On a variant of Tykhonov regularization in optimal control under PDEs

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Motivation

Optimal control problem (too formal):

$$\text{Minimize} \quad I[\mathbf{u},\mathbf{v}] : \mathcal{A} \to \mathbb{R} \quad \text{subject to } \mathbf{F}(\mathbf{u},\mathbf{v}) = \mathbf{0}$$

$\mathcal{A}$: set of feasible pairs, $\mathbf{F}$: differential operator,

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Typical situation: for $\overline{u} \in L^2(\Omega)$ given,

$$\text{Minimize in } (u,v) \in H_0^1(\Omega) \times L^2(\Omega) : \quad I[u,v] = \int_\Omega \left(\frac{1}{2} |u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 \right) d\mathbf{x}$$

where

$$-\operatorname{div}(\nabla u) + u = v \text{ in } \Omega.$$

Motivation (cont'd)

Tikhonov regularization: for $\overline{u} \in L^2(\Omega)$, and $\mu > 0$, given

$$\text{Minimize in } (u, v): \quad I[u, v] = \int_\Omega \left(\frac{1}{2} |u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\mu}{2} |v(\mathbf{x})|^2 \right) d\mathbf{x}$$

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where $(u, v) \in H_0^1(\Omega) \times L^2(\Omega)$, and

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Two main sources of concern that push the analysis to more complicated frameworks:

1. $\mathbf{F}(\mathbf{u}, \mathbf{v}) = \mathbf{0}$ is non-linear in $\mathbf{u}$ for given $\mathbf{v}$;
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E. Casas, I. Lasiecka, P. Neittaanmaki, J. P. Raymond, J. Sprekels, D. Tiba, R. Triggiani, F. Tröltzsch, . . . J. L. LIONS.

Two paradigmatic situations

1. Non-linearities:

$$\text{Minimize in } v \in L^2(\Omega): \quad I[v] = \int_\Omega \left(\frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\mu}{2}|v(\mathbf{x})|^2 \right) d\mathbf{x}$$

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$$-\operatorname{div}(\nabla u) + \phi(u) = v \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega.$$

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2. Pointwise constraints:

Minimize in $(u, v) \in H^1_0(\Omega) \times L^\infty(\Omega)$: $\quad I[u, v] = \int_\Omega \frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 \, d\mathbf{x}$

where

$$-\mathrm{div}\,(\nabla u) + u = v \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega,$$
$$u(\mathbf{x}) \le 0, \quad v_-(\mathbf{x}) \le v(\mathbf{x}) \le v_+(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in \Omega,$$

Idea

Allow for a bit of flexibility so that both variables u and v can be given freely in their respective sets of feasibility, either $H_0^1(\Omega) \times L^2(\Omega)$, or

$$\{u \in H_0^1(\Omega) : v \leq 0\} \times \{v \in L^\infty(\Omega) : v_- \leq v \leq v_+\}.$$

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Introduce a "defect" or residual function $w \in H_0^1(\Omega)$

$$-\text{div}\,(\nabla u + \nabla w) + \phi(u) = v \text{ in } \Omega, \quad -\text{div}\,(\nabla u + \nabla w) + u = v \text{ in } \Omega.$$

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To account for a small size of this defect w, we change the cost functional

$$I[u, v] = \int_\Omega \left(\frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\mu}{2}|v(\mathbf{x})|^2 + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2\right) d\mathbf{x},$$

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and take λ, large.

Our job

Problem 1

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$$\text{under} \qquad (u,v) \in H_0^1(\Omega) \times L^2(\Omega),$$

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Problem 2

$$\text{Minimize}: \quad I[u,v] = \int_\Omega \left(\frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2 \right) \, d\mathbf{x}$$

under

$$(u,v) \in \mathcal{A} = \{(u,v) \in H_0^1(\Omega) \times L^2(\Omega) : u \le 0, v_- \le v \le v_+\},$$

$$-\text{div}\,(\nabla u + \nabla w) + u = v \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Basic issues

1. Existence of optimal solutions.
2. (First-order) Optimality conditions.
3. (Practical) Numerical approximation.
4. Asymptotic behaviour as $\lambda \to +\infty$.

