Optimal control of resources for species survival (current work)

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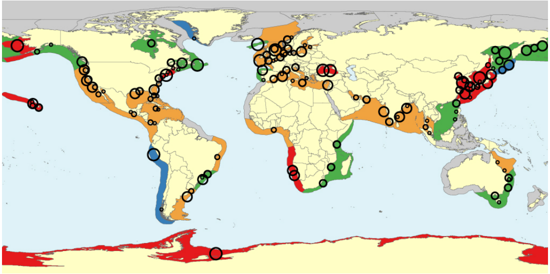
Outline

1. Modeling issues : toward a shape optimization problem
2. Analysis of optimal resources domains
   - Known results
   - New results
3. Conclusion and open problems



J. Lamboley, A. Laurain, G. Nadin, Y. Privat, *Properties of optimizers of the principal eigenvalue with indefinite weight and Robin conditions*, Calc. Var. Partial Differential Equations 55 (2016), no. 6.

Outline

Biological model : population dynamics

Logistic diffusive equation (Fisher-Kolmogorov 1937, Fleming 1975, Cantrell-Cosner 1989)

Introduce

- $\rightsquigarrow$ $\Omega \subset \mathbb{R}^N$: bounded domain with Lipschitz boundary (habitat)
- $\rightsquigarrow$ ω : positive parameter
- $\rightsquigarrow$ $u(t,x)$: density of a species at location x and time t
- $\rightsquigarrow$ $m(x)$: control - intrinsic growth rate of species at location x and
  - $\Omega \cap \{m > 0\}$ (resp. $\Omega \cap \{m < 0\}$) is the favorable (resp. unfavorable) part of habitat
  - $\int_\Omega m$ measures the total resources in the spatially heterogeneous environment Ω
  - After renormalization, one is allowed to assume that

$$-1 \leq m(x) \leq \kappa \quad \text{with } \kappa > 0 \quad \text{and } m \text{ changes sign.}$$

Biological model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Biological model : population dynamics

Choice of boundary conditions

$$\partial_n u + \beta u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+,$$

where β is a non-negative parameter standing for inhospitableness of the region surrounding Ω.

- $\rightsquigarrow$ Case $\beta = 0$: no-flux boundary condition (the boundary acts as a barrier)
- $\rightsquigarrow$ Case $\beta = +\infty$: Dirichlet condition (deadly boundary)
- $\rightsquigarrow$ Intermediate case $\beta > 0$: Ω is surrounded by a partially inhospitable region

Biological model : population dynamics

The complete model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u + \beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

($\rightsquigarrow$ takes into account effects of dispersal and partial heterogeneity)

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u + \beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\begin{cases} \Delta \varphi + \lambda m \varphi = 0 & \text{in } \Omega, \\ \partial_n \varphi + \beta \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{EP}$$

Existence of a positive principal eigenvalue $\lambda(m)$

- In the Robin (with $\beta > 0$) and Dirichlet cases, (EP) has a unique positive principal ($\Leftrightarrow \varphi > 0$) eigenvalue $\lambda(m)$.
- In the Neumann case ($\beta = 0$)
  - if $\int_\Omega m < 0$, then (EP) has a unique principal eigenvalue $\lambda(m)$.
  - if $\int_\Omega m \geq 0$, then 0 is the unique nonnegative principal eigenvalue of (EP).

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u + \beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\begin{cases} \Delta \varphi + \lambda m \varphi = 0 & \text{in } \Omega, \\ \partial_n \varphi + \beta \varphi = 0 & \text{on } \partial\Omega, \end{cases} \qquad (EP)$$

Theorem (Cantrell-Cosner 1989, Berestycki-Hamel-Roques 2005)

Let u^* be the unique positive steady solution of the logistic equation above. One has

- $\omega \leq \lambda(m) \implies u(t,x) \underset{t\to\infty}{\longrightarrow} 0,$
- $\omega > \lambda(m) \implies u(t,x) \underset{t\to\infty}{\longrightarrow} u^*(x).$

Comments on the eigenvalue problem (with a sign changing weight m)

Characterization of $\lambda(m)$

$\lambda(m)$ is the unique principal ($\Leftrightarrow \varphi > 0$) positive eigenvalue of the problem :

$$\begin{cases} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n\varphi + \beta\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

Another characterization of $\lambda(m)$

$\lambda(m)$ is also characterized by the min-formula :

$$\lambda(m) = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2 + \beta\int_{\partial\Omega}\varphi^2}{\int_\Omega m\varphi^2}, \quad \varphi \in H^1(\Omega), \ \int_\Omega m\varphi^2 > 0 \right\}.$$

Optimal arrangements of resources

Conclusion of this part

The species can be maintained iff $\omega > \lambda(m)$.

