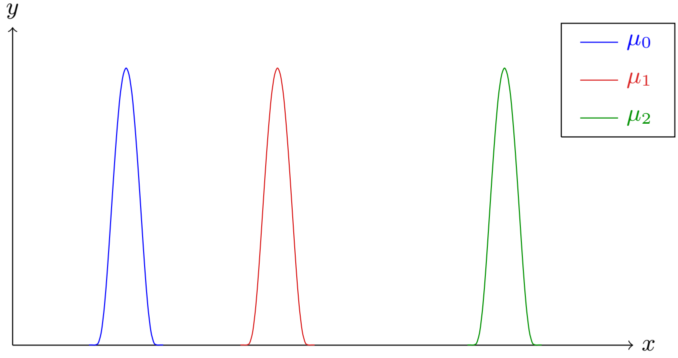
We call **minimal time** the infimum of T for which the approximate/exact controllability holds

Distance between two crowds



Framework : Distance ?

If we represent the population by a **density compactly supported** :

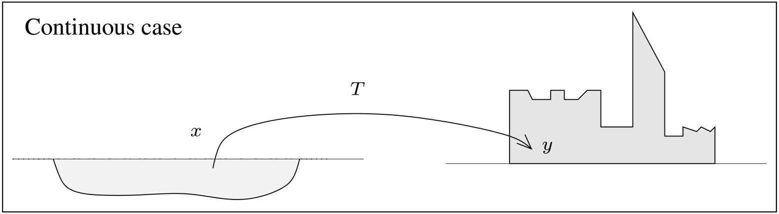


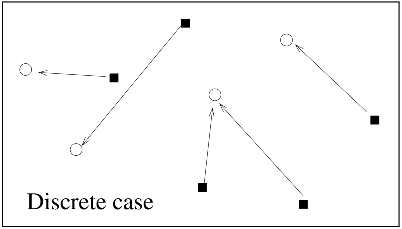
$$\|\mu_0 - \mu_1\|_{L^p} = \|\mu_0 - \mu_2\|_{L^p}$$

The L^p distance is not a good distance for the crowds ! ! !

Monge problem (1781)

Distance : minimal cost to send a mass on an other.





Framework : Wasserstein distance

We denote by :

- Γ the set composed with the Borel applications $\gamma : \mathbb{R}^d \to \mathbb{R}^d$.
- $\mathcal{P}_c(\mathbb{R}^d)$ the set of probability measures which are compactly supported.
- $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the set of measures which are compactly supported and absolute continuously with respect to the Lebesgue measure.

Definition

Let $\gamma \in \Gamma$ and $\mu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. **Push-forward** $\gamma\#\mu$ of μ :

$$(\gamma\#\mu)(E) := \mu(\gamma^{-1}(E)),$$

for all $E \subset \mathbb{R}^d$ such that $\gamma^{-1}(E)$ is μ-measurable.

Definition

Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. **Wasserstein distance** between μ and ν :

$$W_p(\mu, \nu) = \min_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma\#\mu = \nu \right\}.$$

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Framework : Wellposedness

For $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu,\nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} d\pi(x,\cdot) = d\mu(x) \text{ and } \int_{\mathbb{R}^d} d\pi(\cdot,y) = d\nu(y).$$

Definition

Let $p \in [1,\infty)$ and $\mu,\nu \in \mathcal{P}_c(\mathbb{R}^d)$. **Wasserstein distance** between μ and ν :

$$W_p(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left\{ \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\pi \right)^{1/p} \right\}$$

Framework : Wellposedness

Definition

We define the **flow** associated to $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ as $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t), t \geqslant 0, \\ x(0) = x^0. \end{cases}$$

Theorem (Cf Villani 2003)

Let $T > 0$, $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and w a velocity field uniformly bounded, **Lipschitz** in space and measurable in time. Then

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases}$$

admits a unique solution μ in $\mathcal{C}^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$.
Moreover,

$$\mu(t) = \Phi_t^w \# \mu^0,$$

for all $t \in [0, T]$.

