PERVERSE SHEAVES ON FLAG MANIFOLDS AND KAZHDAN-LUSZTIG POLYNOMIALS (AFTER KAZHDAN-LUSZTIG, MACPHERSON, SPRINGER, ...)

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Introduction

0.1. The goal of these notes is to present an easy proof, due to Springer and MacPherson, of the well-known fact that the stalks of simple perverse sheaves on the flag variety $\mathcal{B}$ of a complex semisimple algebraic group G can be described by coefficients of Kazhdan-Lusztig polynomials. These polynomials are defined combinatorially, in terms of the Hecke algebra of the corresponding Weyl group.

We also discuss other proofs of this result, and the related question of a geometric description of the Hecke algebra, and we perform explicit computations.

0.2. The origin of this problem is Kazdan-Lusztig conjecture, which describes characters of simple representations of the complex semisimple Lie algebras Lie(G) in terms of Kazhdan-Lusztig polynomials (see [KL1]). The proof of this conjecture was completed in two steps. First, as described in [Ri], one can describe these characters in terms of stalks of simple perverse sheaves on $\mathcal{B}$. Then, these notes explain how these stalks can be described in terms of Kazhdan-Lusztig polynomials. Note that, historically, the second step was completed (by Kazhdan-Lusztig, see [KL2]) before the first one (by Brylinski-Kashiwara and Beilinson-Bernstein, see [Ri] and references therein).

0.3. Notation. All functors considered are *derived* functors, but for simplicity we will not indicate "R" or "L". We normalize the intersection cohomology complexes so that they are perverse sheaves (as in [Ri]).

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1. Hecke algebras and Kazhdan-Lusztig polynomials

1.1. Hecke algebra. Let (W,S) be a Coxeter group. The associated Hecke algebra $\mathcal{H}_W$ is a $\mathbb{Z}[t,t^{-1}]$-algebra, with a basis $\{T_w,\, w\in W\}$ parametrized by W, whose multiplication is given by the following rules:

$$\begin{cases} T_v\cdot T_w = T_{vw} & \text{if } \ell(vw)=\ell(v)+\ell(w),\\ T_s^2=(t^2-1)T_s+t^2 & \text{if } s\in S.\end{cases}$$

(For details on the construction, see e.g. [Hu].)

The Kazhdan-Lusztig involution $i:\mathcal{H}_W\to\mathcal{H}_W$ is the algebra involution defined by the formulas $i(t)=t^{-1}$, $i(T_w)=T_{w^{-1}}^{-1}$.

1.2. Kazhdan-Lusztig basis. The following fundamental result is due to Kazhdan and Lusztig, see [KL1]. The proof is very elementary. (See [Hu] for a detailled version or the original proof of Kazhdan-Lusztig, or [S2, Theorem 2.1] for a very simple proof avoiding the use of R-polynomials, and an even more general unicity result.)

Theorem 1.2.1. *For any $w\in W$, there exists a unique element $C_w\in\mathcal{H}_W$ which satisfies the following properties:*

1. *$i(C_w)=C_w$*
2. *$C_w = t^{-\ell(w)}\sum_{x\leq w} Q_{x,w}(t)T_x$, where $Q_{w,w}=1$ and for $x<w$, $Q_{x,w}\in\mathbb{Z}[t]$ is a polynomial of degree $\leq \ell(w)-\ell(x)-1$.*

Moreover, for each $x\leq w$, there exists a polynomial $P_{x,w}\in\mathbb{Z}[q]$ (of degree $\leq\frac{1}{2}(\ell(w)-\ell(x)-1)$) such that $Q_{x,w}(t)=P_{x,w}(t^2)$.

The polynomials $P_{x,w}$ are called Kazhdan-Lusztig polynomials. The elements C_w (which are rather denoted C'_w in [KL1] and [Hu], and $\underline{H}_w$ in [S2]) are called Kazhdan-Lusztig elements.

Example 1.2.2. (1) We have $C_1=1$.
(2) For $s\in S$, we have $C_s=t^{-1}(T_s+1)$.
(3) If s,t are different simple reflections, then $C_{st}=t^{-2}(T_{st}+T_s+T_t+1)$.

2. Geometry of the flag variety

For more details (and further results) on this section, the reader may consult [BK].

2.1. Bruhat decomposition. Let G be a complex connected semisimple algebraic group, and let $B\subset G$ be a Borel subgroup and $T\subset B$ be a maximal torus. Let W be the Weyl group of G. The choice of B determines a naturel set of generators $S\subset W$, such that (W,S) is a Coxeter group.

Let $\mathcal{B}=G/B$ be the flag variety. The Bruhat decomposition gives the partition of $\mathcal{B}$ into B-orbits:

$$\mathcal{B} \;=\; \bigsqcup_{w\in W} BwB/B.$$

Moreover, we have

$$X_w \;:=\; \overline{BwB/B} \;=\; \bigsqcup_{y\leq w} ByB/B.$$

We let $\mathcal{T}$ denote the stratification of $\mathcal{B}$ by the BwB/B, $w \in W$. (It is a Whitney stratification.) The orbit BwB/B is isomorphic to the affine space $\mathbb{A}^{\ell(w)}$, in particular it is simply connected. Hence the simple perverse sheaves constructible for the stratification $\mathcal{T}$ are the IC complexes $\mathrm{IC}(X_w)$ associated with the trivial local system on the orbits BwB/B. We write $\mathrm{IC}(X_w)_y$ for the stalk of $\mathrm{IC}(X_w)$ at $yB/B \in \mathcal{B}$. It is a complex of vector spaces. The goal of these notes is to explain how the cohomology of $\mathrm{IC}(X_w)_y$ is described by the Kazhdan-Lusztig polynomial $P_{y,w}$ introduced in §1.2.

It is sometimes convenient to consider a more symmetric situation. Namely, the Bruhat decomposition equivalently describes the orbits of the diagonal G-action on $\mathcal{B} \times \mathcal{B}$:

$$\mathcal{B} \times \mathcal{B} \;=\; \bigsqcup_{w \in W} G \cdot (B/B, wB/B).$$

We also write

$$\mathfrak{X}_w := \overline{G \cdot (B/B, wB/B)}.$$

We denote by $\mathcal{S}$ the stratification of $\mathcal{B} \times \mathcal{B}$ by the $G \cdot (B/B, wB/B)$, $w \in W$. There are natural isomorphisms, for each $w \in W$,

$$G \cdot (B/B, wB/B) \cong G \times^B X_w, \quad \mathfrak{X}_w \cong G \times^B X_w.$$

In particular, these strata are also simply-connected. We write $\mathrm{IC}(\mathfrak{X}_w)_y$ for the stalk of the simple perverse sheaf $\mathrm{IC}(\mathfrak{X}_w)$ at the point $(B/B, yB/B) \in \mathcal{B} \times \mathcal{B}$. Note that, by simply connectedness of the strata, the object $\mathrm{IC}(\mathfrak{X}_w)$ has constant cohomology along $\mathfrak{X}_y$[1]. Moreover, for any $w, y \in W$ and $i \in \mathbb{Z}$ there is a natural isomorphism

$$H^i(\mathrm{IC}(\mathfrak{X}_w)_y) \;\cong\; H^{i+\dim(\mathcal{B})}(\mathrm{IC}(X_w)_y).$$

The strata BwB/B (resp. $G \cdot (B/B, wB/B)$) are called Schubert cells (resp. G-Schubert cells), and their closures X_w (resp. $\mathfrak{X}_w$) are called Schubert varieties (resp. G-Schubert varieties). For any $w \in W$, we denote by

$$j_w : \; G \cdot (B/B, wB/B) \hookrightarrow \mathcal{B} \times \mathcal{B}$$

the inclusion.

Example 2.1.1. (1) We have by definition

$$X_1 = \{B/B\}, \quad \mathfrak{X}_1 = \Delta\mathcal{B},$$

where $\Delta\mathcal{B} \subset \mathcal{B} \times \mathcal{B}$ is the diagonal copy.