Unconstrained situation (Problem 1)

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The other side of the spectrum

$$\text{Minimize in } (u,v) : \quad \int_\Omega \left(\psi(u(\mathbf{x}), v(\mathbf{x}), \mathbf{x}) + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2 \right) d\mathbf{x}$$

subject to $(u,v) \in H_0^1(\Omega) \times L^2(\Omega)$, and

$$-\text{div}\,[\nabla u(\mathbf{x}) + \nabla w(\mathbf{x})] = v(\mathbf{x}) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega,$$

Existence (for the first situation)

Minimizing sequence $\{(u_j, v_j)\}$:

$$u_j \rightharpoonup u\,(L^2(\Omega)), \quad v_j \rightharpoonup v\,(L^2(\Omega)), \quad w_j \rightharpoonup w\,(H_0^1(\Omega)),$$
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Would like to show that, in fact,

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Weak formulation of state law:

$$\int_\Omega (\nabla u_j \cdot \nabla\theta + \nabla w_j \cdot \nabla\theta + \phi(u_j)\theta - v_j\theta)\; d\mathbf{x} = 0.$$

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Limit passage: necessary to strengthened $u_j \rightharpoonup u$ in $H_0^1(\Omega)$.

Proof (cont'd)

Use $\theta = u_j$:

$$\int_\Omega \left(|\nabla u_j|^2 + \nabla w_j \cdot \nabla u_j + \phi(u_j) u_j - v_j u_j \right) \, d\mathbf{x} = 0.$$

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Need a sign condition for nonlinear term:

$$\int_\Omega \phi(u_j)u_j\, d\mathbf{x} \geq 0 : \quad \|\nabla u_j\|^2_{L^2} \leq \|\nabla w_j\|_{L^2}\|\nabla u_j\|_{L^2} + \|v_j\|_{L^2}\|u_j\|_{L^2}.$$

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Theorem

If ϕ has $N/N-2$ growth at infinity, and $\phi(u)u \geq 0$, then there is a unique optimal pair (u, v) for Problem 1.

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Use capital letters to indicate feasible variations of the various functions involved.

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Similar computations for the functional:

$$\int_\Omega \left(\frac{1}{2}|u + \epsilon U - \overline{u}|^2 + \frac{\mu}{2}|v + \epsilon V|^2 + \frac{\lambda}{2}|\nabla w + \epsilon\nabla W|^2\right)\,d\mathbf{x}$$

$$\langle I'[u, v], (U, V)\rangle = \frac{d}{d\epsilon} I((u, v) + \epsilon(U, V))\Big|_{\epsilon=0}$$

$$= \int_\Omega [(u - \overline{u})U + \mu vV + \lambda\nabla w \cdot \nabla W]\,d\mathbf{x}$$

$$w : \quad -\text{div}\,[\nabla u + \nabla w] + \phi(u) = v \text{ in } \Omega,$$

$$W : \quad -\text{div}\,[\nabla U + \nabla W] + \phi'(u)U = V \text{ in } \Omega.$$

Optimality (cont'd)

Use w as a test function in second equation:

$$\int_\Omega \nabla w \cdot \nabla W \, d\mathbf{x} = \int_\Omega [Vw - \phi'(u)Uw - \nabla U \cdot \nabla w] \, d\mathbf{x};$$

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take this identity back to the derivative of the functional:

$$\langle I'[u,v],(U,V)\rangle = \int_\Omega [(u-\overline{u})U + \mu vV + \lambda(Vw - \phi'(u)Uw - \nabla U \cdot \nabla w)] \, d\mathbf{x}.$$

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Conclude: if pair $(u, v) \in H_0^1(\Omega) \times L^2(\Omega)$ is optimal, then previous integral must vanish for arbitrary $U \in H_0^1(\Omega)$ and $V \in L^2(\Omega)$.

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Conclude: if pair $(u,v) \in H^1_0(\Omega) \times L^2(\Omega)$ is optimal, then previous integral must vanish for arbitrary $U \in H^1_0(\Omega)$ and $V \in L^2(\Omega)$.