The continuity equation can be used to reformulate the Wasserstein distance :

Theorem (Benamou-Brenier 00')

Let μ^0, μ^1 be two measures compactly supported with the same total mass. Then

$$W_2(\mu^0, \mu^1) = \min_{(\mu,v)} \left\{ \left(\int_0^1 \int_{\mathbb{R}^d} |v(t)|^2 d\mu(t) dt \right)^{1/2} : \partial_t \mu + \nabla \cdot (v\mu) = 0,\ \mu(0) = \mu^0,\ \mu(1) = \mu^1 \right\},$$

where v is a Borel vector field $v : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$.

Remark : For the minimisers v and the initial data μ^0, the solution of the continuity equation is not necessary unique.

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Existing results

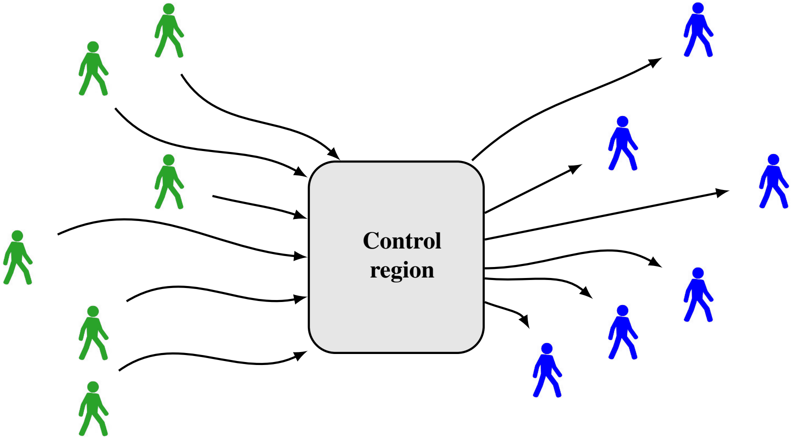
- Optimal control
  - ➥ Colombo-Herty-Mercier 11', Fornasier-Piccoli-Rossi 14'
- Approximate alignment of the Cucker-Smale model
  - ➥ Piccoli-Rossi-Trélat 15'
- Lyapunov-like stabilisation
  - ➥ Caponigro-Piccoli-Rossi-Trélat, 17'

Controllability : Geometrical condition

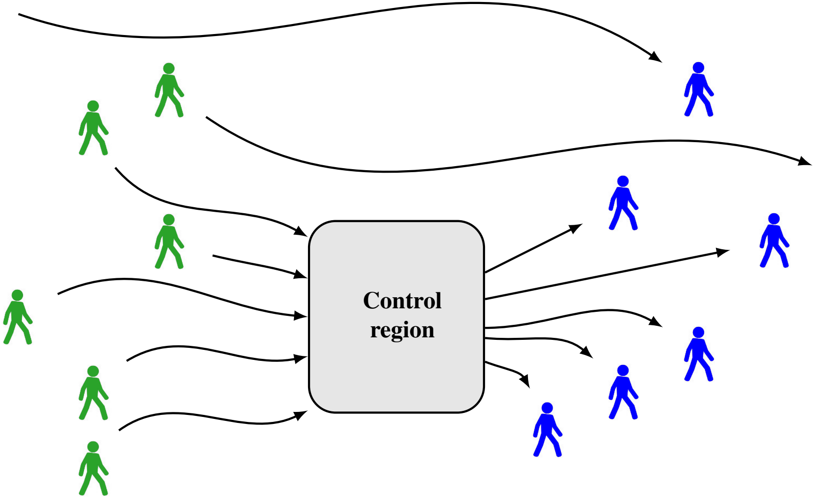
Geometrical condition

(i) $\forall x^0 \in \text{supp}(\mu^0), \exists\, t^0 \in (0,T) : \Phi^v_{t^0}(x^0) \in \omega.$

(ii) $\forall x^1 \in \text{supp}(\mu^1), \exists\, t^1 \in (0,T) : \Phi^v_{-t^1}(x^1) \in \omega.$



Geometric condition does not hold !