(2) The closure X_s coincides with P_s/B, hence is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. In particular it is smooth, and

$$\mathrm{IC}(X_s) = \underline{\mathbb{Q}}_{X_s}[1].$$

Similarly, $\mathfrak{X}_s$ is a $\mathbb{P}^1_{\mathbb{C}}$-fibration over $\mathcal{B}$, and we have

$$\mathrm{IC}(\mathfrak{X}_s) = \underline{\mathbb{Q}}_{\mathfrak{X}_s}[\dim(\mathcal{B}) + 1].$$

[1] Alternatively, this fact can be checked using the G-action.

2.2. Demazure resolutions. In the study of Schubert varieties, a crucial role is played by Demazure resolutions. Let $w \in W$, and let $w = s_1 \cdots s_n$ be a reduced decomposition (so that $n = \ell(w)$). Then consider the variety

$$Y_{(s_1,\cdots,s_n)} := P_{s_1} \times^B P_{s_2} \times^B \cdots \times^B P_{s_n}/B.$$

Here, P_s denotes the minimal parabolic subgroup of B such that $P_s = B \sqcup BsB$. The variety $Y_{(s_1,\cdots,s_n)}$ is an iterated $\mathbb{P}^1$-fibration; in particular it is smooth, of dimension n. There is a natural proper morphism

$$\varpi_{(s_1,\cdots,s_n)} : Y_{(s_1,\cdots,s_n)} \to \mathcal{B}$$

defined by

$$\varpi_{(s_1,\cdots,s_n)}([p_1 : \cdots : p_n B/B]) \ = \ p_1 \cdots p_n B/B.$$

It is well-known that the image of $\varpi_{(s_1,\cdots,s_n)}$ is X_w, and that it is an isomorphism over BwB/B. In particular,

$$\varpi_{(s_1,\cdots,s_n)} : Y_{(s_1,\cdots,s_n)} \to X_w$$

is a resolution of singularities.

Again, it is often convenient to consider a G-equivariant analogue. Namely, for $s \in S$ consider the partial flag variety $\mathcal{P}_s := G/P_s$, and the natural projection morphism $\mathcal{B} \to \mathcal{P}_s$. Let again $w \in W$, and $w = s_1 \cdots s_n$ be a reduced decomposition. Consider the variety

$$\mathfrak{Y}_{(s_1,\cdots,s_n)} := \mathcal{B} \times_{\mathcal{P}_{s_1}} \mathcal{B} \times_{\mathcal{P}_{s_2}} \cdots \times_{\mathcal{P}_{s_n}} \mathcal{B},$$

and the proper morphism

$$\pi_{(s_1,\cdots,s_n)} : \mathfrak{Y}_{(s_1,\cdots,s_n)} \to \mathcal{B} \times \mathcal{B}$$

induced by the projection on the first and the last component. Then as above the image of $\pi_{(s_1,\cdots,s_n)}$ is $\mathfrak{X}_w$, and this morphism is an isomorphism over $\mathfrak{X}_w$. In particular,

$$\pi_{(s_1,\cdots,s_n)} : \mathfrak{Y}_{(s_1,\cdots,s_n)} \to \mathfrak{X}_w$$

is a resolution of singularities. In fact, there are natural isomorphism

$$\mathfrak{Y}_{(s_1,\cdots,s_n)} \cong G \times^B Y_{(s_1,\cdots,s_n)}$$

such that $\pi_{(s_1,\cdots,s_n)}$ is induced by $\varpi_{(s_1,\cdots,s_n)}$.

Let us remark that there is a natural isomorphism

$$\mathfrak{Y}_{(s_1,\cdots,s_n)} \ \cong \ \mathfrak{X}_{s_1} \times_{\mathcal{B}} \mathfrak{X}_{s_2} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathfrak{X}_{s_n}. \tag{2.2.1}$$

Example 2.2.2. When $\ell(w) \leq 2$, the Demazure resolution is an isomorphism. In particular, these Schubert varieties are smooth, and their IC complexes are shifted constant sheaves.

3. Description of perverse sheaves in terms of the Hecke algebra

The proofs in this subsection are due to MacPherson and Springer, see [Sp].

3.1. Crucial lemma. Let $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$ be the bounded derived categories of sheaves of $\mathbb{Q}$-vector spaces on $\mathcal{B}\times\mathcal{B}$, constructible with respect to the stratification $\mathcal{S}$ by G-orbits. To simplify the notation, for any $\mathcal{A}$ in $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$ we write $\mathcal{A}_w$ for the fiber $\mathcal{A}_{(B/B,wB/B)}$ of $\mathcal{A}$ at the point $(B/B,wB/B)$. For any object $\mathcal{A}$ of $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$, we consider the element $h(\mathcal{A})\in\mathcal{H}_W$ defined by the following formula:

$$h(\mathcal{A}) \;=\; \sum_{w\in W}\big(\sum_{i\in\mathbb{Z}}\dim H^i(\mathcal{A}_w)t^i\big)\cdot T_w.$$

Here the fiber $\mathcal{A}_w$ is a complex of vector spaces, and $H^i(\mathcal{A}_w)$ is its i-th cohomology.

Now we define a convolution product on the category $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$. Consider the projection morphism $p_{i,j}:\mathcal{B}^3\to\mathcal{B}^2$ on the i-th and j-th components. Then for $\mathcal{A}_1$ and $\mathcal{A}_2$ in $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$, we set

$$\mathcal{A}_1*\mathcal{A}_2:=(p_{1,3})_*\big(p_{1,2}^*\mathcal{A}_1\otimes_{\mathbb{Q}}p_{2,3}^*\mathcal{A}_2\big)\quad\in D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B}).$$

By the base change formula, this product is associative. Moreover, the object $\underline{\mathbb{Q}}_{\mathfrak{X}_1}$ is a unit. (Recall that $\mathfrak{X}_1$ is the diagonal copy of $\mathcal{B}$ in $\mathcal{B}\times\mathcal{B}$.)

The following crucial lemma is proved in [Sp, Lemme 2.6]. See also [S3, Lemma 3.2.3] for a slightly different argument. The proof does not use the Decomposition Theorem; hence this lemma is true for perverse sheaves with coefficients in any field.

Lemma 3.1.1. *Let $\mathcal{A}$ be an object of $D^b_{\mathcal{S}}(\mathcal{B}\times\mathcal{B})$ such that $\mathcal{H}^i(\mathcal{A})=0$ if i is odd (resp. even), and let $s\in S$. Then the object $\underline{\mathbb{Q}}_{\mathfrak{X}_s}*\mathcal{A}$ has the same property, and we have*

$$h\big(\underline{\mathbb{Q}}_{\mathfrak{X}_s}*\mathcal{A}\big)\;=\;(T_s+1)\cdot h(\mathcal{A}).$$

Proof. An easy computation gives

$$\begin{aligned}(T_s+1)\cdot h(\mathcal{A}) \;=\; &\sum_{sw<w}\big(\sum_{i\in\mathbb{Z}}(\dim H^i(\mathcal{A}_{sw})+\dim H^{i-2}(\mathcal{A}_w))t^i\big)\cdot T_w\\ &+\sum_{sw>w}\big(\sum_{i\in\mathbb{Z}}(\dim H^i(\mathcal{A}_w)+\dim H^{i-2}(\mathcal{A}_{sw}))t^i\big)\cdot T_w.\end{aligned}$$

Hence it is sufficient to prove that

$$(*)\;\;\dim H^i\big((\underline{\mathbb{Q}}_{\mathfrak{X}_s}*\mathcal{A})_w\big)=\begin{cases}\dim H^i(\mathcal{A}_{sw})+\dim H^{i-2}(\mathcal{A}_w) & \text{if } sw<w,\\ \dim H^i(\mathcal{A}_w)+\dim H^{i-2}(\mathcal{A}_{sw}) & \text{if } sw>w.\end{cases}$$

Let $\mathcal{C}$ be the restriction of $p_{1,2}^*\underline{\mathbb{Q}}_{\mathfrak{X}_s}\otimes_{\mathbb{Q}}p_{2,3}^*\mathcal{A}$ to

$$Z_w^s:=\{(B/B,gB/B,wB/B),\;g\in P_s\}\subset\mathcal{B}^3.$$

This variety is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and we have $H^i\big((\underline{\mathbb{Q}}_{\mathfrak{X}_s}*\mathcal{A})_w\big)=H^i(Z_w^s,\mathcal{C})$.