Theorem

If $(u,v) \in H^1_0(\Omega) \times L^2(\Omega)$ with defect $w \in H^1_0(\Omega)$ is optimal then

$$u - \lambda\phi'(u)w = \overline{u} - \lambda\Delta w \ \textit{in}\ \Omega,$$

$$\mu v + \lambda w = 0 \ \textit{in}\ \Omega.$$

Numerical approximation

Steepest descent directions U and V: solutions of variational problems

$$\int_\Omega \left[\frac{1}{2}|\nabla U|^2 + (u - \overline{u})U - \lambda\phi'(u)wU - \lambda\nabla w \cdot \nabla U\right] d\mathbf{x},$$

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Their respective solutions are easily found to be

$$-\text{div}\,[\nabla U - \lambda\nabla w] + (u - \overline{u}) - \lambda\phi'(u)w = 0 \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega,$$

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Choice of step size. For steepest descent direction (U, V), determine value of ϵ minimizing

$$\begin{aligned} g(\epsilon) =& I[(u,v) + \epsilon(U,V)] \\ =& \int_\Omega \left[\frac{1}{2}|u+\epsilon U - \overline{u}|^2 + \frac{\mu}{2}|v+\epsilon V|^2 + \frac{\lambda}{2}|\nabla w + \epsilon\nabla W|^2\right]\,d\mathbf{x}. \end{aligned}$$

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U_j : \quad -\operatorname{div}(\nabla U_j - \lambda \nabla w_j) + u_j - \overline{u} - \lambda \phi'(u_j) w_j = 0 \text{ in } \Omega, \quad U_j = 0 \text{ on } \partial\Omega \\
V_j : \quad V_j = -\mu v_j - \lambda w_j \text{ in } \Omega.
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$$V_j: \quad V_j = -\mu v_j - \lambda w_j \text{ in } \Omega.$$

  - Stopping criterium based on the smallness of the norm

$$\|(U_j, V_j)\|^2 = \int_\Omega \left(|\nabla U_j|^2 + V_j^2\right)\, d\mathbf{x}.$$

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$$\|(U_j, V_j)\|^2 = \int_\Omega \left(|\nabla U_j|^2 + V_j^2\right)\, d\mathbf{x}.$$

  - Solve the additional problem

$$W_j : \quad -\text{div}\,(\nabla U_j + \nabla W_j) + \phi'(u_j) U_j = V_j \text{ in } \Omega, \quad W_j = 0 \text{ on } \partial\Omega;$$

$$\text{step size:} \quad \epsilon_j = -\frac{\int_\Omega \left[(u_j - \overline{u}) U_j + \mu v_j V_j + \lambda \nabla w_j \cdot \nabla W_j\right]\, d\mathbf{x}}{\int_\Omega (U_j^2 + \mu V_j^2 + \lambda |\nabla W_j|^2)\, d\mathbf{x}}.$$

Iterative numerical procedure

- Initialization. Take $u_0 = v_0 = 0$, for instance.
- Main iterative step until convergence. Know (u_j, v_j).
  - Solve successively for

$$w_j : \quad -\mathrm{div}\,(\nabla u_j + \nabla w_j) + \phi(u_j) = v_j \text{ in } \Omega, \quad w_j = 0 \text{ on } \partial\Omega;$$
$$U_j : \quad -\mathrm{div}\,(\nabla U_j - \lambda \nabla w_j) + u_j - \overline{u} - \lambda \phi'(u_j) w_j = 0 \text{ in } \Omega, \quad U_j = 0 \text{ on } \partial\Omega;$$
$$V_j : \quad V_j = -\mu v_j - \lambda w_j \text{ in } \Omega.$$

  - Stopping criterium based on the smallness of the norm

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  - Update rule:

$$u_{j+1} = u_j + \epsilon_j U_j, \quad v_{j+1} = v_j + \epsilon_j V_j.$$

Some numerical simulations

$\Omega = (0,1) \times (0,1)$, $\lambda = 1$, $\phi(u) = -1$ (LINEAR CASE), $\overline{u}(x,y) = \min(x, 1-x)$, decreasing values for μ. FreeFem++ to solve PDEs involved.

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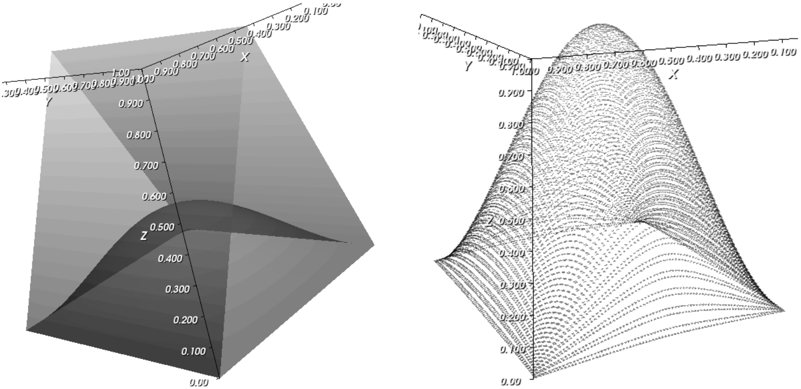


Figure: A comparison between the target and the optimal profile (left) for $\mu = 0.01$. The control v on the right.