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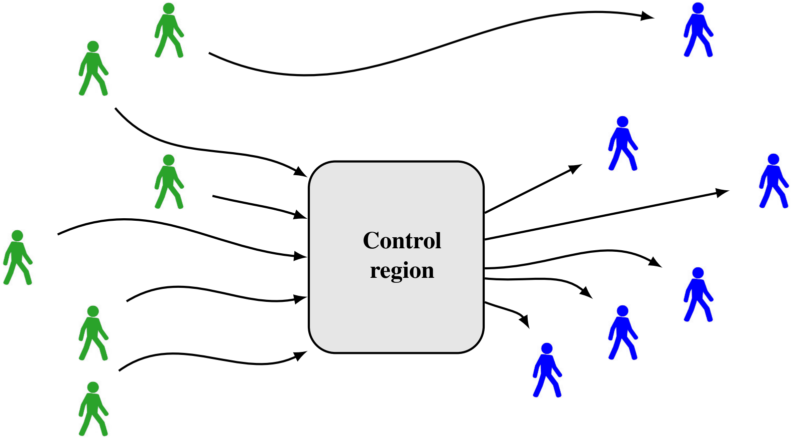


Approximate controllability

Geometrical condition

(i) $\forall x^0 \in \text{supp}(\mu^0), \exists\, t^0 \in (0,T) : \Phi^v_{t^0}(x^0) \in \omega$.

(ii) $\forall x^1 \in \text{supp}(\mu^1), \exists\, t^1 \in (0,T) : \Phi^v_{-t^1}(x^1) \in \omega$.

Theorem (D.-Morancey-Rossi 2017)

Let $\mu^0,\ \mu^1 \in \mathcal{P}^{ac}_c(\mathbb{R}^d)$. Assume the Geometrical Condition.
System (1) is **approximately controllable** with a Lipschitz control.

Approximate controllability in a square

To simplify, we suppose that dim $= 2$.
Let $S \subset\subset \omega$. Assume that

$$\mathrm{supp}(\mu^0),\ \mathrm{supp}(\mu^1) \subset S.$$

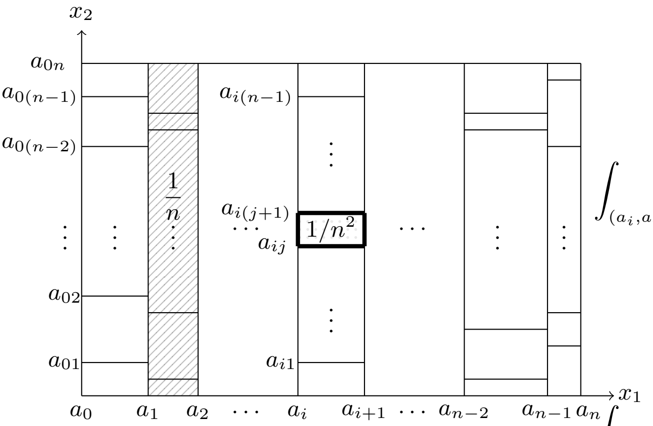
Goal : Find u such that the solution to

$$\partial_t \mu + \mathrm{div}(u\mu) = 0,$$

satisfies $\mathrm{supp}(\mu(t)) \subset S$ and

$$W_2(\mu^1, \mu(T)) \leqslant \varepsilon.$$

Discretization following the mass of μ^0 and μ^1



$$\int_{(a_i,a_{i+1})\times\mathbb{R}} d\mu^0 = \frac{1}{n}$$

$$\int_{(a_i,a_{i+1})\times(a_{ij},a_{i+1,j})} d\mu^0 = \frac{1}{n^2}$$

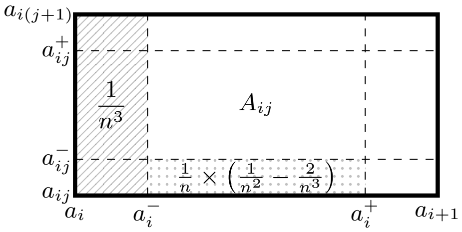
$$\int_{(b_i,b_{i+1})\times\mathbb{R}} d\mu^1 = \frac{1}{n}$$