Assume first that $sw<w$. Then $(gB/B,wB/B)$ is in $G\cdot(B/B,swB/B)$ iff $gB=sB$. In particular, only one point of Z_w^s is in $\mathfrak{X}_s\times_{\mathcal{B}}\big(G\cdot(B/B,swB/B)\big)$; all the other points are in $\mathfrak{X}_s\times_{\mathcal{B}}\big(G\cdot(B/B,wB/B)\big)$. Let

$$i:\{(B/B,sB/B,wB/B)\}\hookrightarrow Z_w^s$$

be the inclusion, and let j be the inclusion of the complement. Consider the exact triangle

$$j_! j^* \mathcal{C} \to \mathcal{C} \to i_* i^* \mathcal{C} \xrightarrow{+1},$$

and the associated long exact sequence of cohomology with compact supports. The complex $i^*\mathcal{C}$ is the constant complex $\mathcal{A}_{sw}$, and $j^*\mathcal{C}$ is the constant complex on $Z_w^s - \{(B/B, sB/B, wB/B)\} \cong \mathbb{A}^1$ with fiber $\mathcal{A}_w$. Using the fact that $H_c^*(\mathbb{C}, \underline{\mathbb{Q}}) = \mathbb{Q}[-2]$, the parity vanishing assumption ensures that the long exact sequence reduces to a collection of short exact sequences. This gives the first case of $(*)$.

Assume now that $sw > w$. Then $(B/B, B/B, wB/B)$ is the only point in Z_w^s which is in $\mathfrak{X}_s \times_{\mathcal{B}} \big(G \cdot (B/B, wB/B)\big)$. All other points are in $\mathfrak{X}_s \times_{\mathcal{B}} \big(G \cdot (B/B, swB/B)\big)$. The same arguments as above give the second line of $(*)$. □

3.2. Computation of fibers of simple perverse sheaves.

Theorem 3.2.1. *For any $w \in W$, we have*

$$h(\mathrm{IC}(\mathfrak{X}_w)) \ = \ t^{-\dim \mathcal{B}} \cdot C_w.$$

In other words, for any $y \leq w$ and any $i \in \mathbb{Z}$, the dimension of $H^i(\mathrm{IC}(X_w)_y)$ is zero if $i + \ell(w)$ is odd, and is the coefficient of $q^{(i+\ell(w))/2}$ in $P_{y,w}(q)$ if $i + \ell(w)$ is even.

Proof. We proceed by induction on $\ell(w)$, the case $w = 1$ being trivial. Let $w = s_1 \cdots s_n$ be a reduced decomposition. (Hence $n = \ell(w)$.) Consider the (G-)Demazure resolution

$$\pi_{(s_1, \cdots, s_n)} : \mathfrak{Y}_{(s_1, \cdots, s_n)} \to \mathfrak{X}_w.$$

Using (2.2.1), a repeated use of the base change theorem gives an isomorphism

$$(\pi_{(s_1, \cdots, s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}} \cong \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1}} * \cdots * \underline{\mathbb{Q}}_{\mathfrak{X}_{s_n}}.$$

By Lemma 3.1.1, we deduce that we have

$$h((\pi_{(s_1, \cdots, s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}}[n]) \ = \ t^{-n}(1 + T_{s_1}) \cdots (1 + T_{s_n}). \tag{3.2.2}$$

As the variety $\mathfrak{Y}_{(s_1, \cdots, s_n)}$ is smooth, we have

$$\mathrm{IC}(\mathfrak{Y}_{(s_1, \cdots, s_n)}) = \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}}[n + \dim \mathcal{B}].$$

Hence the Decomposition Theorem ensures that we have an isomorphism

$$(\pi_{(s_1, \cdots, s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}}[n + \dim \mathcal{B}] \ \cong \ \bigoplus_{y \leq w} \mathrm{IC}(\mathfrak{X}_y) \otimes_{\mathbb{Q}} V_y,$$

where V_y is a graded finite dimensional $\mathbb{Q}$-vector space. Moreover, as the morphism $\pi_{(s_1, \cdots, s_n)}$ is an isomorphism over $G \cdot (B/B, wB/B)$, we have $V_w = \mathbb{Q}$. As $\pi_{(s_1, \cdots, s_n)}$ is proper, the object $(\pi_{(s_1, \cdots, s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}}[n + \dim \mathcal{B}]$ is self-dual (under Verdier duality). Hence $V_x^n = V_x^{-n}$. It follows that we have

$$h((\pi_{(s_1, \cdots, s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1, \cdots, s_n)}}[n + \dim \mathcal{B}]) \ = \ h(\mathrm{IC}(\mathfrak{X}_w)) + \sum_{y < w} Q_y(t) h(\mathrm{IC}(\mathfrak{X}_y)),$$

where Q_y is a polynomial such that $Q_y(t) = Q_y(t^{-1})$. By induction, we know that $h(\mathrm{IC}(\mathfrak{X}_y)) = t^{-\dim \mathcal{B}} C_y$ for any $y < w$. In particular, the element

$t^{\dim \mathcal{B}} \cdot \sum_{y<w} Q_y(t) h(\mathrm{IC}(\mathfrak{X}_y))$ is stable under the involution i of $\mathcal{H}_W$. By (3.2.2), also $t^{\dim \mathcal{B}} \cdot h((\pi_{(s_1,\cdots,s_n)})_* \underline{\mathbb{Q}}_{\mathfrak{Y}_{(s_1,\cdots,s_n)}}[n+\dim \mathcal{B}])$ is stable under i. It follows that $t^{\dim \mathcal{B}} h(\mathrm{IC}(\mathfrak{X}_w))$ is stable under i.

Now, by the general properties of IC complexes, we have, for $y < w$

$$H^{i-\dim \mathcal{B}}(\mathrm{IC}(\mathfrak{X}_w)_y) = 0 \quad \text{if } i \notin [\![-\ell(w), -\ell(y)-1]\!].$$

It follows that $t^{\dim \mathcal{B}} h(\mathrm{IC}(\mathfrak{X}_w))$ satisfies the conditions which characterize C_w, see Theorem 1.2.1. Hence $h(\mathrm{IC}(\mathfrak{X}_w)) = t^{-\dim \mathcal{B}} C_w$. □

3.3. Remarks.

3.3.1. *Positivity.* Let us remark that it follows from Theorem 3.2.1 that the coefficients of $P_{y,w}$ are non-negative. This is far from obvious from the definition, and still a conjecture for a general Coxeter group.