Numerical simulations (cont'd)

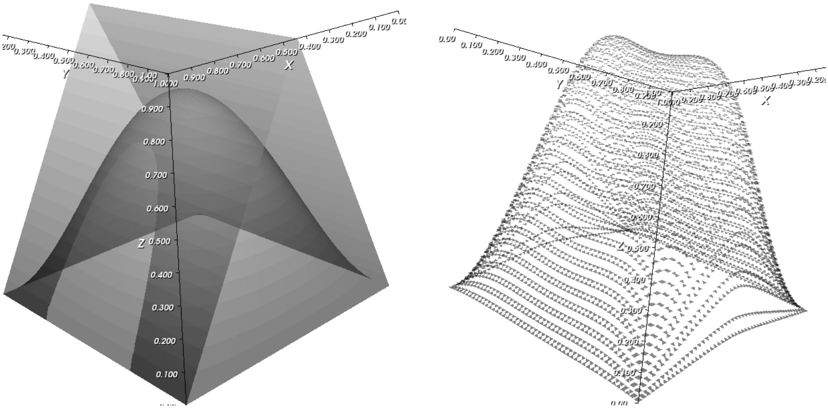


Figure: A comparison between the target and the optimal profile (left) for $\mu = 0.001$. The control v on the right.

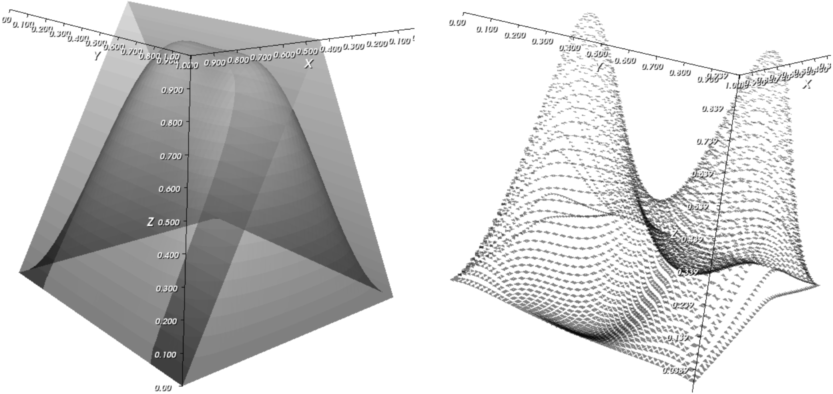


Figure: A comparison between the target and the optimal profile (left) for $\mu = 0.0001$. The control v on the right.

Numerical simulations (cont'd)

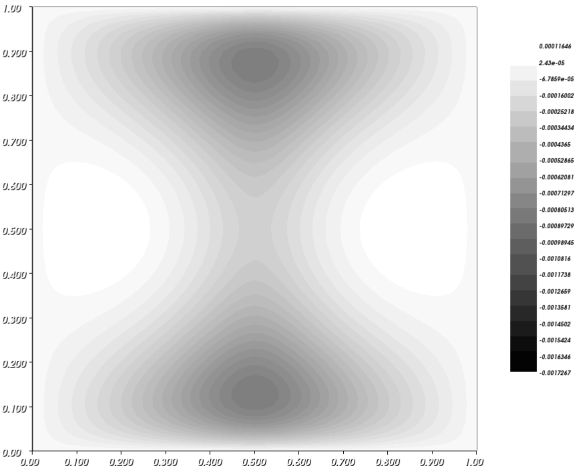


Figure: The contour lines for the residual for $\mu = 0.0001$. The size (L^2-norm of the gradient) is 0.0061237.

Non-linear example

Same data set except $\phi(u) = (u-2)^3$.