Hence, the smaller $\lambda(m)$ is, the more likely the species can survive

$\rightsquigarrow$ among all weights m, which of them yields the smallest principal eigenvalue $\lambda(m)$?

Optimal arrangements of resources

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Optimal control problem

$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda(m), \qquad \text{(P)}$$

with

$$\mathcal{M}_{m_0,\kappa} = \left\{ m \in L^\infty(\Omega, [-1,\kappa]),\ |\{m > 0\}| > 0,\ \int_\Omega m \leq -m_0|\Omega| \right\}$$

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Bang-bang property of minimizers

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

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Shape optimization version of the problem

Consequence : the two following problems

$$\inf \left\{ \lambda(m), \quad m \in L^\infty(\Omega, [-1, \kappa]),\ |\{m > 0\}| > 0,\ \int_\Omega m \leq -m_0|\Omega| \right\} \tag{1}$$

and

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega| \right\}, \tag{2}$$

where $c = c(m_0) \in (0, 1)$, are equivalent. Moreover, each infimum is in fact a minimum.

Bang-bang property of minimizers : elements of proof

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

- Easy : $m \mapsto \lambda(m)$ is continuous for the L^∞ weak-$\star$ topology and the set of admissible weights $\mathcal{M}_{m_0,\kappa}$ is compact.
- Direct computations show that

$$\begin{aligned}\lambda(m) &= \frac{\int_\Omega |\nabla\varphi|^2 + \beta \int_{\partial\Omega} \varphi^2}{\int_\Omega m\varphi^2} \\ &\geq \frac{\int_\Omega |\nabla\varphi|^2 + \beta \int_{\partial\Omega} \varphi^2}{\sup_{\tilde m \in \mathcal{M}_{m_0,\kappa}} \int_\Omega \tilde m \varphi^2} = \frac{\int_\Omega |\nabla\varphi|^2 + \beta \int_{\partial\Omega} \varphi^2}{\int_\Omega (\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E})\varphi^2} \geq \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}),\end{aligned}$$

where E is chosen in such a way that

$$\{\varphi > t\} \subset E \subset \{\varphi \geq t\} \quad \text{and} \quad |E| = c(m_0) \quad \text{(bathtub principle)}$$

for a given $t > 0$.

Bang-bang property of minimizers : elements of proof

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$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

- $\rightsquigarrow$ E is defined in a unique way since the level sets of φ have zero measure.
- $\rightsquigarrow$ The expected conclusion follows.

State of the art

Highly non-exhaustive

- Case $\beta = \infty$, with no sign changement : symmetrization, regularity in case of symmetry [Krein 1955, Friedland 1977, Cox 1990]
- Periodic case : [Hamel-Roques 2007]
- 1D case, $\beta = 0$: solved [Lou-Yanagida 2006]
- 1D case, $\beta > 0$: optimization among intervals [Hintermüller-Kao-Laurain 2012]
- 2D case : regularity [Chanillo-Kenig-To 2008]
- Numerics : [Cox, Hamel-Roques, Hintermüller-Kao-Laurain]

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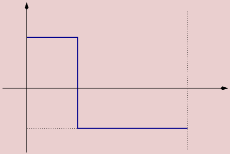
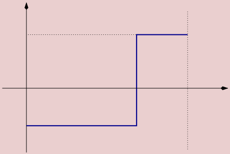
New results : complete solution in dim. 1

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \qquad \text{(P)}$$

Proposition (Hintermüller et al. 2012 - Lamboley et al. 2016)

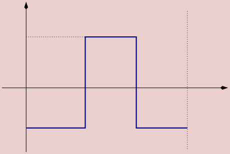
Let $\delta = \frac{1-m_0}{1+\kappa}$. The minimum of λ **among intervals** is

- **(Neumann case)** $(\beta = 0)$ $m = (\kappa+1)\mathbb{1}_{(0,\delta)} - 1$ and $m = (\kappa+1)\mathbb{1}_{(1-\delta,1)} - 1$

are the only solutions.