3.3.2. *Geometric description of the Hecke algebra.* The arguments used above also give a geometric (or topological) construction of the Hecke algebra. More precisely, consider the subcategory $\mathcal{D}$ of $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$ whose objects are the semisimple complexes, i.e. the complexes of the form

$$\bigoplus_{x \in W} \mathrm{IC}(\mathfrak{X}_x) \otimes_{\mathbb{Q}} V_x,$$

where V_x is a graded vector space. (Note that $\mathcal{D}$ is not a triangulated subcategory.) Then one has the application

$$h : \mathrm{Obj}(\mathcal{D}) \to \mathcal{H}_W.$$

One can check easily, using a general result of Goresky-MacPherson and the Decomposition Theorem, that $\mathcal{D}$ is stable under the convolution product (see [Sp, Proposition 2.13]). One can also easily check[2] that

$$h(\mathcal{A}_1 * \mathcal{A}_2) \;=\; h(\mathcal{A}_1) \cdot h(\mathcal{A}_2)$$

for any $\mathcal{A}_1$, $\mathcal{A}_2$ in $\mathcal{D}$ (see [Sp, Corollaire 2.11]). The subcategory $\mathcal{D}$ is stable under (cohomological) shifts, and we have

$$h(\mathcal{A}[1]) \;=\; t^{-1} h(\mathcal{A})$$

for $\mathcal{A}$ in $\mathcal{D}$. It is clear also that $\mathcal{D}$ is stable under Verdier duality $\mathbb{D}$, and that

$$h(\mathbb{D}(\mathcal{A})) \;=\; t^{-2\dim \mathcal{B}} \cdot i(h(\mathcal{A}))$$

for any $\mathcal{A}$ in $\mathcal{D}$. Finally, the image of h is

$$\bigoplus_{w \in W} \mathbb{Z}_{\geq 0}[t, t^{-1}] \cdot C_w.$$

Hence the structure of $\mathcal{H}_W$ is "encoded" in the category $\mathcal{D}$. One can deduce in particular that for $x, y \in W$ we have

$$C_x \cdot C_y \in \bigoplus_{w \in W} \mathbb{Z}_{\geq 0}[t, t^{-1}] \cdot C_w.$$

(This result is stated as [Sp, Corollaire 2.14].)

[2] This property is not clear for arbitrary $\mathcal{A}_1$, $\mathcal{A}_1$ in $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$. This is the reason why one has to restrict to the category $\mathcal{D}$.

4. Examples

4.1. Type $\mathbf{A}_1$. Consider $G = \mathrm{SL}(2,\mathbb{C})$. In this case $W = \{1, s\}$, with $s^2 = 1$. The Kazhdan-Lusztig elements in $\mathcal{H}_W$ are the following:

$$C_1 = 1, \qquad C_s = t^{-1}(T_s + 1).$$

The flag variety is $\mathcal{B} = \mathbb{P}^1_{\mathbb{C}}$. The Schubert cells are

$$X_1 = \{\infty\}, \qquad BsB/B = \mathbb{C}.$$

The Schubert varieties are smooth in this case, hence the simple perverse sheaves are the (shifted) constant sheaves:

$$\mathrm{IC}(X_1) = \underline{\mathbb{Q}}_{X_1}, \qquad \mathrm{IC}(X_s) = \underline{\mathbb{Q}}_{\mathcal{B}}[1].$$

In this case, the Demazure resolutions are isomorphisms.

4.2. Type $\mathbf{A}_2$. Consider $G = \mathrm{SL}(3,\mathbb{C})$. In this case we have

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1 = s_2s_1s_2\}.$$

The first Kazhdan-Lusztig elements are the following:

$$\begin{aligned} C_1 = 1, \quad C_{s_1} &= t^{-1}(T_{s_1} + 1), \quad C_{s_2} \;=\; t^{-1}(T_{s_2} + 1), \\ C_{s_1s_2} \;=\; C_{s_1}C_{s_2} &= t^{-2}(T_{s_1s_2} + T_{s_1} + T_{s_2} + 1), \\ C_{s_2s_1} \;=\; C_{s_2}C_{s_1} &= t^{-2}(T_{s_2s_1} + T_{s_1} + T_{s_2} + 1). \end{aligned}$$

To compute C_{w_0} (with here $w_0 = s_1s_2s_1$), we observe that

$$C_{s_1}C_{s_2}C_{s_1} \;=\; t^{-3}\big(T_{w_0} + T_{s_1s_2} + T_{s_2s_1} + (t^2+1)T_{s_1} + T_{s_2} + (t^2+1)\big).$$

The coefficient of T_{s_1} has degree 2, which is greater than $\ell(w_0) - \ell(s_1) - 1 = 1$. Hence we have to substract a multiple of C_{s_1}. We obtain that $C_{s_1}C_{s_2}C_{s_1} = C_{w_0} + C_{s_1}$, with

$$C_{w_0} \;=\; t^{-3}\big(T_{w_0} + T_{s_1s_2} + T_{s_2s_1} + T_{s_1} + T_{s_2} + 1\big).$$

The Schubert varieties are again all smooth, hence the simple perverse sheaves are the shifted constant sheaves. The Demazure resolution is an isomorphism for all elements except w_0. (In particular $X_{s_1s_2}$ and $X_{s_2s_1}$ are $\mathbb{P}^1_{\mathbb{C}}$-fibrations over $\mathbb{P}^1_{\mathbb{C}}$.) The case of w_0 is more interesting. Choose the (G-)Demazure resolution

$$\mathfrak{Y}_{(s_1,s_2,s_1)} \;=\; \mathcal{B} \times_{\mathcal{P}_{s_1}} \mathcal{B} \times_{\mathcal{P}_{s_2}} \mathcal{B} \times_{\mathcal{P}_{s_1}} \mathcal{B}$$

of the (G-)Schubert variety

$$\mathfrak{X}_{w_0} \;=\; \mathcal{B} \times \mathcal{B}.$$

For simplicity we write π for $\pi_{(s_1,s_2,s_1)}$, and $\mathfrak{Y}$ for $\mathfrak{Y}_{(s_1,s_2,s_1)}$. The restriction of π to $(\mathcal{B} \times \mathcal{B}) - \mathfrak{X}_{s_1}$ is an isomorphism. And we have

$$\pi^{-1}(B, s_1B) \;=\; \{(B, gB, s_1B),\ g \in P_{s_1}/B\} \;\cong\; \mathbb{P}^1_{\mathbb{C}}.$$

Hence we have

$$H^*\big((\pi_*\underline{\mathbb{Q}}_{\mathfrak{Y}})_{(B,s_1B)}\big) \;=\; \mathbb{Q} \oplus \mathbb{Q}[-2].$$

Similarly, we have

$$\pi^{-1}(B, s_1B) \;=\; \{(B, gB, B),\ g \in P_{s_1}/B\} \;\cong\; \mathbb{P}^1_{\mathbb{C}}.$$

Hence we have

$$H^*\big((\pi_*\underline{\mathbb{Q}}_{\mathfrak{Y}})_{(B,B)}\big) \;=\; \mathbb{Q}\oplus\mathbb{Q}[-2].$$

The following table gives the fibers of $\pi_*\mathrm{IC}(\mathfrak{Y}) = \pi_*\underline{\mathbb{Q}}_{\mathfrak{Y}}[6]$:

$\dim(\mathfrak{X}_w)$	w	-6	-5	-4
6	w_0	$\mathbb{Q}$	0	0
5	s_2s_1	$\mathbb{Q}$	0	0
5	s_1s_2	$\mathbb{Q}$	0	0
4	s_2	$\mathbb{Q}$	0	0
4	s_1	$\mathbb{Q}$	0	$\mathbb{Q}$
3	1	$\mathbb{Q}$	0	$\mathbb{Q}$

As $\pi_*\mathrm{IC}(\mathfrak{Y})$ is self-dual, this table shows that it is a perverse sheaf, but that it is not isomorphic to $\mathrm{IC}(\mathfrak{X}_{w_0})$. (Look at the line of s_1.) In fact, the Decomposition Theorem implies that $\pi_*\mathrm{IC}_{\mathfrak{Y}}$ is semisimple. Hence we have an isomorphism

$$\pi_*\mathrm{IC}(\mathfrak{Y}) \;\cong\; \mathrm{IC}(\mathfrak{X}_{w_0})\oplus\mathrm{IC}(\mathfrak{X}_{s_1}).$$

4.3. Type $\mathbf{B}_2$. We have

$$W \;=\; \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2 = s_2s_1s_2s_1\},$$

and $\dim\mathcal{B} = 4$. The Demazure resolutions are isomorphisms for elements of length ≤ 2. Let us consider elements of length 3. We denote by α and β the simple roots, such that $s_\alpha = s_1$, $s_\beta = s_2$.