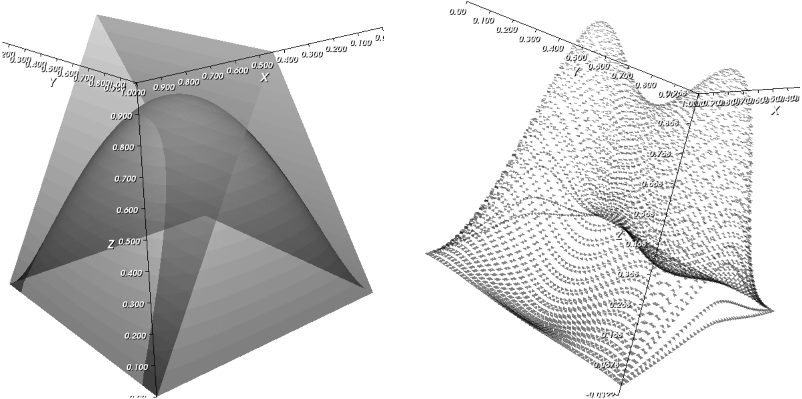
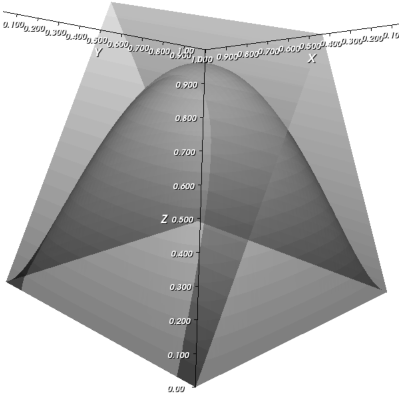


Figure: A comparison between the target and the optimal profile (left) for the non-linear problem and $\mu = 0.01$. The control v on the right. Error: 0.010073

Non-linear example (cont'd)



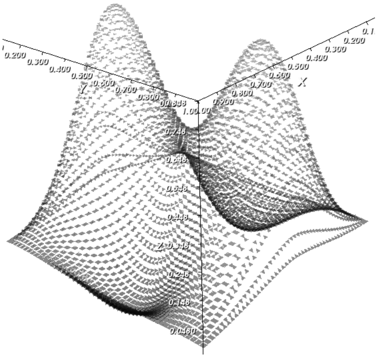


Figure: A comparison between the target and the optimal profile (left) for the non-linear problem and $\mu = 0.001$. The control v on the right. Error: 0.0083419

Non-linear example (cont'd)

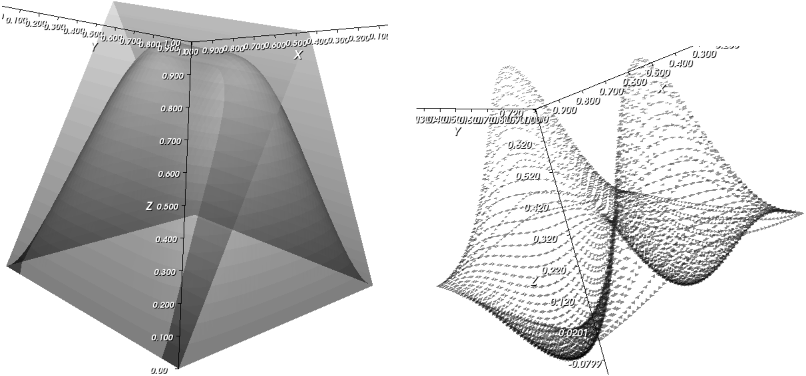


Figure: A comparison between the target and the optimal profile (left) for the non-linear problem and $\mu = 0.0001$. The control v on the right. Error: 0.00541891

Non-linear example (cont'd)

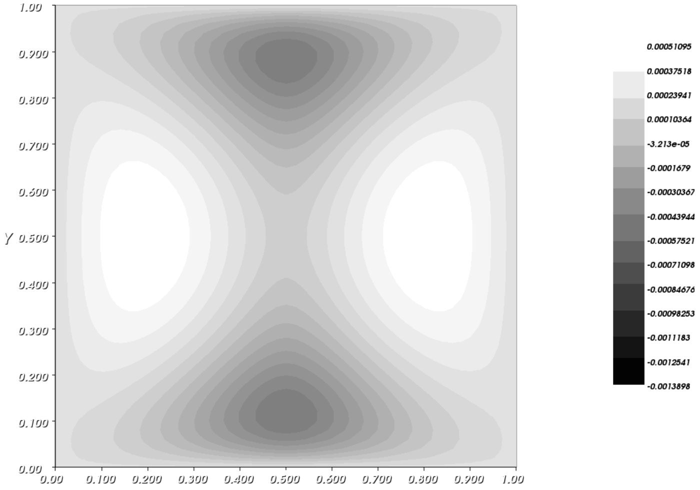


Figure: The contour lines for the residual for $\mu = 0.0001$.