- **(Dirichlet case)** $(\beta = +\infty)$ $\quad m = (\kappa+1)\mathbb{1}_{((1-\delta)/2,(1+\delta)/2)} - 1$



is the only solution.

New results : complete solution in dim. 1

$$\inf\left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\} \qquad (P)$$

Theorem (Lamboley, Laurain, Nadin, YP)

If $\Omega =]0,1[$ and E^* is a solution, then E^* is an interval.

Consequence : there exists $\beta^* = \beta^*(\kappa, c)$ such that

- if $\beta > \beta^*$, same solution as $\beta = \infty$,
- if $\beta < \beta^*$, same solution as $\beta = 0$,
- if $\beta = \beta^*$, solutions are all the intervals of length c.

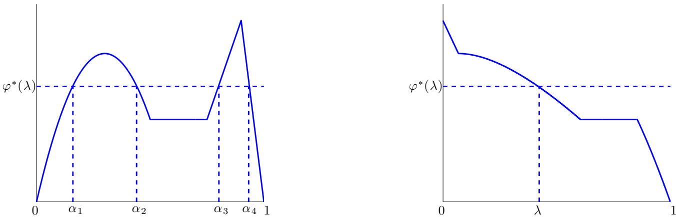
- Dirichlet case $\rightarrow$ Cox, McLaughlin 1990
- Neumann case $\rightarrow$ Lou-Yanagida 2006

Reminders on monotone rearrangements

Definition of monotone rearrangements in 1D

From φ, a measurable nonnegative function, one defines its decreasing rearrangement φ^* by

$$\varphi^*(s) = \inf \{ t \in \mathbb{R} \mid |\{\varphi > t\}| \leq s \} .$$



(a) Graph of a fonction φ

(b) Graph of its monotone decreasing rearrangement φ^*

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Good properties of rearrangements

For φ and ψ as previously and $f : \mathbb{R}_+ \to \mathbb{R}$ is measurable, we have :

$\rightsquigarrow$ $\displaystyle\int_0^1 f \circ \varphi(x)\, dx = \int_0^1 f \circ \varphi^*(s)\, ds$ (equimeasurability prop.)

$\rightsquigarrow$ $\displaystyle\int_0^1 \varphi'(x)^2\, dx \geq \int_0^1 (\varphi^*)'(s)^2\, ds$ (Polyà ineq.)

$\rightsquigarrow$ $\displaystyle\int_0^1 \varphi(x)\psi(x)\, dx \leq \int_0^1 \varphi^*(s)\psi^*(s)\, ds$ (Hardy-Littlewood ineq.)

Proof of Theorem 1

Here, $\Omega =]0,1[$. define the Rayleigh quotient

$$R_m[\varphi] := \frac{\int_0^1 \varphi'^2 + \beta\varphi^2(0) + \beta\varphi^2(1)}{\int_0^1 m\varphi^2},$$

so that the optimization problem rewrites

$$\inf\left\{ R_m[\varphi], \quad \varphi \in H^1(\Omega),\ \int_\Omega m\varphi^2 > 0, -1 \leq m \leq \kappa,\ \int_\Omega m \leq -m_0|\Omega| \right\}.$$

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$\rightsquigarrow$ Two-sided monotone rearrangement : $\alpha = \operatorname{argmax} f$

$$f^r(x) = \begin{cases} f^{\nearrow}(x) & \text{on } (0,\alpha) \\ f^{\searrow}(x) & \text{on } (\alpha,1) \end{cases}$$

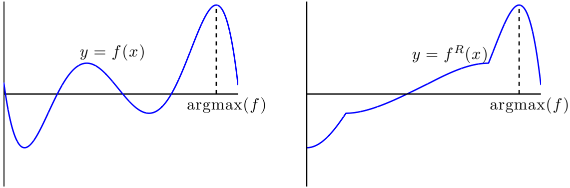

FIGURE – Illustration of the symmetrization procedure

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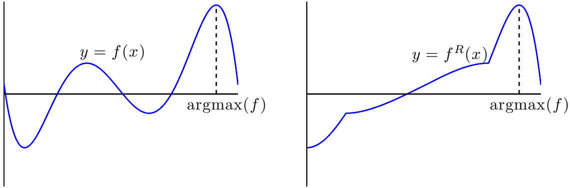
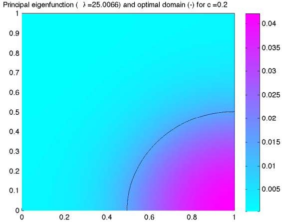