$$\int_{(a_i,a_{i+1})\times(a_{ij},a_{i+1,j})} d\mu^0 = \frac{1}{n^2}$$

Center of the cells

$$\int_{(a_i,a_i^-)\times(a_{ij},a_{i(j+1)})} d\mu^0 = \int_{(a_i^+,a_{i+1})\times(a_{ij},a_{i(j+1)})} d\mu^0 = \frac{1}{n^3}$$

$$\int_{(a_i^-,a_i^+)\times(a_{ij},a_{ij}^-)} d\mu^0 = \int_{(a_i^-,a_i^+)\times(a_{ij}^+,a_{i(j+1)})} d\mu^0 = \frac{1}{n} \times \left(\frac{1}{n^2} - \frac{2}{n^3}\right)$$



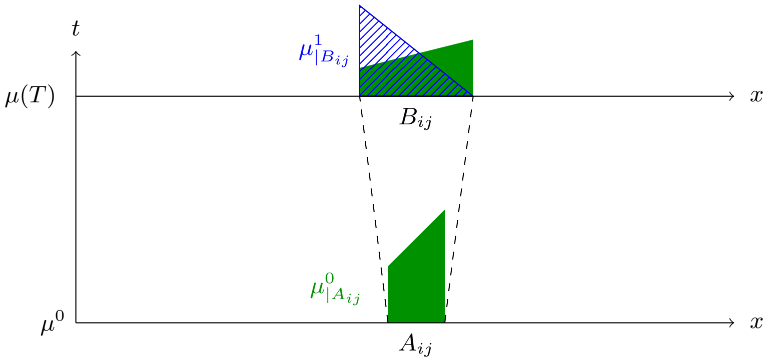
$$\int_{A_{ij}} d\mu^0(x) = \frac{(n-2)^2}{n^4}$$

We define similarly B_{ij}.

Remark : We do not control the mass outside A_{ij}.

Construction of the flow

We send linearly $\mu^0_{|A_{ij}}$ on B_{ij} :



Remark : $|A_{ij}| \underset{n\to\infty}{\longrightarrow} 0.$

Construction of the flow

For all $x^0 = (x_1^0, x_2^0) \in A_{ij}$, we build the flow

$$\Phi_t^u(x^0) := \begin{pmatrix} \dfrac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \dfrac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t) \\ \dfrac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \dfrac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t) \end{pmatrix},$$

where

$$\begin{cases} c_i^-(t) = (b_i^- - a_i^-)t + a_i^-, \\ c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+, \\ c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^-, \\ c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+. \end{cases}$$

Thus

$$\Phi_T^u(A_{ij}) = B_{ij}.$$

Remark : We take a $\mathcal{C}^\infty$ extension outside A_{ij}.

Construction of the control

The corresponding velocity is given by

$$\begin{cases} u_1(x,t) = \alpha_i(t)x_1 + \beta_i(t), \\ u_2(x,t) = \alpha_{ij}(t)x_2 + \beta_{ij}(t), \end{cases}$$

where

$$\begin{cases} \alpha_i(t) = \dfrac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, \quad \beta_i(t) = \dfrac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\ \alpha_{ij}(t) = \dfrac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, \quad \beta_{ij}(t) = \dfrac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}. \end{cases}$$

Estimation of the distance

Define

$$R := (0,1)^2 \setminus \bigcup_{ij} B_{ij},$$

We have

$$W_1(\mu^1, \mu(T)) \leqslant \sum_{i,j=1}^{n} \underbrace{W_1(\mu^1 \times \mathbb{1}_{B_{ij}}, \mu(T) \times \mathbb{1}_{B_{ij}})}_{\text{Included in } B_{ij}} + \underbrace{W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R)}_{\substack{\text{Small mass} \\ \text{No control}}}.$$

Controllability : Sketch of proof

There exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\overline{\gamma} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\gamma_{ij}\#(\mu^1 \times \mathbb{1}_{B_{ij}}) = \mu(T) \times \mathbb{1}_{B_{ij}} \qquad \text{and} \qquad \overline{\gamma}\#(\mu^1 \times \mathbb{1}_R) = \mu(T) \times \mathbb{1}_R.$$