State constraints (Problem 2)

$$\text{Minimize in } (u,v)\in\mathcal{A}: \quad \int_\Omega \left(\frac{1}{2}|u(\mathbf{x})-\overline{u}(\mathbf{x})|^2 + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2\right)\, d\mathbf{x},$$

$$-\text{div}\,[\nabla u(\mathbf{x}) + \nabla w(\mathbf{x})] = v(\mathbf{x}) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega,$$

$$\mathcal{A} = \{(u,v)\in H^1_0(\Omega)\times L^\infty(\Omega) : u(\mathbf{x})\le 0, v_-(\mathbf{x})\le v(\mathbf{x})\le v_+(\mathbf{x})\}.$$

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Existence result: even simpler (linear state equation, strict convexity of functional, uniform bounds for v, convexity of pointwise constraints, etc).

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There is a unique optimal pair $(u,v) \in \mathcal{A}$ for our problem.

Optimality: expressed as variational inequalities through perturbations of the form

$$u \mapsto u + \epsilon(U - u), \quad v \mapsto v + \epsilon(V - v)$$

for arbitrary pairs $(U,V) \in \mathcal{A}$, and demand one-sided conditions.

Optimality

$$\frac{d}{d\epsilon} I[(u,v)+\epsilon(U-u,V-v)]\Big|_{\epsilon=0} =$$

$$\int_\Omega [(u-\overline{u})(U-u)+\lambda((V-v)w-(\nabla U-\nabla u)\cdot\nabla w)]\,d\mathbf{x}\geq 0.$$

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Take $U=u$ (no perturbation for u). Then

$$V=\frac{v_+ + v_-}{2}+\operatorname{sgn} w\,\frac{v_+ - v_-}{2} \quad \text{implies} \quad \int_\Omega (V-v)w\,d\mathbf{x}\geq 0.$$

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Take $V = v$ (no perturbation for v).

Lemma

For given u, the unique solution U of the obstacle problem

$$\textit{Min. in } U \in H_0^1(\Omega): \quad \int_\Omega \left(\frac{1}{2} |\nabla u - \nabla U|^2 + (u - \overline{u})U - \lambda \nabla w \cdot \nabla U \right) d\mathbf{x}$$

under $U \leq 0$, is a descent direction of I at u.

Simple proof

Obstacle problem for U admits a unique solution: standard.

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Derivative of I at (u, v) in the direction $(U - u, \mathbf{0})$:

$$\int_\Omega [(u - \overline{u})(U - u) - \lambda(\nabla U - \nabla u) \cdot \nabla w)] \, d\mathbf{x} \leq - \int_\Omega |\nabla u - \nabla U|^2 \, d\mathbf{x} \leq 0.$$

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$$\int_\Omega \left(\frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2 + \exp(a_{j-1}(\mathbf{x})u(\mathbf{x}))\right.$$

$$\left. + \exp(b_{j-1}^-(\mathbf{x})(v_-(\mathbf{x}) - v(\mathbf{x}))) + \exp(b_{j-1}^+(\mathbf{x})(v(\mathbf{x}) - v_+(\mathbf{x})))\right) d\mathbf{x},$$

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Convergence

Theorem

$\{(u_j, v_j)\} \in H^1_0(\Omega) \times L^2(\Omega)$ iterates of algorithm such that:

$$(u_j, v_j) \to (u, v) \text{ pointwise},$$
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(u, v) *is the solution of the constrained problem.*

1. Feasibility: contradiction

$$u(\mathbf{x}) > 0, \quad u_j(\mathbf{x}) \to u(\mathbf{x}), \quad a_{j-1}(\mathbf{x})u_j(\mathbf{x}) \to 0,$$
$$a_j(\mathbf{x}) = a_{j-1}(\mathbf{x}) \exp(a_{j-1}(\mathbf{x})u_j(\mathbf{x})), \quad a_{j-1}(\mathbf{x}) > 0.$$

2. The possibility of having

$$I[\overline{u}, \overline{v}] < I[u, v]$$

would lead, for j large, to the fact that $(\overline{u}, \overline{v})$ is better than (u_j, v_j) for the modified functional, which is impossible by definition of (u_j, v_j).