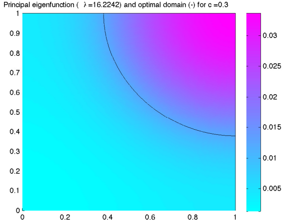
FIGURE – Illustration of the symmetrization procedure

⤳ Then (m^r, φ^r) are admissible and $R_{m^r}[\varphi^r] \leq R_m[\varphi]$.

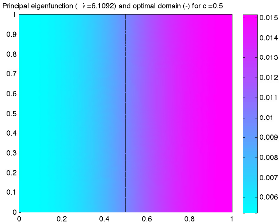
Higher dimensions : $\Omega = (0,1)^2$, $\kappa = 0.5$, $\beta = 0$



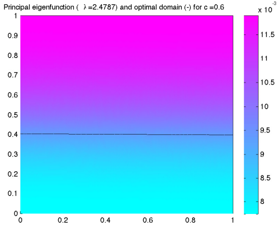
(a) $c = 0.2$



(b) $c = 0.3$

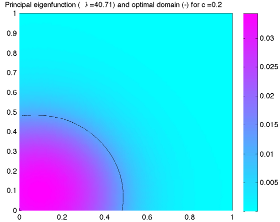


(c) $c = 0.5$

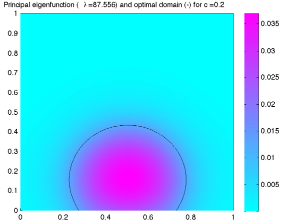


(d) $c = 0.6$

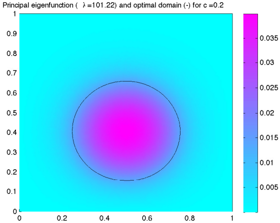
Higher dimensions : $\Omega = (0,1)^2$, $\kappa = 0.5$, $c = 0.2$



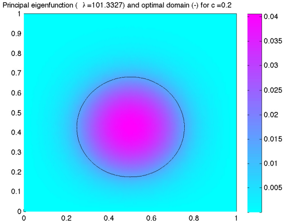
(a) $\beta = 1$



(b) $\beta = 5$



(c) $\beta = 50$



(d) $\beta = 1000$

New results : in dimension $N \geq 2$, is the solution a ball ?

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP)

Let assume that $N \geq 2$ and $\partial\Omega$ is connected and C^1.
Assume E or $\Omega \setminus E$ is invariant by rotation centered on a fixed point O.

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Let assume that $N \geq 2$ and $\partial\Omega$ is connected and C^1.
Assume E or $\Omega\setminus E$ is invariant by rotation centered on a fixed point O. Then

- E is critical $\Rightarrow$ Ω is a ball whose center is O
- If β is large enough,

  E is a minimum $\Rightarrow$ E and Ω are concentric balls.

The wording "critical" means that E satisfies the 1st order optimality conditions, i.e.

shape derivative of λ at E in direction $V = \langle d\lambda(E), V\rangle \geq 0$,

for all smooth vector field $V : \mathbb{R}^N \to \mathbb{R}^N$.
It also rewrites :

E is a level set of φ, i.e. $E = \{\varphi > \alpha\}$.

Steps of the proof of Theorem 2

Assume $E = B(0, r)$.

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$\rightsquigarrow$ φ is radial in E : show that $v_{ij} := x_i \partial_{x_j} \varphi - x_j \partial_{x_i} \varphi$ vanishes $(i \neq j)$; to that end use optimality condition.

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$\rightsquigarrow$ φ is radial in Ω :
Analytic regularity and Cauchy-Kowalevski Theorem.