We have

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_{B_{ij}}, \mu(T) \times \mathbb{1}_{B_{ij}}) &= \int_{B_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ &\leqslant [(b_i^+ - b_i^-) + (b_{ij}^+ - b_{ij}^-)] \int_{B_{ij}} d\mu^1(x) \\ &\leqslant (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4}. \end{aligned}$$

and

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R) &\leqslant \int_R |x - \overline{\gamma}(x)| d\mu^1(x) \\ &\leqslant \operatorname{diam}(S) \left(1 - \frac{(n-2)^2}{n^2}\right) = 4\sqrt{2}\frac{n-1}{n^2}. \end{aligned}$$

Thus

$$W_1(\mu^1, \mu(T)) \underset{n\to\infty}{\longrightarrow} 0.$$

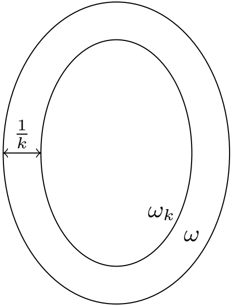
Concentration of the mass of μ^0 in ω

Let $\{u_k\}_{k\in\mathbb{N}^*}$ be a sequence of $\mathcal{C}^\infty$-functions such that

$$\begin{cases} |u_k| \leqslant |v|, & \\ u_k = 0 & \text{in } \omega^c, \\ u_k = -v & \text{in } \omega_k \end{cases}$$

with $\omega_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega^c) > 1/k\}$.
Then for k and T large enough,

$$\text{supp}(\mu(T_1)) \subset \omega.$$



Concentration of the mass of μ^0 in a square

Let $S \subset\subset \omega$ be a square.
From Coron 07', there exists a function $\eta \in \mathcal{C}^2(\overline{\omega})$ satisfying

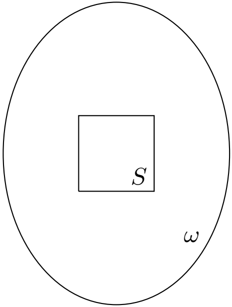
$$\kappa_0 \leqslant |\nabla \eta| \leqslant \kappa_1 \text{ in } \omega \backslash S, \quad \eta > 0 \text{ in } \omega \quad \text{and} \quad \eta = 0 \text{ on } \partial\omega,$$

with $\kappa_0, \kappa_1 > 0$. For $k \in \mathbb{N}^*$, we denote by

$$u_k := \begin{cases} k\nabla\eta & \text{in } \omega, \\ 0 & \text{in } \omega^c. \end{cases}$$

If $\text{supp}(\mu^0) \subset \omega$, then for k large enough

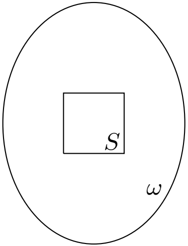
$$\text{supp}(\mu(T_2)) \subset S.$$

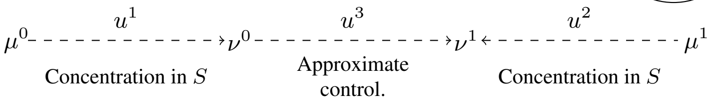


Global strategy

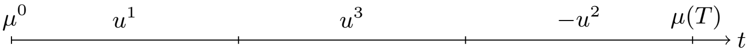
(i) Step 1 : We send μ^0 to ν^0 supported in a square $S \subset\subset \omega$.
We send μ^1 to ν^1 supported in a square $S \subset\subset \omega$.

(ii) Step 2 : We send approximately ν^0 to ν^1.





Final computation :



Exact controllability

Remark

- With a Lipschitz velocity field, the flow is a homeomorphism, then $\text{supp}(\mu^0)$ and $\text{supp}(\mu^1)$ have to be homeomorph. In particular, we can not separate a mass in two parts or bring together to different masses.
- Even with a BV velocity field we cannot bring together to different masses.
- For a Borel velocity field, the solution is not garanted unique.

Theorem (D.-Morancey-Rossi 2017)

Let $\mu^0,\ \mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$. Assume the Geometrical Condition.

- System (1) is **not always exactly contr.** with a Lipschitz control (or BV).
- There exists a couple (μ, u) solution of system (1) such that $\mu(T) = \mu^1$ with a Borel control.