4.3.1. $s_1s_2s_1$. To fix notations, we assume that the roots of the Borel B are the negative ones. Let U^+ be the unipotent subgroup of G whose roots are the positive roots. Let $u_\gamma : \Bbbk \xrightarrow{\sim} U_\gamma$ be isomorphisms, for $\gamma\in R^+$, which satisfy the following commutation relation:

$$u_\alpha(x)u_\beta(y)u_\alpha(-x)u_\beta(-y) \;=\; u_{\alpha+\beta}(xy)u_{2\alpha+\beta}(x^2y).$$

The subset $U^+B/B\subset\mathcal{B}$ is a dense open subset, isomorphic to U^+. We also have an isomorphism

$$U^+ \;\cong\; U_\alpha\times U_\beta\times U_{\alpha+\beta}\times U_{2\alpha+\beta}.$$

To study the singularities of $X_{s_1s_2s_1}$, it is sufficient to consider the intersection $X_{s_1s_2s_1}\cap U^+B/B$.

Consider the Demazure resolution

$$\varpi_{(s_1,s_2,s_1)} :\; Y_{(s_1,s_2,s_1)} \to X_{s_1s_2s_1}.$$

A generic point of $Y_{(s_1,s_2,s_1)}$ is of the form $[u_\alpha(x) : u_\beta(y) : u_\alpha(z)B/B]$. Its image in $\mathcal{B}$ is

$$u_\alpha(x)u_\beta(y)u_\alpha(z) \;=\; u_\alpha(x+z)u_\beta(y)u_{\alpha+\beta}(yz)u_{2\alpha+\beta}(yz^2).$$

We remark that $(yz)^2 = y\times(yz^2)$. Hence a generic point

$$u_\alpha(x_\alpha)u_\beta(x_\beta)u_{\alpha+\beta}(x_{\alpha+\beta})u_{2\alpha+\beta}(x_{2\alpha+\beta})\in U_\alpha\times U_\beta\times U_{\alpha+\beta}\times U_{2\alpha+\beta}$$

is in $X_{s_1s_2s_1}$ iff $x_{\alpha+\beta}^2 = x_\beta x_{2\alpha+\beta}$. It follows that $X_{s_1s_2s_1}\cap U^+B/B$ is included in the hypersurface of U^+ defined by the equation $x_{\alpha+\beta}^2 = x_\beta x_{2\alpha+\beta}$. As both subschemes are reduced, irreducible and of the same dimension, we deduce that $X_{s_1s_2s_1} \cap U^+B/B$ *is* the hypersurface of U^+ defined by the equation $x_{\alpha+\beta}^2 = x_\beta x_{2\alpha+\beta}$.

Consider the equation $f(x_\alpha, x_\beta, x_{\alpha+\beta}, x_{2\alpha+\beta}) = x_{\alpha+\beta}^2 - x_\beta x_{2\alpha+\beta}$. Then the locus where all the partial derivatives of f vanish is defined by $x_\beta = x_{\alpha+\beta} = x_{2\alpha+\beta} = 0$. It follows that the set of singular points of $X_{s_1s_2s_1}$ is X_{s_1}.

Now, let us compute $\mathrm{IC}(X_{s_1s_2s_1})$. Consider the ($G$-)Demazure resolution

$$\pi_{(s_1,s_2,s_1)} : \mathfrak{Y}_{(s_1,s_2,s_1)} \to \mathfrak{X}_{s_1s_2s_1}.$$

As for type $\mathbf{A}_2$, one easily checks that $\pi_{(s_1,s_2,s_1)}$ is an isomorphism over $\mathfrak{X}_{s_1s_2s_1} - \mathfrak{X}_{s_1}$, and that the fibers over $\mathfrak{X}_{s_1}$ are all isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Hence the fibers of $(\pi_{(s_1,s_2,s_1)})_*\mathrm{IC}(\mathfrak{Y}_{(s_1,s_2,s_1)})$ are given by the following table:

$\dim(\mathfrak{X}_w)$	w	-7	-6	-5
7	$s_1s_2s_1$	$\mathbb{Q}$	0	0
6	s_2s_1	$\mathbb{Q}$	0	0
6	s_1s_2	$\mathbb{Q}$	0	0
5	s_2	$\mathbb{Q}$	0	0
5	s_1	$\mathbb{Q}$	0	$\mathbb{Q}$
4	1	$\mathbb{Q}$	0	$\mathbb{Q}$

One deduces that

$$(\pi_{(s_1,s_2,s_1)})_*\mathrm{IC}(\mathfrak{Y}_{(s_1,s_2,s_1)}) \;\cong\; \mathrm{IC}(\mathfrak{X}_{s_1s_2s_1}) \oplus \mathrm{IC}(\mathfrak{X}_{s_1}),$$

and that

$$\mathrm{IC}(\mathfrak{X}_{s_1s_2s_1}) \;=\; \underline{\mathbb{Q}}_{\mathfrak{X}_{s_1s_2s_1}}[7].$$

In particular, the IC complex "does not detect" the singularities of $\mathfrak{X}_{s_1s_2s_1}$.

Correspondingly, in the Hecke algebra we have

$$C_{s_1}C_{s_2}C_{s_1} \;=\; C_{s_1s_2s_1} + C_{s_1},$$

and

$$C_{s_1s_2s_1} \;=\; t^{-3} \sum_{x \leq s_1s_2s_1} T_x.$$

4.3.2. $s_2s_1s_2$. One easily checks that the intersection $X_{s_2s_1s_2} \cap U^+B/B$ is the hypersurface of U^+ defined by the equation $x_{2\alpha+\beta} = x_\alpha x_{\alpha+\beta}$, where here we use the isomorphism

$$U^+ \;\cong\; U_\beta \times U_\alpha \times U_{\alpha+\beta} \times U_{2\alpha+\beta}.$$

Hence $X_{s_2s_1s_2}$ is smooth, and its IC complex is the shifted constant sheaf.

One also checks that the situation for the Demazure resolution is similar to that for $s_1s_2s_1$.

4.3.3. $s_1s_2s_1s_2$. Now, consider the longest elements $w_0 = s_1s_2s_1s_2$, and the Demazure resolution

$$\mathfrak{Y} := \mathfrak{Y}_{(s_1,s_2,s_1,s_2)}$$

of the (smooth) Schubert variety $\mathfrak{X}_{w_0} = \mathcal{B} \times \mathcal{B}$, and write π for $\pi_{(s_1,s_2,s_1,s_2)}$. The morphism π is an isomorphism over $\mathfrak{X}_{w_0} - \mathfrak{X}_{s_1s_2}$, and all the non-trivial fibers are isomorphic to two copies of $\mathbb{P}^1_{\mathbb{C}}$ glued along one point. The latter variety has a cell decomposition with one cell isomorphic to $\{\mathrm{pt}\}$, and two cells isomorphic to $\mathbb{A}^1$. Hence its comology is $\mathbb{Q} \oplus \mathbb{Q}^2[-2]$. The following table gives the fibers of $\pi_* \mathrm{IC}(\mathfrak{Y})$:

$\dim(\mathfrak{X}_w)$	w	-8	-7	-6
8	w_0	$\mathbb{Q}$	0	0
$\vdots$	$\vdots$	$\mathbb{Q}$	0	0
6	s_1s_2	$\mathbb{Q}$	0	$\mathbb{Q}^2$
5	s_2	$\mathbb{Q}$	0	$\mathbb{Q}^2$
5	s_1	$\mathbb{Q}$	0	$\mathbb{Q}^2$
4	1	$\mathbb{Q}$	0	$\mathbb{Q}^2$

It follows that we have

$$\pi_* \mathrm{IC}(\mathfrak{Y}) \;\cong\; \mathrm{IC}(\mathfrak{X}_{w_0}) \oplus \mathrm{IC}(\mathfrak{X}_{s_1s_2})^2.$$

4.4. Type $\mathbf{G}_2$. We will not make any computation in this case. We only mention that, according to [KL2, §1.6], all Schubert varieties in rank ≤ 2 (hence in particular in type $\mathbf{G}_2$) are rationally smooth. It follows that their IC complexes are shifted constant sheaves. (See [HTT, p. 211] for generalities on rational smoothness and relation with perverse sheaves.)