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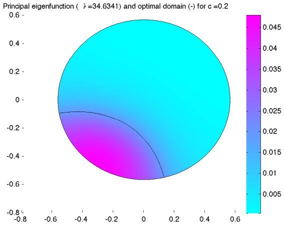
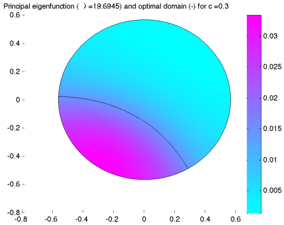
$\rightsquigarrow$ φ is radial in Ω :
Analytic regularity and Cauchy-Kowalevski Theorem.

$\rightsquigarrow$ Ω is a centered ball.
Easy if $\beta = \infty$; study the contact with the inscribed and circumscribed balls otherwise.

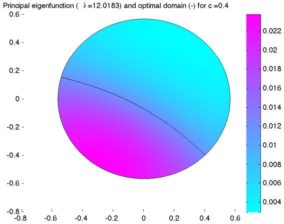
Other numerical computations : $\Omega = B(0,1), \beta = 0, \kappa = 0.5$



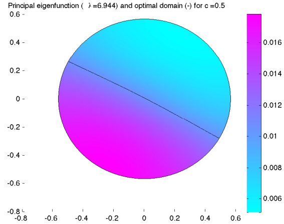
(e) $c = 0.2$



(f) $c = 0.3$



(g) $c = 0.4$



(h) $c = 0.5$

New results : Neumann case in dimension $N = 2, 3, 4$: non-optimality of the centered ball in a ball

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

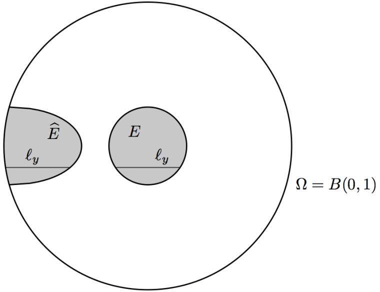
Theorem (Lamboley, Laurain, Nadin, YP)

Let $N \in \{2, 3, 4\}$, $\beta = 0$ and $\Omega = B(0,1) \subset \mathbb{R}^N$.
Then the centered ball of volume $c|\Omega|$ is not a minimizer for Problem (P).

Ideas of the proof

$\Omega = B(0,1)$, E rotationnally symmetric :

Disymmetrization procedure :



One proves : $\lambda(\widehat{E}) < \left(\frac{5N-4}{4N}\right)\lambda(E)$.

Outline

Conclusion and open questions

$$\text{On the problem } \inf\left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\} \qquad \text{(P)}$$

- If Ω is a ball, is E a concentric ball ?
  - $\rightsquigarrow$ Solved if $N = 1$: yes if β is large enough, no else.
  - $\rightsquigarrow$ Yes if $\beta = \infty$, No if $\beta = 0$ and $N \in \{2, 3, 4\}$
- Can $\partial E \cap \Omega$ be a piece of sphere ?
  - $\rightsquigarrow$ No if $\beta = 0$ and Ω is a square/cube
- Find sufficient conditions so that $\partial E \cap \partial\Omega \neq \emptyset$,
  - $\rightsquigarrow$ Expected to be always true if $\beta = 0$

Conclusion and open questions

Ongoing work (1) - Same problem with an improved model

$\rightsquigarrow$ We enrich the model by adding an advection term along the gradient of the habitat quality (Belgacem and Cosner)

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \alpha u \nabla m) + \lambda u(m-u) & \text{in } \Omega \times (0,\infty), \\ e^{\alpha m}(\partial_n u - \alpha u \partial_n m) + \beta u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

This models the tendency of the population to move up along the gradient of m.

New shape optimization problem

$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda(m),$$

$$\text{with} \quad \lambda(m) = \inf_{\varphi \in \mathcal{S}_0} \frac{\int_\Omega e^{\alpha m} |\nabla \varphi|^2}{\int_\Omega m e^{\alpha m} \varphi^2} \quad \text{and} \quad \mathcal{S}_0 = \{\varphi \in H^1(\Omega), \int_\Omega m e^{\alpha m} \varphi^2 > 0\}$$

$\rightsquigarrow$ with T. Deheuvels (ENS Rennes) & F. Caubet (univ. Toulouse)

F. Caubet, T. Deheuvels, Y. Privat, *Optimal location of resources for biased movement of species : the 1D case*, To appear in SIAM J. Applied Math.