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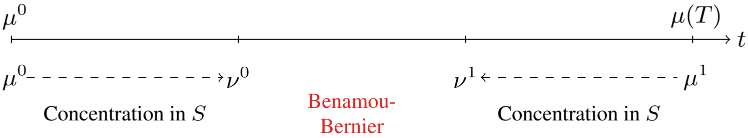


$$W_2(\nu^0,\nu^1) = \min_{(\mu,v)\in\mathcal{B}} \left\{ \left(\int_0^1 \int_{\mathbb{R}^d} |v(t)|^2 d\mu(t) dt \right)^{1/2} : \partial_t\mu + \nabla\cdot(v\mu) = 0,\ \mu(0) = \nu^0,\ \mu(1) = \nu^1 \right\},$$

Outline

Minimal time : Discrete case

Let

$$\mu^0 := \sum_{i=1}^{n} \frac{1}{n}\delta_{x_i^0} \quad \text{and} \quad \mu^1 := \sum_{i=1}^{n} \frac{1}{n}\delta_{x_i^1} \quad (x_i^k \neq x_j^k).$$

System (1) is equivalent to

$$\begin{cases} \dot{x}(t) = (v + \mathbb{1}_\omega u)(x(t), t), \ t \geqslant 0, \\ x(0) = x_i^0. \end{cases}$$

Define

$$\begin{cases} t_i^0 := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x_i^0) \in \omega\}, \\ t_i^1 := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x_i^1) \in \omega\}, \\ y_i^0 := \Phi_{t_i^0}^v(x_i^0), \\ y_i^1 := \Phi_{-t_i^1}^v(x_i^1). \end{cases}$$

Theorem (D.-Morancey-Rossi 2017)

Assume the Geometrical Condition and ω convex.
Minimal time to steer μ^0 to μ^1 :

$$T_0 = \min_{\sigma \in S_n} \max_{i \in \{1,\ldots,n\}} |t_i^0 + t_{\sigma(i)}^1|.$$

Computation of the optimal permutation

Define

$$K_{ij} := \begin{cases} \|(y_i^0, t_i^0) - (y_j^1, T - t_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } t_i^0 < T - t_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimisation problem

$$\inf_{\pi \in \mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij} \right\},$$

where $\mathcal{B}_n$ is the set of bistochastic matrices $\pi := (\pi_{ij})_{1 \leqslant i,j \leqslant n}$, *i.e.*

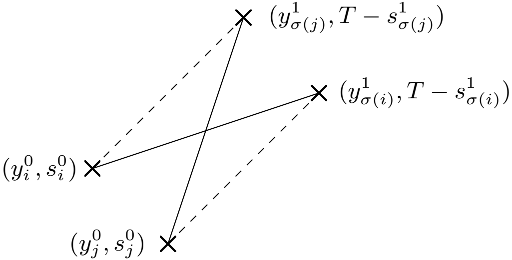
$$\sum_{i=1}^{n} \pi_{ij} = 1, \ \sum_{j=1}^{n} \pi_{ij} = 1, \ \pi_{ij} \geqslant 0.$$

The infimum is reached.

Since $\mathcal{B}_n$ is convex, there exists a minimum which is**a permutation matrix**.

No intersection of the trajectories

By contradiction : no intersection of the trajectories

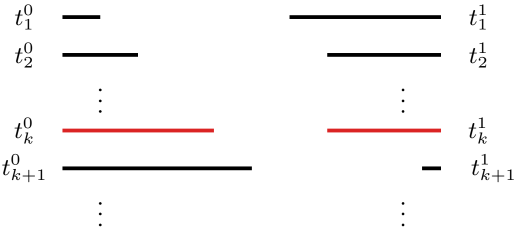


Computation of the optimal time

Corollary

Assume the Geometrical Condition and ω convex.
Assume the $\{t_i^0\}_{i\in\{1,\dots,n\}}$ et $\{t_i^1\}_{i\in\{1,\dots,n\}}$ are increasingly and decreasingly ordered respectively, then

$$T_0 := \max_{i\in\{1,\dots,n\}} \{t_i^0 + t_i^1\}.$$



Minimal time : continuous case

Consider for all $t \geq 0$

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \text{Supp}(\mu^0) : t^0(x^0) \leqslant t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \text{Supp}(\mu^0) : t^1(x^1) \leqslant t\}), \end{cases}$$

where

$$\begin{cases} t^0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}, \\ t^1(x^1) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}. \end{cases}$$