4.5. Type $\mathbf{A}_3$. Consider $G = \mathrm{SL}(4, \mathbb{C})$. The Weyl group W has three generators, denoted s_1, s_2, s_3. Here s_1 and s_3 commute. The flag variety has dimension 6.

In this case, all the Schubert varieties are smooth, except for two of them, corresponding to $s_2s_1s_3s_2$ and $s_1s_3s_2s_3s_1$. We will only consider these cases.

4.5.1. $s_2s_1s_3s_2$. One easily checks that

$$\begin{aligned} C_{s_2}C_{s_1}C_{s_3}C_{s_2} \;=\; t^{-4}\big(&T_{s_2s_1s_3s_2} + T_{s_2s_1s_3} + T_{s_2s_1s_2} + T_{s_2s_3s_2} + T_{s_1s_3s_2} + T_{s_2s_1} \\ &+ T_{s_1s_2} + T_{s_1s_3} + T_{s_2s_3} + T_{s_2s_3} + T_{s_1} + T_{s_3} + (t^2+1)T_{s_2} + (t^2+1)\big). \end{aligned}$$

Hence we have $C_{s_2s_1s_3s_2} = C_{s_2}C_{s_1}C_{s_3}C_{s_2}$.

Now, consider the Demazure resolution

$$\mathfrak{Y}^1 := \mathfrak{Y}_{(s_2,s_1,s_3,s_2)} = \mathcal{B} \times_{\mathcal{P}_{s_2}} \mathcal{B} \times_{\mathcal{P}_{s_1}} \mathcal{B} \times_{\mathcal{P}_{s_3}} \mathcal{B} \times_{\mathcal{P}_{s_2}} \mathcal{B}$$

of the Schubert variety $\mathfrak{X}^1$, where $\mathfrak{X}^1 := \mathfrak{X}_{s_2s_1s_3s_2}$. We write π for $\pi_{(s_2,s_1,s_3,s_2)}$. Then π is an isomorphism over $\mathfrak{X}^1 - \mathfrak{X}_{s_2}$. Moreover, we have

$$\pi^{-1}(B, s_2B) \;=\; \{(B, gB, gB, gB, s_2B),\ g \in P_{s_2}/B\} \;\cong\; \mathbb{P}^1_{\mathbb{C}}.$$

Similarly we have

$$\pi^{-1}(B, B) \;=\; \{(B, gB, gB, gB, B),\ g \in P_{s_2}/B\} \;\cong\; \mathbb{P}^1_{\mathbb{C}}.$$

The following table gives the fibers of $\pi_* \mathrm{IC}(\mathfrak{Y}^1) = \pi_*(\underline{\mathbb{Q}}_{\mathfrak{Y}}[10])$:

$\dim(\mathfrak{X}_w)$	w	-10	-9	-8
	$\mathcal{B}^2 - \mathfrak{X}^1$	0	0	0
10–7	$\mathfrak{X}^1 - \mathfrak{X}_{s_2}$	$\mathbb{Q}$	0	0
7	s_2	$\mathbb{Q}$	0	$\mathbb{Q}$
6	1	$\mathbb{Q}$	0	$\mathbb{Q}$

It follows from this table that $\pi_* \mathrm{IC}(\mathfrak{Y}^1)$ is a perverse sheaf, and even that

$$\pi_* \mathrm{IC}(\mathfrak{Y}^1) \ \cong \ \mathrm{IC}(\mathfrak{X}^1).$$

(In fact, in this case the morphism π is small.) In particular, in this case the IC complex detects the singularities.

More concretely, if $\mathcal{B}$ is identified with the variety of complete flags

$$\{0\} \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4,$$

and if $V_\bullet^0$ is the flag fixed by B, then $\overline{X_{s_2 s_1 s_3 s_2}}$ is identified with the subvariety of $\mathcal{B}$ defined by the conditions

$$V_1 \subset V_3^0, \quad V_0^1 \subset V_3.$$

4.5.2. $s_1 s_3 s_2 s_3 s_1$. Consider the Demazure resolution

$$\mathfrak{Y}^2 := \mathfrak{Y}_{(s_1,s_3,s_2,s_3,s_1)}$$

of the Schubert variety $\mathfrak{X}^2$, where $\mathfrak{X}^2 := \mathfrak{X}_{s_1 s_3 s_2 s_3 s_1}$. As usual, we write π for $\pi_{(s_1,s_3,s_2,s_3,s_1)}$. The morphism π is an isomorphism over $\mathfrak{X}^2 - \mathfrak{X}_{s_1 s_3}$. And all the non-trivial fibers are isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, whose cohomology is $\mathbb{Q} \oplus \mathbb{Q}^2[-2] \oplus \mathbb{Q}[-4]$. For example, we have

$$\pi^{-1}(B/B, B/B) \ = \\ \{(B/B, gB/B, ghB/B, ghB/B, gB/B, B/B), \ g \in P_{s_3}, \ h \in P_{s_1}\}.$$

The following table gives the fibers of $\pi_* \mathrm{IC}(\mathfrak{Y}^2) = \pi_* \underline{\mathbb{Q}}_{\mathfrak{Y}^2}[11]$:

$\dim(\mathfrak{X}_w)$	w	-11	-10	-9	-8	-7
	$\mathcal{B}^2 - \mathfrak{X}^2$	0	0	0	0	0
11–7	$\mathfrak{X}^1 - \mathfrak{X}_{s_1 s_3}$	$\mathbb{Q}$	0	0	0	0
8	$s_1 s_3$	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0	$\mathbb{Q}$
7	s_1	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0	$\mathbb{Q}$
7	s_3	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0	$\mathbb{Q}$
6	1	$\mathbb{Q}$	0	$\mathbb{Q}^2$	0	$\mathbb{Q}$

We deduce that

$$\pi_* \mathrm{IC}(\mathfrak{Y}^2) \ = \ \mathrm{IC}(\mathfrak{X}^2) \oplus \mathrm{IC}(\mathfrak{X}_{s_1 s_3})[1] \oplus \mathrm{IC}(\mathfrak{X}_{s_1 s_3})[-1],$$

and that the fibers of $\mathrm{IC}(\mathfrak{X}^2)$ are given by the following table:

$\dim(\mathfrak{X}_w)$	w	-11	-10	-9
	$\mathcal{B}^2 - \mathfrak{X}^2$	0	0	0
11–7	$\mathfrak{X}^1 - \mathfrak{X}_{s_1 s_3}$	$\mathbb{Q}$	0	0
8	$s_1 s_3$	$\mathbb{Q}$	0	$\mathbb{Q}$
7	s_1	$\mathbb{Q}$	0	$\mathbb{Q}$
7	s_3	$\mathbb{Q}$	0	$\mathbb{Q}$
6	1	$\mathbb{Q}$	0	$\mathbb{Q}$

Notice that, in this case, the resolution π is not semismall.

More concretely, if $\mathcal{B}$ is identified with the variety of complete flags

$$\{0\} \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4,$$

and if $V_\bullet^0$ is the flag fixed by B, then $X_{s_1s_2s_3s_2s_1}$ is identified with the subvariety of $\mathcal{B}$ defined by the conditions

$$V_2 \cap V_2^0 \neq \{0\}.$$

5. Other approaches and generalizations

5.1. Kazhdan-Lusztig. The first proof of Theorem 3.2.1 was due to Kazhdan and Lusztig ([KL2]), and was quite different from the one described in Section 3. The main step is the proof of pointwise purity of simple perverse sheaves on $\mathcal{B}$ (or rather on the similar variety defined over a finite field).

First, by general results explained in [BBD, §6], the dimension of fibers of simple $\mathbb{Q}$-perverse sheaves on $\mathcal{B}$ (or $\mathcal{B} \times \mathcal{B}$) is the same as the dimension of fibers of simple $\mathbb{Q}_l$-perverse sheaves on $\mathcal{B}_{\mathbb{F}_q}$ (or $\mathcal{B}_{\mathbb{F}_q} \times \mathcal{B}_{\mathbb{F}_q}$), for $q \gg 0$. Here $\mathcal{B}_{\mathbb{F}_q}$ is the flag variety defined over the finite field $\mathbb{F}_q$, and l does not divide q. Hence one can work with l-adic sheaves on $\mathcal{B}_{\mathbb{F}_q}$, and consider weights of the Frobenius. We use the definitions and notations from [BBD].