Numerical experiment

$$\overline{u}(x,y) = \frac{1}{4}(\min(x, 1-x) - \frac{1}{4}), \quad v_- \equiv 5., v_+ \equiv -3.,$$

$$\lambda = .1, \quad a_0 \equiv b_0^- \equiv b_0^+ \equiv .1$$

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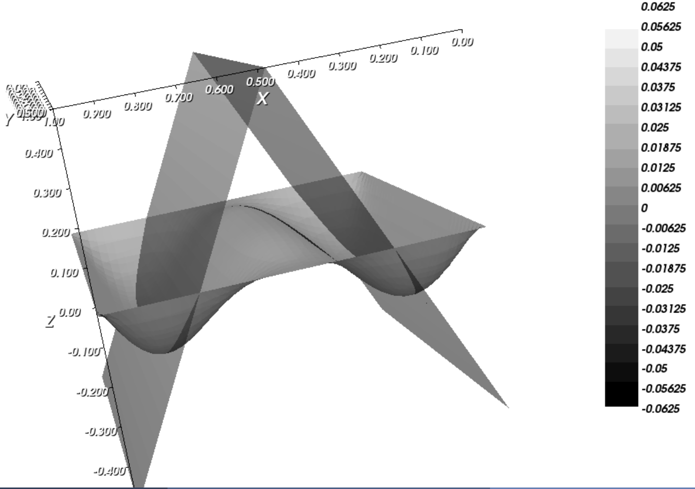

Numerical experiment (cont'd)

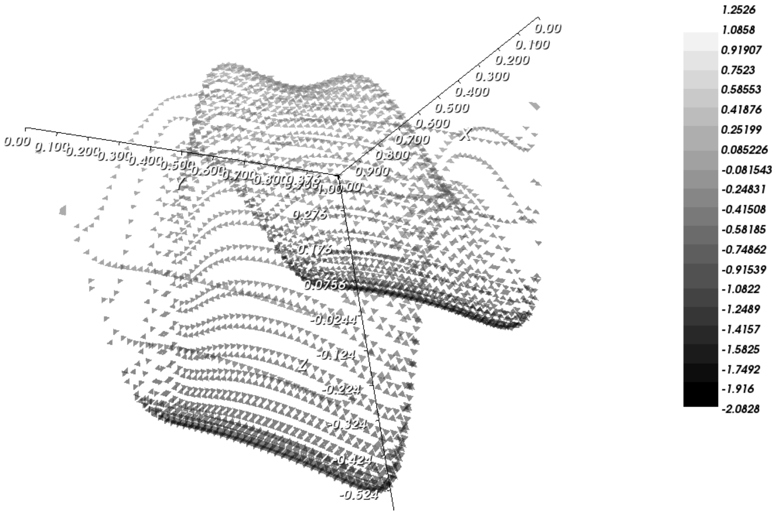

Figure: The control

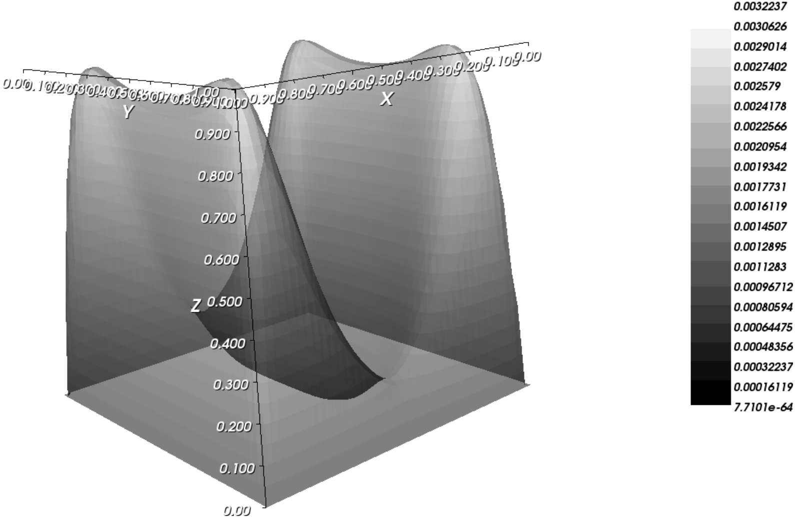

Figure: The residual

Numerical experiment (cont'd)