We define for all $m \in [0, 1]$

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geqslant m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geqslant m\}. \end{cases}$$

Consider also

$$\begin{cases} T_0^* := \sup\{t^0(x^0) : x^0 \in \text{Supp}(\mu^0)\}, \\ T_1^* := \sup\{t^1(x^1) : x^1 \in \text{Supp}(\mu^1)\}, \\ T_2^* := \max\{T_0^*, T_1^*\}. \end{cases}$$

Minimal time : continuous case

Theorem (D.-Morancey-Rossi 17')

Let $\mu^0, \ \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Assume the Geometrical Condition and ω convex.

$$T_0 := \max_{m\in[0,1]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-m)\}.$$

Then

(i) For all $T > T_0$, System (1) is **approximately controllable** from μ^0 to μ^1 at time T with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

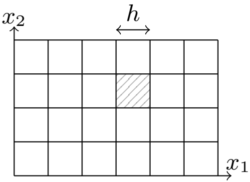
(ii) For all $T \in (T_2^*, T_0)$, System (1) is **not approximately controllable** from μ^0 to μ^1.

Remark

If $T \in (0, T_2^*)$, then we cannot act on all the measure,
but the measure can reach alone the desired configuration.

Step 1 : Uniform discretization

We discretize uniformly supp(μ^0) :



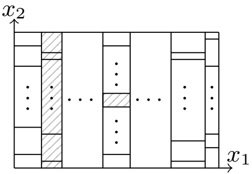
We take h small enough such that the cells K_h satisfies

$$\Phi^v_{t^*}(K_h) \subset\subset \omega,$$

for a $t^* > 0$.

Step 2 : Discretization following the mass

We discretize each cell following the mass



- Each cell will the same mass $1/n$.
- The rest will be negligible.

Step 3 : Association of the masses

We use the results of the discrete case to associate the masses

- We approximate the measure by a sum of Dirac.
- We control this discrete approximation.
- We follow the trajectory of the Dirac masses, up to a concentration of the mass.

Outline

1. **Framework**
2. **Controllability**
3. **Minimal time**
4. **Numerical simulation**
5. **Perspectives**

Algorithm 1

Step 1 : Discretisation of μ^0 and μ^1
(i) Construction of the uniform mesh
(ii) Computation of the cell A_{ij} following the mass
Step 2 : Computation of the minimal time

$$T_0 := \max_{i\in\{1,\ldots,n\}} \{t^0_{\sigma_0(i)} + t^1_{\sigma_1(i)}\}$$

where $(t^0_{\sigma_0(i)})_i$ increasing and $(t^1_{\sigma_1(i)})_i$ decreasing.
Step 4 : Computation of the optimal permutation

$$\inf_{\pi\in\mathcal{B}_n} \left\{ \frac{1}{n} \sum_{i,j=1}^{n} K_{ij}\pi_{ij} \right\}$$

Step 5 : Concentration of the masses (if necessary)
Step 6 : Final computation

Numerical simulation

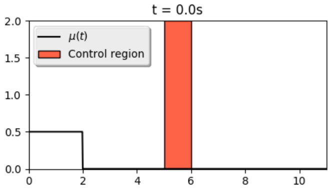
Consider the initial data μ^0 and the target μ^1 defined by

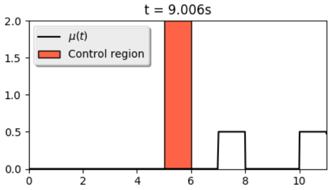
$$\mu^0 := \begin{cases} 0.5 & \text{if } x \in (0,2), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu^1 := \begin{cases} 0.5 & \text{if } x \in (7,8) \cup (10,11), \\ 0 & \text{otherwise.} \end{cases}$$

We fix the velocity field $v := 1$ and the control region $\omega := (5,6)$.
The **minimal time** is equal to : $8s$.