5.1.1. *Pointwise purity: definition.* Let X_0 be a scheme of finite type over $\mathbb{F}_q$, and let X be the scheme over $\overline{\mathbb{F}_q}$ obtained by extension of scalars. Let $\mathrm{Fr}_q : X \to X$ be the Frobenius morphism. Let $\mathcal{F}_0$ be a $\overline{\mathbb{Q}_l}$-sheaf on X_0, and $\mathcal{F}$ be the sheaf on X obtained by extension of scalars. Then $\mathcal{F}$ is naturally endowed with an isomorphism $F_q^* : \mathrm{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$. We denote by $F_{q^n}^*$ the n-th power of F_q^*.

Definition 5.1.1. (i) The sheaf $\mathcal{F}_0$ is said to be *pointwise pure* of weight $w \in \mathbb{Z}$ if for any $n \geq 1$ and any point x of X defined over $\mathbb{F}_{q^n}$ (i.e. fixed by Fr_q^n), the eigenvalues of the isomorphism $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_x$ induced by $F_{q^n}^*$ are algebraic numbers, all of whose complex conjugates have absolute value $q^{nw/2}$.

(ii) An object $\mathcal{F}_0$ of $D_c^b(X_0, \overline{\mathbb{Q}_l})$ is said to be *pointwise pure* of weight w if for any $i \in \mathbb{Z}$, the sheaf $\mathcal{H}^i(\mathcal{F}_0)$ is pointwise pure of weight $w + i$.

Pointwise purity is not a general property of simple perverse sheaves. However, simple perverse sheaves are pointwise pure for most of the situations encountered in Geometric Representation Theory.

5.1.2. *Pointwise purity on the flag variety.* Let us assume in this subsection that G, B, W are defined over $\mathbb{F}_q$, and let $X_w^{\mathbb{F}_q}$ be the Schubert cell considered over $\mathbb{F}_q$.

Theorem 5.1.2. *For any $w \in W$, the perverse sheaf* $\mathrm{IC}(X_w^{\mathbb{F}_q})$ *is pointwise pure.*

This result was first proved by Kazhdan and Lusztig, see [KL2, Theorem 4.2]. Their arguments are reproduced (in the language of Hodge modules), perhaps more clearly, in [HTT, Proposition 13.2.9]. The same arguments are used in [S1] to prove a slightly more general result, see [S1, Parabolic purity theorem]. In [G1], Ginzburg simplifies these arguments by using general results on weights contained in [BBD]. All these proofs are based on the fact that if $y \leq w$, ByB/B is, locally in X_w, the fixed points set of a contracting $\mathbb{C}^\times$-action. Finally, Haines gives a different proof of Theorem

5.1.2 in [Ha], based on the fact (also proved in [Ha]) that fibers of Demazure resolutions are paved by affine spaces.

In fact, let us mention that Kazhdan and Lusztig prove more precisely that the eigenvalues of the Frobenius action on $\mathcal{H}^i(\mathrm{IC}(X_w))_x$, where x is defined over $\mathbb{F}_{q^n}$, are equal to $q^{in/2}$. The proof of [Ha] also gives this more precise result (but not that of [G1]). Even more precisely, it is proved in [BGS, §4.4] that the action of the Frobenius on stalks of simple perverse sheaves is semisimple.

5.1.3. *End of the proof.* The end of the proof of Theorem 3.2.1 given by Kazhdan and Lusztig uses the Lefschetz fixed point formula (which allows to relate the trace of the Frobenius action on global intersection cohomology and traces of the Frobenius on stalks of IC sheaves), and Poincaré duality (which implies that a certain element of the Hecke algebra is fixed by the involution i). Let us remark that Poincaré duality is the "global counterpart" of the fact that Verdier duality fixes IC-sheaves. Hence the proof of the fact that a certain element is fixed by i relies on arguments similar to those given in Section 3.

Let us mention also that the arguments of [KL2] are (essentially) reproduced in [Ha, §5].

5.2. Lusztig-Vogan. In [LV], Lusztig and Vogan give a different realization of $\mathcal{H}_W$ in terms of perverse sheaves on $\mathcal{B}$, which allows them to give a different proof of Theorem 3.2.1. (In fact, they consider in [LV] a more general geometric situation, encountered in representation theory of real Lie groups. Let us mention also that the proofs of [LV] are not purely geometric.) More precisely, they consider a certain category $\mathcal{C}$ of sheaves on $\mathcal{B}_{\overline{\mathbb{F}_q}} \times \mathcal{B}_{\overline{\mathbb{F}_q}}$, which are certain pointwise pure, G-equivariant Weil sheaves[3] (see [LV, Definition 2.2] for the precise definition, which is not quite natural), and construct an isomorphism

$$\mathcal{H}_W \;\cong\; K^0(\mathcal{C}).$$

Under this isomorphism, the product on $\mathcal{H}_W$ corresponds to a convolution product similar to the one considered in Section 3, the multiplication by t corresponds to a Tate twist, and the involution i corresponds essentially to Verdier duality.

To deduce Theorem 3.2.1, they use *purity* of IC complexes on $\mathcal{B}_{\mathbb{F}_q} \times \mathcal{B}_{\mathbb{F}_q}$. This property (which does not imply pointwise purity, nor is implied by it) is a general property of simple perverse sheaves on schemes over $\mathbb{F}_q$.

5.3. Tanisaki.

5.3.1. *Definitions.* The most satisfactory geometric construction of $\mathcal{H}_W$ has been obtained by Tanisaki using the theory of mixed Hodge modules on $\mathcal{B} \times \mathcal{B}$. (Here, as in Section 3, we consider the *complex* flag variety.) We will not review the theory of Hodge modules here; the interested reader may consult [HTT, §8.3] for a detailled overview. Let us only recall that a mixed Hodge module is a $\mathcal{D}$-module endowed with extra structures, including a

[3]Recall that, if X_0 is defined over $\mathbb{F}_{||}$ and X is obtained by extension of scalars to $\overline{\mathbb{F}_q}$, a Weil sheaf on X is a $\overline{\mathbb{Q}_l}$-sheaf $\mathcal{F}$ together with an isomorphism $\Phi : \mathrm{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

good filtration. For the basics of $\mathcal{D}$-modules, see [Ri] and the references therein.

Let us denote by $\mathrm{MHM}^G(\mathcal{B} \times \mathcal{B})$ the abelian category of G-equivariant mixed Hodge modules on $\mathcal{B} \times \mathcal{B}$. We will consider the objects

$$\mathcal{M}_w^H := (j_w)_!^H \mathcal{O}^H_{G\cdot(B/B,wB/B)}, \qquad \mathcal{L}_w^H := \mathrm{IC}^H(\mathfrak{X}_w).$$

Here, $\mathcal{O}^H_{G\cdot(B/B,wB/B)}$ is the mixed Hodge module on $\mathfrak{X}_w$ associated to the natural $\mathcal{D}_{G\cdot(B/B,wB/B)}$-module $\mathcal{O}_{G\cdot(B/B,wB/B)}$, $(j_w)_!^H$ is the !-direct image for mixed Hodge modules, and $\mathrm{IC}^H(\mathfrak{X}_w)$ is the Hodge version of $\mathrm{IC}(\mathfrak{X}_w)$.