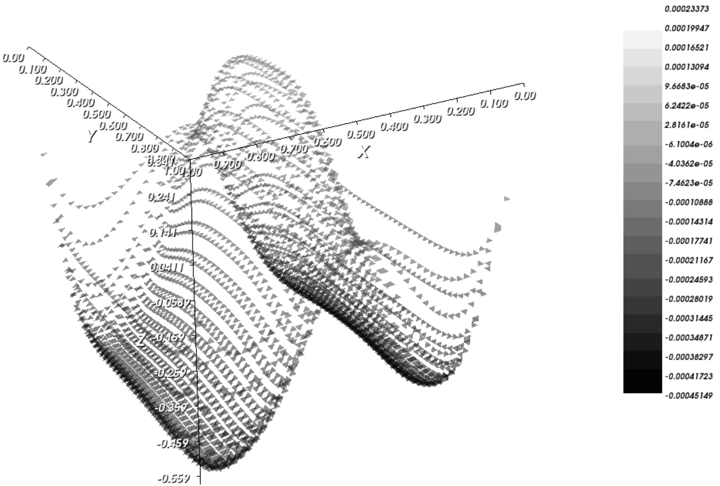

Figure: Product $a_{j-1}u_j$.

Numerical experiment (cont'd)

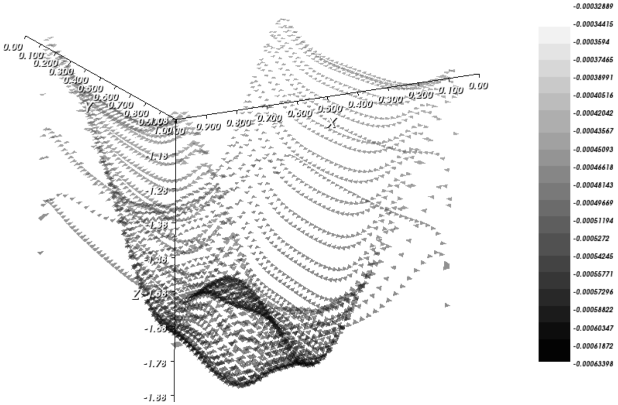

Figure: The product $b_{j-1}^{-}(v_{-} - v_j)$).

Numerical experiment (cont'd)

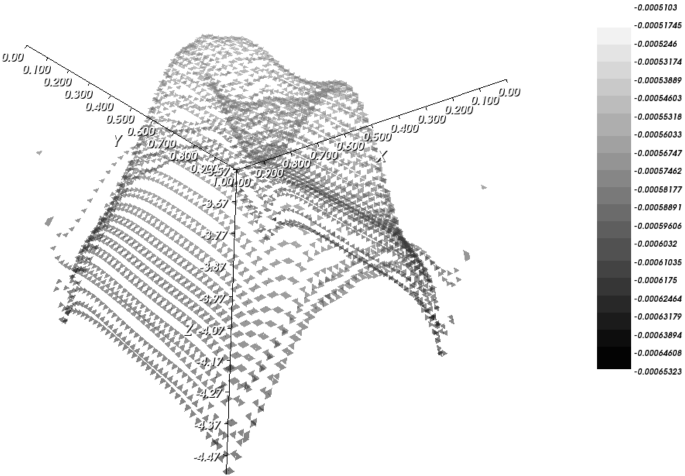

Figure: The product $b^- + j - 1(v_j - v_+))$.

Final issue: asymptotic behavior as $\lambda \to \infty$

Minimize in (u, v) : $$\int_\Omega \left(\frac{1}{2}|u(\mathbf{x}) - \overline{u}(\mathbf{x})|^2 + \frac{\mu}{2}|v(\mathbf{x})|^2 + \frac{\lambda}{2}|\nabla w(\mathbf{x})|^2 \right) d\mathbf{x},$$

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Theorem

Non-linearity ϕ, non-decreasing: $(\phi(u_1) - \phi(u_2))(u_1 - u_2) \geq 0$. Then

$$u_\lambda \to \tilde{u} \text{ in } H_0^1(\Omega), \quad v_\lambda \to \tilde{v} \text{ in } L^2(\Omega).$$