Numerical simulation

Numerical simulation

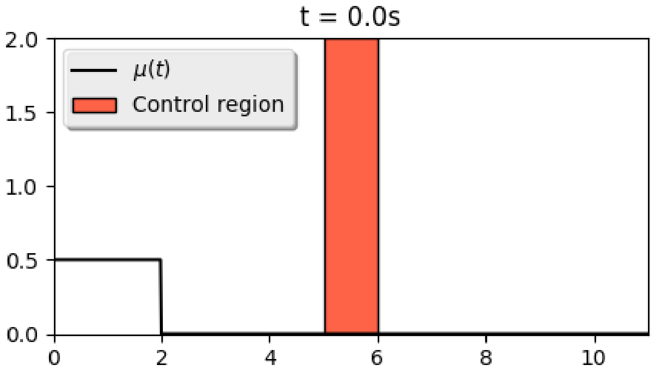


FIGURE – Solution at time $t = 0.0$.

Numerical simulation

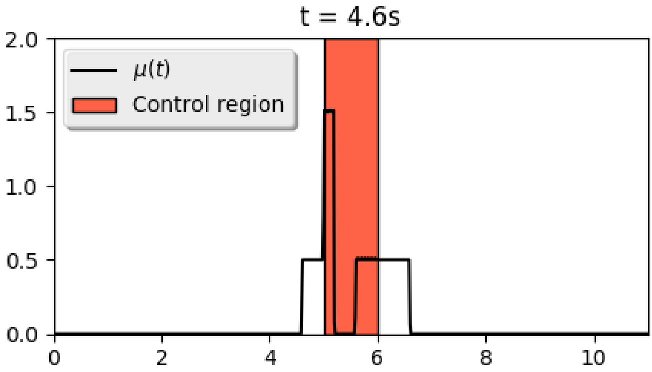


FIGURE – Solution at time $t = 4.6$.

Numerical simulation

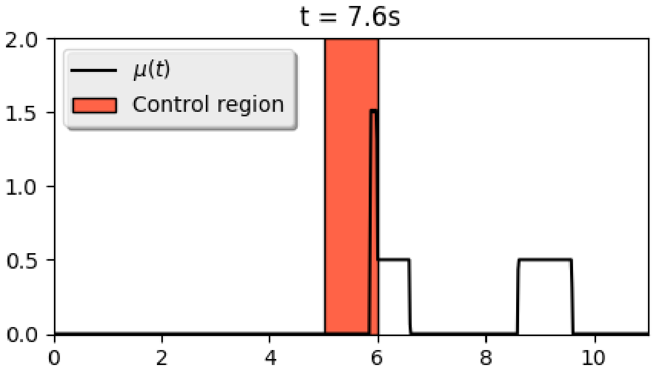


FIGURE – Solution at time $t = 7.6$.

Numerical simulation

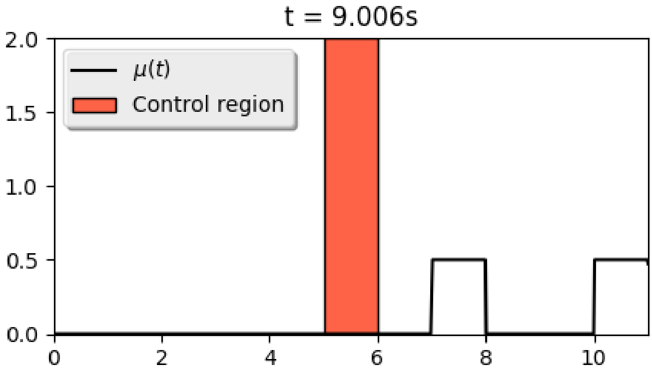


FIGURE – Solution at time $t = 9.0$.

Outline

Perspectives

- Control with interactions
- Optimal control with interactions
- Pontryagin Maximum Principle

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} V(x_i - x_j) \quad \underset{N\to\infty}{\longrightarrow} \quad \partial_t \mu + \operatorname{div}(v[\mu]\mu) = 0$$

Flock of starlings

Thank you for your attention !