Conclusion and open questions

Ongoing work (2)

Effects of dispersal and spatial heterogeneity of the environment on the total population size

$\rightsquigarrow$ Consider the steady-state

$$\begin{cases} \mu\Delta\bar{u} + \bar{u}(m - \bar{u}) = 0 & \text{in } \Omega, \\ \partial_n \bar{u} + \beta\bar{u} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mu = \text{migration rate})$$

This problem has a unique positive solution in $W^{2,p}(\Omega)$, for every $p \geq 1$.

New optimization problem

$$\sup_{m \in \mathcal{M}_{m_0,\kappa}} \int_\Omega \bar{u}(x)\, dx, \qquad \text{(total population size of the species)}$$

or

$$\sup_{m \in \mathcal{M}_{m_0,\kappa}} \int_\Omega \bar{u}(x)^3\, dx, \qquad \text{(natural energy of the population)}$$

$\rightsquigarrow$ Ph.D. thesis of I. Mazari (univ. Paris 6)

Conclusion and open questions

Ongoing work (3)

Optimal release of *Wolbachia*-infected mosquitoes in a 2 populations model

- Joint work with L. Almeida, M. Strugarek and N. Vauchelet.
- Population densities of uninfected (n_1) and infected (n_2) mosquitoes.

$$\begin{cases} \partial_t \mathbf{n} - \Delta \mathbf{n} = \mathbf{F}(\mathbf{n}) + \begin{pmatrix} 0 \\ u(t,x) \end{pmatrix} & \text{in } (0,T)\times\Omega, \\ \mathbf{n}(0,\cdot) = \mathbf{n}_0(\cdot) & \text{in } \Omega, \\ \frac{\partial \mathbf{n}(t,\cdot)}{\partial \nu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } (0,T)\times\varphi\Omega, \end{cases}$$

where

$$F_1(n_1,n_2) = B_1 \tfrac{n_1}{n_1+n_2}((1-s_h)n_2 + n_1)\left(1 - \tfrac{n_1+n_2}{K_1}\right) - d_1 n_1,$$

$$F_2(n_1,n_2) = B_2 n_2\left(1 - \tfrac{n_1+n_2}{K_2}\right) - d_2 n_2,$$

and

- artificial infection performed by the means of releases of infected mosquitoes, in time and space, which we denote by $u(t,x) \geq 0$.

Conclusion and open questions

Ongoing work (3)

Optimal release of *Wolbachia*-infected mosquitoes in a 2 populations model

$\rightsquigarrow$ Setting of the optimization problem.

Minimize the discrepancy functional

$$J : u \mapsto \frac{1}{2}\int_\Omega \left(\min\{n_I^* - n_2(T,x), 0\}^2 + n_1(T,x)^2\right) dx,$$

w.r.t. the control variable u, for some fixed positive number T.

$\rightsquigarrow$ Design of the control ?

We consider controls that are localized in time, *i.e.* of the form $u(t,x) = \sum_{i=1}^{N} \delta_{t=t_i} u_i(x)$,

with

- # of mosquitoes released at each time t_i bounded, i.e. $\int_\Omega u_i(x)dx \leq U_0$, $i = 1, \dots, N$
- mosquitoes released and not removed, and the density of released mosquitoes bounded, i.e. $u_i \geq 0$ and $u_i \leq M$, a.e. in Ω

Conclusion and open questions

Ongoing work (3)

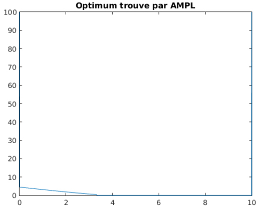
Optimal release of *Wolbachia*-infected mosquitoes in a 2 populations model

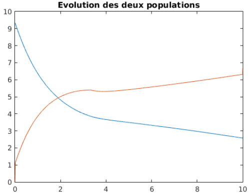
⤳ Design of the control ?

We consider the simpler system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad \mathbf{x}(0) = \mathbf{x}^0 \in \mathbb{R}^2_+,$$

and minimize the discrepancy functional $J : u \mapsto \frac{1}{2}x_1(T)^2 + \frac{1}{2}\big(\max(X_2 - x_2(T), 0)\big)^2$. w.r.t. the control variable u, for some fixed positive number T, under the same constraints as before.





Issues :

- optimal control = sum of Dirac masses ?

- bang-bang character with L^∞ constraint ?

Thank you for your attention