Next, one can define a (convolution) product on the Grothendieck group $K^0(\mathrm{MHM}^G(\mathcal{B} \times \mathcal{B}))$ by the following formula:

$$[\mathcal{M}_1] * [\mathcal{M}_2] \ := \ (-1)^{\dim(\mathcal{B})} \sum_{j \in \mathbb{Z}} (-1)^j [H^j\big((p_{1,3})_!^H r_H^*(\mathcal{M}_1 \boxtimes \mathcal{M}_2)\big)].$$

Here, $p_{1,3} : \mathcal{B}^3 \to \mathcal{B}^2$ is the projection on the first and third components, $r : \mathcal{B}^3 \to \mathcal{B}^4$ is defined by $r(a,b,c) = (a,b,b,c)$, and $(p_{1,3})_!^H$ and r_H^* are the (derived) !-direct and inverse images for Hodge modules. Note that this formula is again very similar to that used in Section 3[4].

Finally, let us denote by R the Grothendieck group of the category of mixed Hodge modules on a point. Note that we have an embedding

$$\mathbb{Z}[q, q^{-1}] \hookrightarrow R$$

given by $q^n \mapsto [\mathcal{O}^H_{\mathcal{B}\times\mathcal{B}}(-n)]$. (Here, (1) is the Tate twist.) The product defined above is R-bilinear.

5.3.2. *Hodge-theoretic description of $\mathcal{H}_W$.* The following result is due to Tanisaki ([Ta]). His arguments are reproduced in [HTT, §13.2]. We define $q := t^2$, and denote by $\mathcal{H}_W^q \subset \mathcal{H}_W$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra

$$\bigoplus_{w \in W} \mathbb{Z}[q, q^{-1}] \cdot T_w.$$

Theorem 5.3.1. *There exists an isomorphism of R-algebras*

$$F : \ K^0(\mathrm{MHM}^G(\mathcal{B} \times \mathcal{B})) \xrightarrow{\sim} R \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_W^q,$$

which satisfies

$$F(\mathcal{M}_w^H) = (-1)^{\ell(w)} T_w, \qquad F(\mathcal{L}_w^H) = (-t)^{\ell(w)} C_w.$$

Under this isomorphism, the involution i corresponds to $q^{\dim \mathcal{B}}$ times the morphism induced by Verdier duality.

[4]The only difference is the term $(-1)^{\dim \mathcal{B}}$. This difference can be explained the following considerations. In Section 3 it is natural to define the product so that the unit is $\underline{\mathbb{Q}}_{\mathfrak{X}_1}$. But the natural unit for the product on $K^0(\mathrm{MHM}^G(\mathcal{B}\times\mathcal{B}))$ is the class of the mixed Hodge module on $\mathfrak{X}_1$, which is associated with the perverse sheaf $\mathrm{IC}(\mathfrak{X}_1) = \underline{\mathbb{Q}}_{\mathfrak{X}_1}[\dim \mathcal{B}]$.

5.3.3. *Consequences.* From Theorem 5.3.1 easily following a result which is slightly weaker than Theorem 3.2.1 (but still sufficient to prove the Kazhdan-Lusztig conjecture on characters of simple modules over Lie(G), see [Ri]), namely equality

$$P_{y,w}(1) \;=\; (-1)^{\ell(w)} \cdot \sum_{j \in \mathbb{Z}} (-1)^j \dim(H^j(\mathrm{IC}(X_w)_y)).$$

This proof again uses purity of simple mixed Hodge modules.

Then, it is not difficult to deduce Theorem 3.2.1 from pointwise purity (in the sense of Hodge modules) of IC sheaves, see [HTT, Theorem 13.2.11]. This property can be proved exactly as for perverse sheaves over $\mathbb{F}_q$. One even obtains a complete description of restrictions of $\mathrm{IC}^H(\mathfrak{X}_w)$ to G-orbits as Hodge modules (and not only weights).

5.3.4. *Relation to the geometric description of the extended affine Hecke algebra.* Another advantage of this description of $\mathcal{H}_W$ is that it is possible to relate it to the description of the extended affine Hecke algebra $\widetilde{\mathcal{H}}$ associated with G (i.e. the Hecke algebra of the group $\widetilde{W}_{\mathrm{aff}}$ defined below) in terms of coherent sheaves on the Steinberg variety due to Kazhdan-Lusztig and Ginzburg (see [CG, Chapter 7]). Indeed the inclusion $\mathcal{H}_W \hookrightarrow \widetilde{\mathcal{H}}$ is induced by the functor which sends a mixed Hodge module on $\mathcal{B} \times \mathcal{B}$ to the associated graded of the underlying D-module with respect to the filtration (which is given as part of the structure of mixed Hodge module). This associated graded is a coherent sheaf on $T^*\mathcal{B} \times T^*\mathcal{B}$, supported on the Steinberg variety. For details, see [Ta].

5.4. Generalizations and related results.

5.4.1. *Affine flag manifold.* All the proofs mentioned above generalize to the case where $\mathcal{B}$ is replaced by the affine flag variety, and W is replaced by the affine Weyl group W_{aff}, see e.g. [KL2, §5]. In [KT], Kashiwara and Tanisaki even generalize some of the constructions of §5.3 to give a description of the intersection cohomology groups of the Schubert varieties in partial flag manifolds over symmetrizable Kac-Moody Lie algebras in terms of parabolic Kazhdan-Lusztig polynomials (introduced by Deodhar, see e.g. [S2]).

5.4.2. *Affine Grassmannian.* Let $W_{\mathrm{aff}} = W \ltimes Q$, resp. $\widetilde{W}_{\mathrm{aff}} = W \ltimes P$ be the semi-direct product of W with the root lattice Q, resp. the weight lattice P. It is well-known that W_{aff} is a Coxeter group, with a natural set of generators S_{aff} containing S. Let Ω be the normalizer of S_{aff} in $\widetilde{W}_{\mathrm{aff}}$. Then we have a natural isomorphism $\widetilde{W}_{\mathrm{aff}} \cong \Omega \ltimes W_{\mathrm{aff}}$ (induced by multiplication). We extend the length function ℓ and the Bruhat order $\leq$ of W_{aff} by setting for $w, w' \in W_{\mathrm{aff}}$ and $\omega, \omega' \in \Omega$

$$\ell(\omega w) \;=\; \ell(w),$$
$$\omega w \leq \omega' w' \quad \text{iff } \omega = \omega' \text{ and } w \leq w'.$$

With these conventions, the definition of Kazhdan-Lusztig elements C_w and Kazhdan-Lusztig polynomials $P_{y,w}(q)$ ($y, w \in \widetilde{W}_{\mathrm{aff}}$) generalize. For λ a dominant weight, we denote by n_λ the minimal length representative of $W \cdot \lambda \cdot W$ in $\widetilde{W}_{\mathrm{aff}}$.

Let $\mathrm{Gr} := \check{G}(\mathbb{C}((x)))/\check{G}(\mathbb{C}[[t]])$ be the affine Grassmannian of the semisimple group $\check{G}$ of adjoint type whose root system is dual to that of G (i.e. $\check{G}$ is the dual of the simply-connected cover of G in the sense of Langlands). There is a natural injection $P \hookrightarrow \mathrm{Gr}$. We denote by Gr^λ the $\check{G}(\mathbb{C}[[x]])$-orbit of λ.

In [Lu, §11], Lusztig explains that, for dominant weights λ, μ, the coefficients of P_{n_μ,n_λ} compute the dimension of cohomology spaces of the stalk at μ of the intersection cohomology object $\mathrm{IC}(\overline{\mathrm{Gr}^\lambda})$. In [Lu, Theorem 6.1], he also proves that $P_{n_\mu,n_\lambda}(1)$ is the dimension of the μ-weight space of the simple $\mathrm{Lie}(G)$-module with highest weight λ (again for dominant λ, μ). The main step is to prove a "q-analog" of Weyl's character formula. These results were the starting point for the study of the "Geometric Satake isomorphism", see [G2] (and in particular [G2, §5]).

For this situation, the pointwise purity results of [G1] and [BGS] can also be applied.

5.4.3. *Kazhdan-Lusztig polynomials.* In [Po], Polo proves, using Theorem 3.2.1, that any polynomial with non-negative integral coefficients and constant term 1 is a Kazhdan-Lusztig polynomial for certain (explicit) elements of a symmetric group.

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